DIFFERENTIAL SYSTEMS OF TYPE (1, 1) 
ON HERMITIAN SYMMETRIC SPACES AND THEIR SOLUTIONS

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Abstract. This paper concerns $G$-invariant systems of second order differential operators on irreducible Hermitian symmetric spaces $G/K$. The systems of type $(1, 1)$ are obtained from $K$-invariant subspaces of $p_+ \otimes p_-$. We show that all such systems can be derived from a decomposition $p_+ \otimes p_- = \mathcal{H}' \oplus \mathcal{L} \oplus \mathcal{H}'^c$. Here $\mathcal{L}$ gives the Laplace-Beltrami operator and $\mathcal{H} = \mathcal{H}' \oplus \mathcal{L}$ is the celebrated Hua system, which has been extensively studied elsewhere. Our main result asserts that for $G/K$ of rank at least two, a bounded real-valued function is annihilated by the system $\mathcal{L} \oplus \mathcal{H}'^c$ if and only if it is the real part of a holomorphic function. In view of previous work, one obtains a complete characterization of the bounded functions that are solutions for any system of type $(1, 1)$ which contains the Laplace-Beltrami operator.

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1. Introduction

Let $G/K$ be a non-compact irreducible Hermitian symmetric space of rank $r$. The algebra $\mathcal{D}(G/K)$ of left-$G$-invariant differential operators on $G/K$ has $r$ algebraically

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independent generators. $\mathbf{D}(G/K)$ contains no first degree operators and has only one second degree generator, the Laplace-Beltrami operator. In order to fully exploit the virtues of $G$-invariance in low degrees, one is lead naturally to consider invariant systems of differential operators.

An abstract formalism for such systems (on any homogeneous space) can be found in [BV]. (See also Chapter V, Section 4 in [H3].) Following [BV], a $G$-invariant system is determined by a representation of $K$ in some vector space $V$ together with a $K$-equivariant map $d : V^* \to \mathbf{D}(G)$ from $V^*$ to the algebra of left-$G$-invariant differential operators on $G$. The associated system maps smooth functions on $G/K$ to smooth sections in the vector bundle $G \times_K V$ over $G/K$. We loose no generality by assuming that $d$ is injective and can replace the data $(V, d)$ by the image $W = d(V^*)$, a $K$-invariant subspace in $\mathbf{D}(G)$. The corresponding system becomes a map $D_W$ from $C^\infty(G/K)$ to $\Gamma(G \times_K W^*)$. We will describe this construction in greater detail below in Section 2.

In the present context, there is a natural notion of type for systems $D_W$. Roughly speaking, we say that $D_W$ has type $(a, b)$ if each operator in $W$ has the form

$$\sum_{|\alpha| = a, |\beta| = b} c_{\alpha, \beta} \partial^\alpha \overline{\partial}^\beta$$

at the identity, where $\alpha, \beta$ are multi-indices and "$\partial$" denotes derivatives with respect to holomorphic coordinates in directions tangent to $G/K$. This will be made precise below.

The holomorphic (type $(a, 0)$) and anti-holomorphic (type $(0, b)$) systems on irreducible Hermitian symmetric spaces are completely classified in view of results of Johnson [J]. In general, the classification of all systems of specified type $(a, b)$ on a given $G/K$ reduces to a problem in Invariant Theory. Our focus here is the systems of type $(1, 1)$, determined by $K$-invariant subspaces in $p^+ \otimes p^-$. To our knowledge, such systems have not been the subject of any systematic study.

Proposition 4.2, formulated below, classifies the systems of type $(1, 1)$ on any non-compact irreducible Hermitian symmetric space. In general, the possibilities are quite limited. Apart from the Laplace-Beltrami operator, these include the so-called Hua system. This system, which we denote by $D_H$, is given by a canonical $K$-invariant subspace $\mathcal{H}$ in $p_+ \otimes p_-$. The action of $K$ on $\mathcal{H}$ is equivalent to its complexified adjoint representation on $\mathfrak{g} \subset \mathfrak{p}$. The space $\mathcal{H}$ further decomposes under the action of $K$ as

$$\mathcal{H} = \mathcal{L} \oplus \mathcal{H}'$$

where $\mathcal{L}$ corresponds to the (necessarily one dimensional) center in $\mathfrak{g} \subset \mathfrak{p}$ and $\mathcal{H}'$ to the (semi-simple) derived algebra $\mathfrak{g}' \subset \mathfrak{p}$. If $G/K$ has type $A$ III and rank at least two then $\mathfrak{g}'$ has two simple factors and $\mathcal{H}' = \mathcal{H}'_1 \oplus \mathcal{H}'_2$ is the sum of two irreducible subspaces. In all other cases $\mathfrak{g}'$ is simple and hence $\mathcal{H}'$ is irreducible. The one dimensional space $\mathcal{L}$ consists of all multiples of the Laplace-Beltrami operator, which we also denote by $\mathcal{L}$, the meaning being clear from the context.
One can also consider the subspace $\mathcal{H}^c$ orthogonal to $\mathcal{H}$ in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. This is necessarily $K$-invariant and we call $D_{\mathcal{H}^c}$ the complementary Hua system. When $G/K$ has rank one, $\mathcal{H}^c = 0$. We will show that for $\text{rank}(G/K) \geq 2$, $\mathcal{H}^c$ is always non-zero and $K$-irreducible. We will provide two additional abstract characterizations of $\mathcal{H}^c$:

- $\mathcal{H}^c$ is the kernel of the linear map $\mathfrak{p}_+ \otimes \mathfrak{p}_- \to \mathfrak{k}_C$ given by the Lie bracket.
- $\mathcal{H}^c$ is the Cartan component in the tensor product $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ of the irreducible $K$-modules $\mathfrak{p}_\pm$.

These facts support the viewpoint that the complementary Hua system is a natural object for study. For the classical families of non-compact irreducible Hermitian symmetric spaces, concrete descriptions are given in Section 3 for the spaces $\mathcal{H}$ and $\mathcal{H}^c$.

The solutions for a $G$-invariant system $D_W$ are of particular interest. We say that a smooth function $f$ on $G/K$ is $W$-harmonic when $D_Wf = 0$. The aim of this paper is to characterize such functions for each invariant system of type $(1, 1)$ which contains the Laplace-Beltrami operator $\mathcal{L}$.

In [JK] it is shown that for $G/K$ of tube type, a smooth bounded function is $\mathcal{H}$-harmonic if and only if it is a Poisson-Szegö integral over the Shilov boundary. It was previously known (see [Hua]) that for $G/K = SU(n, n)/S(U(n) \times U(n))$, Poisson-Szegö integrals $f$ necessarily satisfy $D_{\mathcal{H}}f = 0$. Hua harmonicity on tube domains is also the subject of [L]. The Hua system for non-tube domains has been studied in [BBDHPT] and [B]. In this context, the real valued functions $f$ on $G/K$ satisfying $D_{\mathcal{H}}f = 0$ are the pluriharmonic functions [B]. That is, $f$ is the real part of a holomorphic function on $G/K$. To describe Poisson-Szegö integrals on general non-tube domains a third order system $\mathcal{B}V$ of type $(2, 1)$ is needed [BV].

Our principal object of study is the system $D_{\tilde{\mathcal{H}}^c}$, where

$$\tilde{\mathcal{H}}^c = \mathcal{L} \oplus \mathcal{H}^c.$$  

That is, we augment the complementary Hua system to include the Laplace-Beltrami operator. This ensures that $\tilde{\mathcal{H}}^c$-harmonic functions are harmonic in the usual sense. Theorem 5.3 below asserts that for $\text{rank}(G/K) \geq 2$ a bounded real valued function on $G/K$ is $\tilde{\mathcal{H}}^c$-harmonic if and only if it is pluriharmonic. This is the main result in the current work.

Our proof of Theorem 5.3 is based essentially on an interplay between the “$G/K$”-picture and “$S$” - picture of a Hermitian symmetric space, $S$ being a solvable Lie group acting simply transitively on the corresponding Siegel domain $\mathcal{D}$. In Section 6 we use classical structure theory for Hermitian symmetric spaces to exhibit some operators on $G/K$ that belong to the system $\mathcal{L} \oplus \mathcal{H}^c$. In Section 7 we leave the “$G/K$-picture” and express these operators in terms of $S$. This allows us to apply techniques and results from [BBDHPT] and [B] to the proof of Theorem 5.3 in Section 8. We show, in particular, that a bounded function $f$ annihilated by $\mathcal{L} \oplus \mathcal{H}^c$ is a Poisson-Szegö integral. In the tube case, it follows from [JK] that $f$ is $\mathcal{H}$-harmonic.
and we immediately conclude that \( f \) is pluriharmonic, since \( \mathcal{H}' \oplus \mathcal{L} \oplus \mathcal{H}^c = p_+ \otimes p_- \).

In the non-tube case, in view of [BV], \( f \) is annihilated by the third order system \( BV \).

We use this fact and a technique from [B] to complete the proof for non-tube spaces.

As explained in Section 5, Theorem 5.3 together with previously known results concerning the Hua system yields a complete characterization for the bounded solutions of all possible type \((1, 1)\) systems which contain the Laplace-Beltrami operator.

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2. Invariant systems of differential operators

Given a Lie group \( G \), we let \( D(G) \) (\( \cong \mathcal{U}(\mathfrak{g}^C) \)) denote the algebra of left-\( G \)-invariant differential operators on \( G \). The group \( G \) acts on \( D(G) \) via the adjoint representation:

\[
\text{Ad}(g)X(f) = X(f \circ r_g^{-1}) \circ r_g \quad \text{for} \; g \in G, \; X \in D(G), \; f \in C^\infty(G),
\]

where \( r_g : G \to G \) denotes right multiplication. We recall that the symmetrization map

\[
\lambda : S(\mathfrak{g}^C) \to D(G)
\]

is a canonical \( \text{Ad}(G) \)-equivariant vector space isomorphism from the symmetric algebra on the complexified Lie algebra \( \mathfrak{g}^C \) of \( G \) to \( D(G) \). We refer the reader to §4 in Chapter II of [H2] for details.

Now suppose \( K \) is a closed Lie subgroup of \( G \). For \( f \in C^\infty(G/K) \), we let \( \tilde{f} \in C^\infty(G) \) denote the map

\[
\tilde{f}(g) = f(gK).
\]

If \( X \in D(G) \) is \( \text{Ad}(K) \)-invariant then \( X \) yields a left-\( G \)-invariant differential operator on \( G/K \) (also denoted \( X \)) via the rule

\[
(Xf)\tilde{} = X\tilde{f}.
\]

More generally, an \( \text{Ad}(K) \)-invariant subspace \( W \) of \( D(G) \) yields a left-\( G \)-invariant system \( D_W \) of differential operators on \( G/K \) as explained below.

To describe \( D_W \), one forms the (complex) vector bundle \( G \times_K W^* \) associated to the principle bundle \( G \to G/K \) via the representation \( \text{Ad}^* : K \to GL(W^*) \) contragredient to \( \text{Ad}|_K \) on \( W \) \((\text{Ad}^*(k)\alpha = \alpha \circ \text{Ad}(k^{-1}))\). The total space is

\[
G \times_K W^* = (G \times W^*)/\sim
\]

where \( \sim \) is the equivalence relation on \( G \times W^* \) given by

\[
(gk, \alpha) \sim (g, \text{Ad}^*(k)\alpha)
\]

for all \( k \in K \).

The projection map \( \pi : G \times_K W^* \to G/K \) is just

\[
\pi([g, \alpha]) = gK,
\]
where \([g, \alpha] \in G \times_K W^*\) denotes the equivalence class of \((g, \alpha) \in G \times W^*\). We let 
\[ \Gamma(G \times_K W^*) \]
 denote the space of smooth sections in the bundle \(G \times_K W^*\). Smooth sections \(s \in \Gamma(G \times_K W^*)\) are in one-to-one correspondence with smooth maps \(\tilde{s} : G \to W^*\) satisfying
\[
(2.1) \quad \tilde{s}(gk) = \text{Ad}^*(k^{-1})\tilde{s}(g) = \tilde{s}(g) \circ \text{Ad}(k),
\]
for \(g \in G, k \in K\). The correspondence \(s \leftrightarrow \tilde{s}\) is given by
\[
s(gK) = [g, \tilde{s}(g)].
\]

**Definition 2.2.** The system \(D_W\) of differential operators on \(G/K\) determined by an \(\text{Ad}(K)\)-invariant subspace \(W \subset \mathcal{D}(G)\) is the map
\[
D_W : C^\infty(G/K) \to \Gamma(G \times_K W^*)
\]
defined via
\[
(D_W f) \tilde{\varphi}(g)(X) = X(\tilde{f})(g),
\]
for \(f \in C^\infty(G/K), g \in G, X \in W\).

To justify Definition 2.2 one verifies (easily) that the map \((D_W f)\tilde{\varphi}\) satisfies the \(K\)-equivariance property (2.1). The \(\text{Ad}(K)\)-invariance of \(W\) is needed here. When \(W\) is finite dimensional we can write
\[
(D_W f) \tilde{\varphi}(g) = \sum_{j=1}^n X_j(\tilde{f})(g)X_j^*,
\]
where \(\{X_j : j = 1, \ldots, n\}\) is any basis for \(W\) and \(\{X_j^*\}\) is the dual basis for \(W^*\). There is a natural left action of \(G\) on \(\Gamma(G \times_K W^*)\):
\[
(s \circ L_g) \tilde{\varphi}(h) = \tilde{s}(gh).
\]
The system \(D_W\) is left-\(G\)-invariant in the sense that
\[
(D_W f) \circ L_g = D_W(f \circ \ell_g),
\]
where \(\ell_g\) is left multiplication by \(g\) on \(G/K\). This follows immediately from the fact that each operator \(X \in W\) is left-\(G\)-invariant.

It is of interest to study zeros for systems of the form \(D_W\). Observe that \(f \in C^\infty(G/K)\) satisfies \(D_W f = 0\) if and only if \(X \tilde{f} = 0\) for all \(X \in W\). In this case, we will say that \(f\) is a \(W\)-harmonic function and often write \(W(f) = 0\) in place of \(D_W f = 0\).

### 2.1. Invariant systems on symmetric spaces.

Now suppose that \(G/K\) is a symmetric space of non-compact type and let
\[
g = \mathfrak{k} \oplus \mathfrak{p}
\]
denote the Cartan decomposition for the Lie algebra \(g\) of \(G\). Since \(\mathfrak{p}\) is \(\text{Ad}(K)\)-invariant, we have
\[
\mathcal{D}(G) = \mathcal{D}(G) \mathfrak{k} \oplus \lambda(S(p_C)),
\]
where \( \lambda : S(\mathfrak{g}_C) \to \mathbf{D}(G) \) is the symmetrization map. (See Lemma 4.7 in Chapter II of [H2].) Both \( \mathbf{D}(G)\mathfrak{k} \) and \( \lambda(S(\mathfrak{p}_C)) \) are Ad(\( K \))-invariant subspaces of \( \mathbf{D}(G) \), because \( \lambda \) is Ad(\( G \))-equivariant. For \( f \in C^\infty(G/K) \) one has \( X \tilde{f} = 0 \) for all \( X \in \mathbf{D}(G)\mathfrak{k} \). Thus we obtain:

Lemma 2.3. Let \( W \) be an Ad(\( K \))-invariant subspace of \( \mathbf{D}(G) \) and \( W' \) be the projection of \( W \) onto \( \lambda(S(\mathfrak{p}_C)) \) with respect to the decomposition \( \mathbf{D}(G) = \mathbf{D}(G)\mathfrak{k} \oplus \lambda(S(\mathfrak{p}_C)) \). Then \( W' \) is Ad(\( K \))-invariant, and for any \( f \in C^\infty(G/K) \):

\[
  f \text{ is } W\text{-harmonic if and only if } f \text{ is } W'\text{-harmonic.}
\]

Lemma 2.3 shows that if we wish to study the zeros of invariant systems of differential operators on \( G/K \), we can restrict attention to systems obtained from Ad(\( K \))-invariant subspaces \( W \) of \( \lambda(S(\mathfrak{p}_C)) \).

2.2. Invariant systems on Hermitian symmetric spaces. Next suppose that \( G/K \) is a Hermitian symmetric space of non-compact type. The complex structure on \( G/K \) yields an almost complex structure \( J \) on \( T_{eK}(G/K) \cong \mathfrak{p} \). This extends to a complex linear map \( J : \mathfrak{p}_C \to \mathfrak{p}_C \) and one has

\[
  \mathfrak{p}_C = \mathfrak{p}_+ \oplus \mathfrak{p}_-
\]
where \( \mathfrak{p}_\pm \) are the \((\pm i)\)-eigenspaces for \( J \). The spaces \( \mathfrak{p}_\pm \) are Ad(\( K \))-invariant abelian subalgebras of \( \mathfrak{g}_C \).

In this context, the canonical algebra isomorphism \( S(\mathfrak{p}_C) \cong S(\mathfrak{p}_+) \otimes S(\mathfrak{p}_-) \) is also an isomorphism of \( K \)-modules. (The group \( K \) acts on \( S(\mathfrak{p}_C) \) and \( S(\mathfrak{p}_\pm) \) by symmetric powers of the Adjoint representation.) It will here be convenient to replace the symmetrization map \( \lambda : S(\mathfrak{p}_C) \to \mathbf{D}(G) \) by

\[
  \lambda^\otimes : S(\mathfrak{p}_+) \otimes S(\mathfrak{p}_-) \to \mathbf{D}(G), \quad \lambda^\otimes = \lambda \otimes \lambda.
\]
Explicitly,

\[
  \lambda^\otimes((X_1 \cdots X_a) \otimes (\overline{Y}_1 \cdots \overline{Y}_b)) = \frac{1}{a!b!} \sum_{\sigma \in S_a, \sigma' \in S_b} X_{\sigma(1)} \cdots X_{\sigma(a)} \overline{Y}_{\sigma'(1)} \cdots \overline{Y}_{\sigma'(b)};
\]
for \((X_1 \cdots X_a) \otimes (\overline{Y}_1 \cdots \overline{Y}_b) \) in \( S^a(\mathfrak{p}_+) \otimes S^b(\mathfrak{p}_-) \). The map \( \lambda^\otimes \) is Ad(\( K \))-equivariant and Lemma 2.3 remains true if we replace \( \lambda(S(\mathfrak{p}_C)) \) by \( \lambda^\otimes(S(\mathfrak{p}_+) \otimes S(\mathfrak{p}_-)) \).

Definition 2.4. If \( W \) is a \( K \)-invariant subspace of \( \lambda^\otimes(S^a(\mathfrak{p}_+) \otimes S^b(\mathfrak{p}_-)) \) then we say that \( D_W \) is a system of type \((a, b)\) on \( G/K \).

Thus if \( D_W \) has type \((a, b)\) then each element of \( W \) is a linear combination of terms of the form \( X_1 \cdots X_a \overline{Y}_1 \cdots \overline{Y}_b \) where \( X_i \in \mathfrak{p}_+ \) and \( \overline{Y}_j \in \mathfrak{p}_- \).
3. Preliminaries on Hermitian symmetric spaces and Siegel domains

For the remainder of this paper, $G/K$ will denote a Hermitian symmetric space of non-compact type. For our purposes we can assume, moreover, that $G/K$ is irreducible, since any Hermitian symmetric space is a direct product of irreducible factors. Thus $G$ is a connected non-compact simple Lie group with trivial center and $K$ is a maximal compact subgroup of $G$ with center analytically isomorphic to $\mathbb{T}$. (See Theorem 6.1 and Proposition 6.2 of Chapter VIII in [H1].)

In this section we collect notation and background material concerning the structure of $G/K$ and its realizations as bounded and unbounded domains. For details we refer the reader to any of the standard references. See for example, [H1], [KW] and [W].

3.1. Algebraic preliminaries. As in the preceding section, $g = \mathfrak{k} \oplus \mathfrak{p}$ denotes the Cartan decomposition for the Lie algebra of $G$ and $\mathfrak{p}_C = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ is the eigenspace decomposition determined by the complex structure. We have

$$[\mathfrak{k}_C, \mathfrak{p}_C] \subset \mathfrak{p}_C, \quad [\mathfrak{p}_+, \mathfrak{p}_+] = 0 = [\mathfrak{p}_-, \mathfrak{p}_-], \quad [\mathfrak{p}_+, \mathfrak{p}_-] = \mathfrak{k}_C.$$ 

The Lie algebra $u = \mathfrak{k} \oplus i\mathfrak{p}$ is a compact real form of $g_C$. For $Z \in g_C$ we let $\overline{Z}$ and $\tau(Z)$ denote the complex conjugates for $Z$ with respect to the real forms $g$ and $u$. The two conjugation operators are related via

$$\tau(\theta(Z)) = Z = \theta(\tau(Z))$$

where $\theta$ is the complexified Cartan involution. If $B$ is the Killing form on $g_C$, then the bilinear form defined by

$$B_\tau(X, Y) = -B(X, \tau Y)$$

is a positive definite Hermiian inner product on $g_C$. Recall that $J$ denotes the almost complex structure on the tangent space $\mathfrak{p}$ to $G/K$ at $eK$ and its complexification $\mathfrak{p}_C \to \mathfrak{p}_C$. Let $\mathfrak{c}$ denote the (one dimensional) center of $\mathfrak{k}$. It is a key fact that there exists an element $Z_0 \in \mathfrak{c}$ with

$$J = ad(Z_0)|_{\mathfrak{p}_C}.$$ 

Choose a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{k}$. Then $\mathfrak{h}_C$ is a Cartan subalgebra of $g_C$. Define $\Delta$ to be the system of roots of $g_C$ with respect to $\mathfrak{h}_C$. Then each root $\alpha \in \Delta$ is real valued on $i\mathfrak{h}$. We specify an ordering on $\Delta$ as follows: for two roots $\alpha$, $\beta$ we say that $\alpha$ is bigger than $\beta$ when $-i(\alpha - \beta)(Z_0) > 0$. In this way we obtain the sets $\Delta^+$ and $\Delta^-$ of positive and negative roots.

Each root space $g^\alpha$ is contained either in $\mathfrak{k}_C$ or in $\mathfrak{p}_C$. In the first case $\alpha$ is called compact and in the second case noncompact. We write

$$\Delta = C \cup Q$$

where $C$ is the set of compact roots and $Q$ is the set of noncompact roots and let $Q^\pm = \Delta^\pm \cap Q$ denote the sets of positive and negative noncompact roots.
For each $\alpha \in \Delta$ we associate elements $H_{\alpha}$, $E_{\alpha}$ and $E_{-\alpha}$ which span a subalgebra of $g_{\mathbb{C}}$, isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. We do this in a standard way as follows. The Killing form $B$ is positive definite on $i\mathfrak{h}$. Thus for each $\alpha \in \Delta$ there is a unique element $\tilde{H}_{\alpha} \in i\mathfrak{h}$ for which

$$\alpha(H) = B(H, \tilde{H}_{\alpha}).$$

For roots $\alpha, \beta \in \Delta$ let

$$(\alpha, \beta) = B(\tilde{H}_{\alpha}, \tilde{H}_{\beta})$$

and set

$$H_{\alpha} = \frac{2}{(\alpha, \alpha)} \tilde{H}_{\alpha},$$

so that $\alpha(H_{\alpha}) = 2$.

Now for $\alpha \in \Delta^+$ choose $E_{\alpha} \in g^\alpha$ with $B_\tau(E_{\alpha}, E_{\alpha}) = 2/(\alpha, \alpha)$ and set $E_{-\alpha} = -\tau(E_{\alpha})$. With these conventions, we have

$$[E_{\alpha}, E_{-\alpha}] = H_{\alpha}, \quad [H_{\alpha}, E_{\alpha}] = 2E_{\alpha}, \quad [H_{\alpha}, E_{-\alpha}] = -2E_{-\alpha},$$

$$\overline{E}_{\alpha} = E_{-\alpha} \quad \text{for } \alpha \in Q, \quad \overline{E}_{-\alpha} = -E_{-\alpha} \quad \text{for } \alpha \in C,$$

and

$$\mathfrak{p}_+ = \sum_{\alpha \in Q^+} g^\alpha = \sum_{\alpha \in Q^+} \mathbb{C}E_{\alpha},$$

$$\mathfrak{p}_- = \sum_{\alpha \in Q^+} g^{-\alpha} = \sum_{\alpha \in Q^+} \mathbb{C}E_{-\alpha}.$$  

(3.1)

For $\alpha \in Q^+$, let

$$X_{\alpha} = E_{\alpha} + E_{-\alpha},$$

$$Y_{\alpha} = i(E_{\alpha} - E_{-\alpha}).$$

Then the set $\{X_{\alpha}, Y_{\alpha}\}_{\alpha \in Q^+}$ is a basis for the real vector space $\mathfrak{p}$. One has

$$\mathcal{J}X_{\alpha} = Y_{\alpha},$$

$$\mathcal{J}Y_{\alpha} = -X_{\alpha},$$

(3.3)

$$E_{\alpha} = \frac{1}{2}(X_{\alpha} - iY_{\alpha}),$$

$$E_{-\alpha} = \frac{1}{2}(X_{\alpha} + iY_{\alpha}).$$

3.2. **Restricted roots.** Two roots $\alpha, \beta \in \Delta$ are called strongly orthogonal if neither $\alpha + \beta$ nor $\alpha - \beta$ are roots. This implies orthogonality in the usual sense: $(\alpha, \beta) = 0$.

Let

$$\Gamma = \{\gamma_1, \ldots, \gamma_r\} \subset Q^+$$

(3.4)
be a maximal set of strongly orthogonal positive noncompact roots. Then
\[(3.5) \quad a = \sum_{\gamma \in \Gamma} \mathbb{R}X_{\gamma},\]
is a maximal abelian subalgebra of \(p\) and \(r = \dim(a)\) is the rank of \(G/K\).

Take \(h^-\) to be the real span of the elements \(iH_\gamma\) \((\gamma \in \Gamma)\), and \(h^+\) to be the orthogonal complement of \(h^-\) in \(h\) via the Killing form \(B\):
\[(3.6) \quad h^- = \sum_{\gamma \in \Gamma} \mathbb{R}iH_\gamma, \quad h = h^- \oplus_B h^+.
\]

For \(\alpha, \beta \in \Delta\) write \(\alpha \sim \beta\) if and only if \(\alpha|_{h^-} = \beta|_{h^-}\) and define:
\[(3.7) \begin{align*}
C_i &= \{ \alpha \in C : \alpha \sim -\frac{1}{2}\gamma_i \} \quad \text{for } i = 1, \ldots, r, \\
C_{ij} &= \{ \alpha \in C : \alpha \sim \frac{1}{2}(\gamma_j - \gamma_i) \} \quad \text{for } 1 \leq i < j \leq r, \\
Q_i &= \{ \alpha \in Q : \alpha \sim \frac{1}{2}\gamma_i \} \quad \text{for } i = 1, \ldots, r, \\
Q_{ij} &= \{ \alpha \in Q : \alpha \sim \frac{1}{2}(\gamma_i + \gamma_j) \} \quad \text{for } 1 \leq i < j \leq r.
\end{align*}\]

Some important properties of the above sets are contained in the following theorem:

**Theorem 3.8** (Restricted Roots Theorem, [H3]). The map \(\alpha \mapsto \gamma_i + \alpha\) is a bijection of \(C_i\) onto \(Q_i\) (for \(1 \leq i \leq r\)) and \(C_{ij}\) onto \(Q_{ij}\) (for \(1 \leq i < j \leq r\)). \(Q^+\) is the disjoint union of the sets \(\Gamma, Q_i, Q_{ij}\).

In addition, we will need these facts:

- All elements \(\alpha \in Q^+\) have a common length \((\alpha, \alpha)^{\frac{1}{2}}\).
- For \(\alpha, \beta\) in \(Q^+\), \(\alpha(H_\beta) = \beta(H_\alpha)\).
- The sets \(Q_{ij}\) have a common cardinality for \(1 \leq i < j \leq r\). Likewise, the sets \(Q_i, \ldots, Q_r\) have a common cardinality. We let \(q_1\) denote the common cardinality of the \(Q_i\)’s and \(q_2\) the common cardinality of the \(Q_{ij}\)’s.
- For \(\alpha \in Q_{ij}\) set \(\tilde{\alpha} = \gamma_i + \gamma_j - \alpha\).

Proposition 8 in [L] shows that \(\tilde{\alpha}\) is a noncompact positive root. So clearly \(\tilde{\alpha} \in Q_{ij}\).

### 3.3. Harish-Chandra realization.

Let \(G_C\) denote the adjoint group for \(g_C\) and \(K_C\) be the analytic subgroup corresponding to \(k_C\). The analytic subgroups of \(G_C\) corresponding to subalgebras \(p_+\), \(p_-\) will be denoted by \(P_+\) and \(P_-\) respectively. They are abelian. The exponential map from \(p_\pm\) to \(P_\pm\) is biholomorphic, so \(P_\pm\) is biholomorphically equivalent with \(\mathbb{C}^n\) for some \(n\).
The mapping \((p_1, k, p_2) \mapsto p_1 kp_2\) is a diffeomorphism of \(P_+ \times K_C \times P_-\) onto an open submanifold of \(G_C\) containing \(G\). For \(g \in G\) let \(p_+(g)\) denote the unique element in \(p_+\) such that \(g \in \exp(p_+(g))K_CP_-\). Then \(p_+(g) = p_+(gk)\) and \(p_+\) is a diffeomorphism of \(G/K\) onto a bounded domain \(D \subset p_+\). \(G\) acts biholomorphically on \(D\) by \(g \cdot p_+(\tilde{g}) = p_+(g\tilde{g})\). Let \(o = p_+(e)\), then \(D\) is the \(G\)-orbit of \(o\) and the group \(K\) is the stabilizer of the point \(o\). This is the Harish–Chandra embedding and in fact realizes \(G/K\) as a bounded symmetric domain.

3.4. **Siegel domains.** Let \(D\) denote the bounded realization of \(G/K\) described above. Set 
\[ X_\Gamma = \sum_{\gamma \in \Gamma} X_\gamma = \sum_{\gamma \in \Gamma} (E_\gamma + E_{-\gamma}), \]
where \(E_\Gamma = \sum_{\gamma \in \Gamma} E_\gamma\), and define an element of \(G_C\) called the Cayley transform:
\[ c = \exp\left(\frac{\pi}{4} iX_\Gamma\right). \]
Let
\[ ^cG = Ad(c)G, \]
\[ ^cK = Ad(c)K, \]
\[ ^cG = Ad(c)g. \]
For \(g \in G\), \(c \exp(p_+(g)) \in P_+^C K_CP_-\) and by [KW], the mapping \(x \mapsto c \cdot x\), where
\[ c \cdot p_+(g) = p_+(c \exp p_+(g)), \]
defines a biholomorphism of \(D\) onto a domain \(^cD \subset p_+\). Clearly, \(^cD\) is the orbit of the point \(c \cdot o = iE_\Gamma\) under the action of the group \(^cG\) and \(^cK\) is the isotropy group of \(iE_\Gamma\).

It was proved in [KW] that \(^cD\) is a Siegel domain. We briefly recall the definition and notation of Siegel domains. The reader is referred to the book of J. Faraut and A. Koranyi [FK] for more details.

Let \(V\) be a Euclidean Jordan algebra and \(\Omega\) be an irreducible symmetric cone contained in \(V\). We denote by \(L(x)\) the self-adjoint endomorphism of \(V\) given by left multiplication by \(x\), i.e. \(L(x)y = xy\). We fix a Jordan frame \(\{c_1, \ldots, c_r\}\) in \(V\). The Peirce decomposition of \(V\) related to the Jordan frame \(\{c_1, \ldots, c_r\}\) ([FK], Theorem IV.2.1) may be written as
\[ V = \bigoplus_{1 \leq i \leq j \leq r} V_{ij}. \]
It is given by the common diagonalization of the self-adjoint endomorphisms \(L(c_j)\) with respect to their only eigenvalues \(0, \frac{1}{2}, 1\). In particular \(V_{jj} = \mathbb{R} c_j\) is the eigenspace of \(L(c_j)\) related to 1, and, for \(i < j\), \(V_{ij}\) is the intersection of the eigenspaces of \(L(c_i)\) and \(L(c_j)\) related to \(\frac{1}{2}\). All \(V_{ij}\), for \(i < j\), have the same dimension \(d\).
Suppose that we are given a complex vector space $\mathcal{Z}$ and a Hermitian symmetric bilinear mapping

$$\Phi : \mathcal{Z} \times \mathcal{Z} \mapsto V^C.$$ 

We assume that

- $\Phi(\zeta, \zeta) \in \Omega$, $\zeta \in \mathcal{Z}$,
- $\Phi(\zeta, \zeta) = 0$ implies $\zeta = 0$.

The associated Siegel domain is then

\begin{equation}
\tilde{D} = \{(\zeta, z) \in \mathcal{Z} \times V^C : \Im z - \Phi(\zeta, \zeta) \in \Omega\}.
\end{equation}

One says that $\tilde{D}$ is of type I (or has tube type) when $\mathcal{Z} = \{0\}$. Otherwise, $\tilde{D}$ is said to be of type II (non-tube type).

The data $V$, $\mathcal{Z}$ and $\Phi$ can be defined in terms of some subspaces of $\mathfrak{g}_C$ so that

$$p_+ = \mathcal{Z} \times V^C \quad \text{and} \quad \tilde{D} = \mathcal{D}.$$

(For details we refer to [KW], [B]). Moreover it is known that $\mathcal{D}$ has tube type if and only if the sets $Q_i$ in (3.7) are all empty.

### 3.5. Iwasawa decomposition of $^cG$.

Consider the Iwasawa decomposition of $^cG = N^c\tilde{A}^cK$ and denote by $S$ its solvable part: $S = N^c\tilde{A}$. Let $\mathfrak{n}$, $\tilde{\mathfrak{a}}$, $\mathfrak{s}$ be the corresponding Lie algebras. Then $\tilde{\mathfrak{a}}$ can be chosen as a subalgebra of $\mathfrak{g}$ consisting of elements $H = L(a)$, where

$$a = \sum_{j=1}^r a_j c_j.$$

We let $\lambda_j$ denote the linear form on $\tilde{\mathfrak{a}}$ given by $\lambda_j(H) = a_j$. All endomorphisms of $\mathfrak{s}$ having the form: $adH$ for $H \in \tilde{\mathfrak{a}}$ admit joint diagonalization. Therefore $\mathfrak{s}$ can be decomposed as a direct sum of corresponding root spaces. The forms of all roots are well-known:

\begin{equation}
\mathfrak{s} = \left( \bigoplus_{j} \mathfrak{n}_{-\lambda_j} \right) \oplus \left( \bigoplus_{1 \leq i \leq j \leq r} \mathfrak{n}_{\lambda_i + \lambda_j} \right) \oplus \left( \bigoplus_{1 \leq i < j \leq r} \frac{\mathfrak{n}_{\lambda_j - \lambda_i}}{2} \right) \oplus \tilde{\mathfrak{a}}
\end{equation}

To simplify our notation put

- $\mathcal{Z}_j = \mathfrak{n}_{\lambda_j}$,
- $\mathfrak{n}_{ij} = \mathfrak{n}_{\lambda_j - \lambda_i}$,
- $V_{ij} = \mathfrak{n}_{\lambda_i + \lambda_j}$.

Then it is known that

$$\mathcal{Z} = \bigoplus_j \mathcal{Z}_j, \quad V = \bigoplus_{i,j} V_{ij},$$
and we set
\[ n_0 = \bigoplus_{1 \leq i < j \leq r} n_{ij}. \]

Denote by \( N(\Phi) \) and \( N_0 \) the subgroups of \( S \) corresponding to the subalgebras \( \mathcal{Z} \oplus V \) and \( n_0 \) of \( \mathfrak{s} \). Then \( S = N(\Phi)N_0 \tilde{A} \), \( N(\Phi) \) is two-step nilpotent with center \( V \) and each of \( N(\Phi) \), \( N_0 \) and \( N = N(\Phi)N_0 \) is normal in \( S \). Since \( S \) acts simply transitively on the domain \( ^cD \), we may identify \( S \) and \( ^cD \):
\[ (3.15) \quad S \ni s \sim s \cdot (c \cdot o) \in ^cD. \]

Now we describe an orthonormal basis of \( \mathfrak{s} \) corresponding to the decomposition (3.14). This will be the same basis as in [DHMP], [BDH] and [B]. By \( Q_{ij} \) we shall denote a subset of \( Q_{ij} \) that contains exactly one from each pair of roots \( (\alpha, \tilde{\alpha} = \gamma_i + \gamma_j - \alpha) \) when \( \tilde{\alpha} \neq \alpha \). Let us define
\[ (3.16) \quad X_i = E_{\gamma_i} \quad \text{for} \quad \gamma_i \in \Gamma, \]
\[ X_\alpha^1 = \frac{1}{\sqrt{2}}(E_\alpha - \varepsilon_\alpha E_{\tilde{\alpha}}) \quad \text{for} \quad \alpha \in Q_{ij} \text{ and } \alpha \neq \tilde{\alpha}, \]
\[ X_\alpha^2 = \frac{1}{\sqrt{2}}(E_\alpha + \varepsilon_\alpha E_{\tilde{\alpha}}) \quad \text{for} \quad \alpha \in Q_{ij} \text{ and } \alpha \neq \tilde{\alpha}, \]
\[ X_\alpha = \frac{1}{\sqrt{2}}\psi(E_\alpha) \quad \text{for} \quad \alpha \in Q_i, \]
where \( \psi = I + Ad(c^2)\tau \), and \( \varepsilon_\alpha = \pm 1 \) (the precise value was determined in [B]). If \( \alpha = \tilde{\alpha} \) then instead of \( X_\alpha^1, X_\alpha^2 \) we define \( X_\alpha^1 = E_\alpha \).

In the rest of the paper we will use the notation \( X_\alpha^k \) without specifying the set of indices \( k \)'s. In particular, the summation over \( \alpha \in Q_{ij} \) will always denote the summation over \( \alpha \) and \( k \) together.

Then by [B]
\[ V_{ii} = \text{span}\{X_i\}, \]
\[ V_{ij} = \text{span}\{X_\alpha^k\}_{\alpha \in Q_{ij}}, \]
\[ Z_j = (\text{span}\{X_\alpha\}_{\alpha \in Q_j})^c. \]

Next we transport the complex structure \( \mathcal{J} \) from \( ^cD \) to \( \mathfrak{s} \) and define:
\[ H_j = \mathcal{J}(X_j), \]
\[ Y_\alpha^k = \mathcal{J}(X_\alpha^k), \]
\[ Y_\alpha = \mathcal{J}(X_\alpha). \]

It was proved in [B] that
\[ X_j, X_\alpha^k, X_\alpha, H_j, Y_\alpha^k, Y_\alpha \]
form an orthonormal basis of \( \mathfrak{s} \) with respect to the Hermitian product \( B_\tau \).
Finally,
\begin{align*}
Z_j &= X_j - iH_j, \quad (1 \leq j \leq r), \\
Z^k_\alpha &= X^k_\alpha - iY^k_\alpha, \quad (\alpha \in \bigcup Q_{ij}, k = 1, 2), \\
Z_\alpha &= X_\alpha - iY_\alpha \quad (\alpha \in \bigcup Q_j)
\end{align*}
(3.18)
is a basis of $S$-invariant holomorphic vector fields.

4. Systems of type $(1,1)$

We suppose here, as in the previous section, that $G/K$ is an irreducible Hermitian symmetric space of non-compact type. To determine all systems $D_W$ of a given type $(a, b)$ on $G/K$, one needs to find all $K$-invariant subspaces $W$ of $S^a(p_+) \otimes S^b(p_-)$ (and apply the modified symmetrization map $\lambda^\otimes$). Recall that $K$ acts on $S(p_\pm)$ by symmetric powers of the adjoint representation. Since $G/K$ is irreducible, $p_\pm$ are irreducible $K$-modules. Moreover, it is known that the representations of $K$ on $S(p_\pm)$ are multiplicity free. The decompositions for $S(p_\pm)$ under the action of $K$ are described in [J] on a case-by-case basis using the classification of irreducible Hermitian symmetric spaces. Thus, in principle, all systems of types $(a, 0)$ and $(0, b)$ are known.

The current work concerns systems of type $(1,1)$. Thus, we must describe the $K$-invariant subspaces of $p_+ \otimes p_-$. We begin with some observations concerning the adjoint representations of $K$ on $p_\pm$, which we now denote by $\sigma_\pm$:
\[ \sigma_+(k) = Ad(k)|_{p_+}, \quad \sigma_-(k) = Ad(k)|_{p_-}. \]
These representations are unitary with respect to the positive definite Hermitian inner product $B_+$ on $p_+$. The conjugation map $Z \mapsto \overline{Z}$ (with respect to the real form $g \subset g_C$) interchanges $p_+$ with $p_-$, as shown by Equations 3.1. Since
\[ \sigma_-(k)(\overline{Z}) = \overline{\sigma_+(k)(Z)} \]
for $k \in K, Z \in p_+$, we see that $\sigma_-$ is (unitarily) equivalent to $\overline{\sigma}_+$, the conjugate representation for $\sigma_+$. (Recall that the conjugate for a complex representation is obtained by replacing the complex structure on the representation space by its conjugate. The conjugate for a matrix representation is obtained by conjugation of matrix entries.)

The Hermitian inner product $B_+$ on $p_+$ yields a further isomorphism of complex vector spaces:
\[ \overline{p}_+ \to p^*_+, \quad Z \mapsto B_+(\cdot, Z), \]
establishing a unitary equivalence of $\overline{\sigma}_+$ with $\sigma^*_+$, the contragredient representation for $\sigma_+$. In summary, we have canonical unitary equivalences
\[ \sigma_- \simeq \overline{\sigma}_+ \simeq \sigma^*_+. \]
We will denote elements of \( p_+ \) as "\( Z \)" for \( Z \in p_+ \). One can choose to interpret this literally, as the result of applying the conjugation map on \( g_C = g + ig \) to \( Z \), or as simply a notation for \( Z \) itself, but viewed as an element of \( p_+ \) with the conjugate complex structure.

The representation \( \sigma_+ \otimes \sigma_- \) always contains a copy of \( Ad_K : K \to U(V_k) \), the complexified adjoint representation of \( K \) on \( V_k \). Indeed, the linear map \( \varphi : p_+ \otimes p_- \to V_k \) determined by
\[
\varphi(X \otimes Y) = [X, Y]
\]
intertwines \( \sigma_+ \otimes \sigma_- \) with \( Ad_K \) and is surjective. Thus, the dual map
\[
\varphi^* : V_k^* \to (p_+ \otimes p_-)^* \cong (p_+^* \otimes p_-^* \cong p_- \otimes p_+ \cong p_+ \otimes p_-)
\]
is injective and intertwines the contragredient representation \( Ad_K^* \) for \( Ad_K \) with \( \sigma_+ \otimes \sigma_- \). In fact, \( Ad_K^* \) and \( Ad_K \) are equivalent representations, since \( K \) is compact. So
\[
H = \varphi^*(V_k^*)
\]
is a \( K \)-invariant subspace of \( p_+ \otimes p_- \) and \( (\sigma_+ \otimes \sigma_-)|H \) is equivalent to \( Ad_K \).

As explained in the Introduction, \( D_H \) is the Hua system. It is the subject of works including [Hua], [JK], [BV], [L] and [BBDHPT]. The Hua system is not irreducible. Indeed, the Lie algebra \( V_k \) decomposes as
\[
V_k = V'_k \oplus c_k
\]
where \( c \) is the (one-dimensional) center of \( V \) and the derived algebra \( V'_k \) is semi-simple. Thus, the \( K \)-invariant subspace \( H = \varphi^*(V'_k) \) in \( p_+ \otimes p_- \) further decomposes under \( \sigma_+ \otimes \sigma_- \) as
\[
H = \varphi^*(V'_k*) \oplus \varphi^*(c_k^*) = H' \oplus L
\]
The group \( K \) acts trivially on the one-dimensional space \( L = \varphi^*(c_k^*) \) and \( H' = \varphi^*(V'_k*) \) contains one \( K \)-irreducible subspace for each simple factor in \( V'_k \). The classification of irreducible Hermitian symmetric spaces of non-compact type, discussed below, shows that \( V'_k \) has at most two simple factors. The elements of \( L \) yield scalar multiples of the Laplace-Beltrami operator on \( G/K \). These are the only second order left-\( G \)-invariant differential operators on \( G/K \), so \( p_+ \otimes p_- \) contains no further copies of the trivial representation.

We now have
\[
p_+ \otimes p_- = H \oplus H_c
\]
(4.1)
is the orthogonal complement to \( H \) in \( p_\oplus p_- \) and also the kernel of \( \varphi : p_+ \otimes p_- \to V_k \). The space \( H_c \) is \( K \)-invariant and, as will be shown below, is non-zero except when \( G/K \) has rank one. (The rank one cases are the complex hyperbolic spaces, up to isomorphism.) \( D_{H_c} \) is the complementary Hua system. We will show that \( H_c \) is \( K \)-irreducible and inequivalent to any of the irreducible constituents of \( H \).
Moreover, we will show that \( H^c \) is the Cartan component in \( \sigma_+ \otimes \sigma_- \). That is, a highest weight vector in \( H^c \) is obtained as the tensor product of highest weight vectors in \( p_+ \) and \( p_- \). If \( X, Y \in p_+ \) are highest and lowest weight vectors for \( \sigma_+ \) then \( H^c = \text{Span}\{ (\sigma_+ \otimes \sigma_-)(K)(X \otimes Y) \} \) and \( X \otimes Y \) is a highest weight vector for \( (\sigma_+ \otimes \sigma_-)|H^c \). This provides another characterization of \( H^c \). We summarize the preceding discussion as follows:

**Proposition 4.2.** Let \( G/K \) be an irreducible Hermitian symmetric space of non-compact type. Then:

- The representation \( \sigma_+ \otimes \sigma_- \) of \( K \) on \( p_+ \otimes p_- \) is multiplicity free, with a canonical decomposition
  \[
  p_+ \otimes p_- = \mathcal{H} \oplus H^c
  \]
  as an orthogonal direct sum of \( K \)-invariant subspaces. (The space \( \mathcal{H} \) yields the Hua system and \( H^c \) the complementary Hua system.)
- The representation \( (\sigma_+ \otimes \sigma_-)|\mathcal{H} \) is equivalent to \( \text{Ad}_K \), the complexified adjoint representation of \( K \) on \( \mathfrak{k}_C \) and decomposes as
  \[
  \mathcal{H} = \mathcal{H}' \oplus \mathcal{L},
  \]
  where \( \mathcal{L} \) is a copy of the trivial representation of \( K \) on \( \mathbb{C} \) and \( \mathcal{H}' \) contains an irreducible subspace for each simple factor in \( \mathfrak{k}_C \);
- The space \( H^c \) is \( H^c = \text{Ker}(\varphi) \) where \( \varphi : p_+ \otimes p_- \to \mathfrak{p}_C \) is the linear map with \( \varphi(X \otimes Y) = [X,Y] \).
- If \( G/K \) has rank at least two then \( H^c \) is non-zero, \( (\sigma_+ \otimes \sigma_-)|H^c \) is irreducible and is the Cartan component in \( \sigma_+ \otimes \sigma_- \).

Note that since \( \sigma_+ \otimes \sigma_- \) is multiplicity free, any \( K \)-invariant subspace of \( p_+ \otimes p_- \) is a sum of the irreducibles described above. Thus, Proposition 4.2 determines all systems of type \((1,1)\) on \( G/K \).

To complete the proof of Proposition 4.2 we employ case-by-case analysis using the classification for irreducible Hermitian symmetric spaces of non-compact type. We refer the reader to [H1], Chapter X, for details concerning this classification and for definitions of the Lie groups that arise. The spaces in question fall into 4 classical families (types \( \text{A III}, \text{C I}, \text{D III}, \text{BD I} \)) and two exceptional cases (types \( \text{E III}, \text{E VII} \)). For the classical families, we will exhibit the spaces \( \mathcal{H} \) and \( H^c \) in Proposition 4.2 explicitly.

### 4.1. Type A III

Here \( G/K = SU(n,m)/S(U(n) \times U(m)) \) with \( 1 \leq n \leq m \). The rank of \( G/K \) is \( n \). One can realize \( p_+ \) as \( p_+ = M_{n,m}(\mathbb{C}) \) with Hermitian inner product \( \langle X,Y \rangle = \text{tr}(XY^\ast) \), where \( Y^\ast \) denotes conjugate-transpose. The bounded realization for \( G/K \) as a domain in \( p_+ \) is

\[
\mathcal{D} = \{ Z \in p_+ : Z^t \bar{Z} < I_m \} = \{ Z \in p_+ : ZZ^\ast < I_n \}.
\]
The group $K = S(U(n) \times U(m))$ acts on $p_+$ via
\[
\sigma_+(k_1, k_2) Z = k_1 Z k_2^*,
\]
$(k_1 \in U(n), k_2 \in U(m), det(k_1) det(k_2) = 1)$. Writing elements of $p_+ = \bar{p}_+$ as $Z$ for $Z \in p_+$, one has
\[
(\sigma_+ \otimes \sigma_-)(k_1, k_2)(X \otimes \overline{Y}) = (k_1 X k_2^*) \otimes (k_1 \overline{Y} k_2^t).
\]
Let $E_{a,b}$ $(1 \leq a \leq n, 1 \leq b \leq m)$ denote the $(n \times m)$-matrix with a one in position $(a, b)$ and set
\[
L_{i,j} = \sum_{\ell=1}^m E_{i,\ell} \otimes \overline{E}_{j,\ell} \quad (1 \leq i, j \leq n)
\]
\[
R_{i',j'} = \sum_{\ell=1}^n E_{\ell,i'} \otimes \overline{E}_{\ell,j'} \quad (1 \leq i', j' \leq m).
\]
One verifies that
\[
(\sigma_+ \otimes \sigma_-)(k_1, k_2) L_{i,j} = (k_1 L k_2^*)_{i,j}, \quad (\sigma_+ \otimes \sigma_-)(k_1, k_2) R_{i',j'} = (k_2 R k_2^*)_{i',j'},
\]
where "$k_1 L k_2^*$" and "$k_2 R k_2^*$" mean formal matrix multiplication. Hence
\[
\mathcal{H} = \text{Span}\{L_{i,j}, R_{i',j'} : 1 \leq i, j \leq n, 1 \leq i', j' \leq m\}
\]
is $K$-invariant and $K$ acts on $\mathcal{H}$ by a copy of $Ad_K$.

The space $\mathcal{H}$ decomposes under $K$ as $\mathcal{H} = \mathcal{L} \oplus \mathcal{H}_1' \oplus \mathcal{H}_2'$. Here $K$ acts trivially on $\mathcal{L}$ of dimension 1 and $\mathcal{H}_1'$, $\mathcal{H}_2'$ correspond to the simple factors $sl(n, \mathbb{C})$, $sl(m, \mathbb{C})$ in $\mathfrak{sl}_c = sl(n, \mathbb{C}) \oplus sl(m, \mathbb{C})$. Explicitly:
\[
\mathcal{L} = \mathbb{C} \left( \sum_{i=1}^n L_{i,i} \right) = \mathbb{C} \left( \sum_{i'=1}^m R_{i',i'} \right),
\]
\[
\mathcal{H}_1' = \left\{ \sum_{i,j=1}^n c_{i,j} L_{i,j} : \sum_i c_{i,i} = 0 \right\},
\]
\[
\mathcal{H}_2' = \left\{ \sum_{i',j'=1}^m c_{i',j'} R_{i',j'} : \sum_{i'} c_{i',i'} = 0 \right\}.
\]
The space $\mathcal{H}_1' = 0$ whenever $n = 1$ and $\mathcal{H}_2' = 0$ when $n = m = 1$. Note that for $n = m = 1$, $\mathcal{D} \cong SU(1,1)/SU(1) \times U(1))$ is the unit ball in $\mathbb{C}$. When $n = 1$ and $m \geq 2$, $SU(1, m)/SU(1) \times U(m)$, is a complex hyperbolic space of dimension $m$. These are the rank one cases.

One can easily exhibit bases for the spaces $\mathcal{H}_1'$, $\mathcal{H}_2'$. For example,
\[
\{ L_{i,j} : i \neq j \} \cup \{ L_{i,i} - L_{i+1,i+1} : 1 \leq i \leq n - 1 \}
\]
Note that $\mathcal{H}^c = 0$ when $n = 1$. That is, when $G/K$ has rank one. The action of $K' = SU(n) \times SU(m)$ on $\mathcal{H}^c$ is equivalent to the (exterior) tensor product of the complexified adjoint representations for $SU(n)$, $SU(m)$ on $sl(n, \mathbb{C})$, $sl(m, \mathbb{C})$. In particular, $\mathcal{H}^c$ is $K$-irreducible. Highest weight vectors in $\mathfrak{p}_\pm$ are given by $E_{1,1}$ and $\overline{E}_{n,m}$. As $E_{1,1} \otimes \overline{E}_{n,m}$ belongs to $\mathcal{H}^c$, we see that $\mathcal{H}^c$ is the Cartan component in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. The space $\mathcal{H}^c$ has dimension $(n^2 - 1)(m^2 - 1)$. The following elements form a basis:

- $E_{i,i'} \otimes \overline{E}_{j,j'}$ with $i \neq j$; $i' \neq j'$,
- $E_{i,i'} \otimes \overline{E}_{j,j'} - E_{i,i'+1} \otimes \overline{E}_{j,j'+1}$ with $i \neq j$; $1 \leq i' \leq m - 1$,
- $E_{i,i'} \otimes \overline{E}_{i+1,j'} - E_{i+1,i'} \otimes \overline{E}_{i,j'+1}$ with $i' \neq j'$; $1 \leq i \leq n - 1$,
- $E_{1,1} \otimes \overline{E}_{1,1} - E_{i,1} \otimes \overline{E}_{i,1} - E_{1,j'} \otimes \overline{E}_{1,j'} + E_{i,j'} \otimes \overline{E}_{i,j'}$ with $2 \leq i \leq n$; $2 \leq j' \leq m$.

### 4.2. Type C I

These are the spaces $G/K = Sp(n, \mathbb{R})/U(n)$. Since $Sp(1, \mathbb{R})/U(1) \cong U(1, 1)/S(U(1) \times U(1))$, we can assume that $n \geq 2$. The rank of $G/K$ is $n$. The space $\mathfrak{p}_+$ is realized as the space of $n \times n$ symmetric matrices:

$$\mathfrak{p}_+ = \{ Z \in M_{n,n}(\mathbb{C}) : Z^t = Z \}$$

with Hermitian inner product $\langle X, Y \rangle = tr(XY^*) = tr(YX^*)$. The bounded realization for $G/K$ as a domain in $\mathfrak{p}_+$ is

$$\mathcal{D} = \{ Z \in \mathfrak{p}_+ : ZZ^t < I_n \}.$$

This is a tube domain.

The group $K$ acts on $\mathfrak{p}_+$ via

$$\sigma_+(k)Z = kZk^t,$$

and on $\mathfrak{p}_+ \otimes \mathfrak{p}_- = \mathfrak{p}_+ \otimes \overline{\mathfrak{p}}_+$ by

$$(\sigma_+ \otimes \sigma_-)(k)(X \otimes \overline{Y}) = (kXk^t) \otimes (\overline{kYk^*}).$$

Define elements $F_{i,j} \in \mathfrak{p}_+$ ($1 \leq i, j \leq n$) as

$$F_{i,j} = E_{i,j} + E_{j,i}.$$

Note that $F_{i,j} = F_{j,i}$ and that the $F_{i,j}$’s are pair-wise orthogonal with

$$\langle F_{i,j}, F_{i,j} \rangle = \begin{cases} 4 & \text{for } i = j \\ 2 & \text{for } i \neq j \end{cases}.$$
Define
\[ T_{i,j} = \sum_{\ell=1}^{n} F_{i,\ell} \otimes F_{\ell,j}, \quad (1 \leq i, j \leq n). \]
Then one has
\[ (\sigma_+ \otimes \sigma_-)(k)T_{i,j} = (kT^*)_{i,j}. \]
Thus
\[ \mathcal{H} = \text{Span}\{T_{i,j} : 1 \leq i, j \leq n\} \]
is \( K \)-invariant and \( K \) acts on \( \mathcal{H} \) by a copy of \( Ad_K \). The space \( \mathcal{H} \) contains two irreducible components, corresponding to the decomposition \( \mathfrak{t}_C = \mathbb{C} \oplus \mathfrak{sl}(n, \mathbb{C}) \). This can be written as \( \mathcal{H} = \mathcal{L} \oplus \mathcal{H}' \) where
\[ \mathcal{L} = \mathbb{C} \left( \sum_{i=1}^{n} T_{i,i} \right) = \mathbb{C} \left( \sum_{i,\ell} F_{i,\ell} \otimes F_{\ell,i} \right), \]
\[ \mathcal{H}' = \left\{ \sum_{i,j=1}^{n} c_{i,j} T_{i,j} : \sum_{i=1}^{n} c_{i,i} = 0 \right\}. \]
A basis for \( \mathcal{H}' \) is given by
\[ \{ T_{i,j} : i \neq j \} \cup \{ T_{i,j} - T_{i+1,j+1} : 1 \leq i \leq n - 1\}. \]

Let \( \mathcal{H}^c \) denote the orthogonal complement to \( \mathcal{H} \) in \( \mathfrak{p}_+ \otimes \mathfrak{p}_- \). As representations of \( SU(n) \), \( \sigma_\pm \) have highest weights \( (2,0,\ldots,0) \) and \( (2,2,\ldots,2,0) \), and \( F_{1,1}, F_{n,n} \) are highest weight vectors. An easy application of the Littlewood-Richardson rules (see [FH], Appendix A) shows that \( \sigma_+ \otimes \sigma_- \) has exactly three irreducible components. Thus \( \mathcal{H}^c \) is determined by \( \mathcal{H} \). The highest weight for \( \mathcal{H}^c \) is \((4,2,\ldots,2,0)\). Since \( F_{1,1} \otimes F_{n,n} \) belongs to \( \mathcal{H}^c \) (it is orthogonal to \( \mathcal{H} \)), \( \mathcal{H}^c \) is the Cartan component in \( \mathfrak{p}_+ \otimes \mathfrak{p}_- \).

The dimension of \( \mathcal{H}^c \) is \[ |n(n+1)/2|^2 - n^2 = n^2(n-1)(n+3)/4. \] Working from the above description of \( \mathcal{H} \) and using the fact that the \( F_{i,j} \otimes F_{k,l} \)'s are pair-wise orthogonal in \( \mathfrak{p}_+ \otimes \mathfrak{p}_- \), we see that the following vectors form a basis for \( \mathcal{H}^c \):
- \( F_{i,i} \otimes F_{j,j} \) with \( i \neq j \),
- \( F_{i,i} \otimes F_{j,k} \) with \( i \neq j, k; j < k \),
- \( F_{i,j} \otimes F_{k,k} \) with \( k \neq i, j; i < j \),
- \( F_{i,j} \otimes F_{k,\ell} \) with \( \{i,j\} \cap \{k,\ell\} = \emptyset; i < j; k < \ell \),
- \( \|F_{i,k}\|^2\|F_{k,j}\|^2F_{i,1} \otimes F_{1,j} - \|F_{i,1}\|^2\|F_{1,j}\|^2F_{i,k} \otimes F_{k,j} \) with \( i \neq j \); \( 2 \leq k \leq n \),
- \( F_{i,i} \otimes F_{i,i} + F_{j,j} \otimes F_{j,j} - 4F_{i,j} \otimes F_{j,i} \) with \( i < j \).

4.3. Type D III. Here \( G/K = SO^*(2n)/U(n) \). Since \( SO^*(4)/U(2) \) is non-irreducible and \( SO^*(6)/U(3) \approx SU(1,3)/S(U(1) \times U(3)) \), we can take \( n \geq 4 \) here. Thus \( G/K \) has rank \( [n/2] \geq 2 \). The space \( \mathfrak{p}_+ \) can be realized as the space of skew-symmetric \( n \times n \)-matrices:
\[ \mathfrak{p}_+ = \{ Z \in M_{n,n}(\mathbb{C}) : Z^t = -Z \} \]
with Hermitian inner product $\langle X, Y \rangle = tr(XY^*) = -tr(YX)$. The bounded realization of $G/K$ is

$$\mathcal{D} = \{ Z \in \mathfrak{p}_+ : Z'\overline{Z} < I_n \}.$$ 

$\mathcal{D}$ is a tube domain if and only if $n$ is even.

As in the preceding case, $K = U(n)$ acts on $\mathfrak{p}_+$ via $\sigma_+(k)Z = kZk^t$. The description of $\mathcal{H} \subset \mathfrak{p}_+ \otimes \mathfrak{p}_- = \mathfrak{p}_+ \oplus \overline{\mathfrak{p}}_+$ parallels that for type C I. We set

$$F'_{i,j} = E_{i,j} - E_{j,i}$$

and note that $F'_{i,j} = -F'_{j,i}$, $F'_{i,i} = 0$. The $F'_{i,j}$’s are pair-wise orthogonal with $\langle F'_{i,j}, F'_{i,j} \rangle = 2$ for $i \neq j$. Letting

$$T'_{i,j} = \sum_{\ell=1}^n F'_{i,\ell} \otimes F'_{\ell,j},$$

and

$$\mathcal{H} = \text{Span}\{ T'_{i,j} : 1 \leq i, j \leq n \},$$

we see that $(\sigma_+ \otimes \sigma_-)|\mathcal{H}$ is a copy of $Ad_K$. The space $\mathcal{H}$ decomposes as $\mathcal{H} = \mathcal{L} \oplus \mathcal{H}'$, were $\mathcal{L}, \mathcal{H}'$ are defined as for type C I.

An application of the Littlewood-Richardson rules shows that the orthogonal complement $\mathcal{H}^c$ to $\mathcal{H}$ in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ must be irreducible with highest weight $(2, 2, 1, \ldots, 1, 0)$ and dimension $[n(n-1)/2]^2 - n^2 = n^2(n+1)(n-3)/4$. The space $\mathcal{H}^c$ contains the tensor product $F'_{1,2} \otimes F'_{n-1,n}$ of highest weight vectors in $\mathfrak{p}_ \pm$. Thus $\mathcal{H}^c$ is the Cartan component in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. Working from the description of $\mathcal{H}$ one obtains the following spanning set for $\mathcal{H}^c$:

- $F'_{i,j} \otimes F'_{k,\ell}$ with $\{i, j\} \cap \{k, \ell\} = \emptyset$; $i < j$; $k < \ell$,

- $F'_{i,1} \otimes F'_{1,j} - F'_{i,k} \otimes F'_{k,j}$ with $k, i, j > 1$; $i \neq j$; $k \neq i, j$,

- $F'_{i,j} \otimes F'_{j,i} + \frac{1}{n-1} \sum_{\ell=1}^{n-1} T'_{\ell,i} - \frac{1}{n-2} (T'_{i,i} + T'_{j,j})$ with $i < j$.

This spanning set for $\mathcal{H}^c$ is not, however, linearly independent. For example, summing the elements of the third kind over $j \neq i$ for fixed $i$ gives zero.

4.4. Type BD I. Here $G/K = SO_o(2, n)/(SO(2) \times SO(n))$. We can take $n \geq 5$, in view of isomorphisms in low dimensions. $(SO_o(2, 2)/(SO(2) \times SO(2))$ is not irreducible, $SO_o(2, 3)/(SO(2) \times SO(2)) \cong Sp(2, \mathbb{R})/U(2)$, and $SO_o(2, 4)/(SO(2) \times SO(4)) \cong SU(2, 2)/SU(2) \times U(2))$. The rank of $G/K$ is 2. In this case, $\mathfrak{p}_+$ is realized as $\mathfrak{p}_+ = M_{2,n}(\mathbb{R})$ with complex structure

$$J \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix},$$

for row vectors $x, y \in \mathbb{R}^n$. The map $T : M_{2,n}(\mathbb{R}) \to \mathbb{C}^n$ defined as

$$T(Z) = x^t + iy^t,$$

for $Z = \begin{bmatrix} y \\ x \end{bmatrix}$.
is an isomorphism from the complex vector space \((p_+, J)\) to \(\mathbb{C}^n\) (column vectors). The bounded realization for \(G/K\) is
\[
\mathcal{D} = \{Z \in p_+ : ZZ^t < I_2\}.
\]
This is a tube domain.

The group \(K = SO(2) \times SO(n)\) acts on \(p_+\) via
\[
\sigma_+(k_1, k)Z = k_1Zk^t, \quad (k_1 \in SO(2), k \in SO(n)).
\]
For \(k_1 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}\) one computes
\[
T((\sigma_+ \otimes \sigma_-)(k_1, k)Z) = e^{i\theta}kT(Z).
\]
That is, the action of \(K\) on \(p_+\) coincides, via \(T\), with the standard action of \(\mathbb{T} \times SO(n)\) on \(\mathbb{C}^n\). In particular, \(SO(2)\) acts on \(p_+\) by (complex) scalars, so the irreducible subspaces in \(p_+ \otimes p_-\) under the actions of \(K\) and \(K' = SO(n)\) agree. Thus, we need to decompose \(\mathbb{C}^n \otimes (\mathbb{C}^n)^*\), or equivalently \(\mathbb{C}^n \otimes \mathbb{C}^n\), under the diagonal action of \(SO(n)\). Identifying \(\mathbb{C}^n \otimes \mathbb{C}^n\) with \(M_{n,n}(\mathbb{C})\), \(SO(n)\) acts via \(k \cdot A = kAk^t, \quad (k \in SO(n), A \in M_{n,n}(\mathbb{C}))\).

From this viewpoint, the decomposition into \(SO(n)\)-irreducible subspaces is transparent:
\[
M_{n,n}(\mathbb{C}) = \mathbb{C}I_n \oplus \{A : A^t = -A\} \oplus \{A : A^t = A, tr(A) = 0\}.
\]
One can use the above isomorphism \(p_+ \otimes p_- \cong M_{n,n}(\mathbb{C})\) to obtain the corresponding decomposition for our original model. Letting \(X_j = E_{2,j} \in (p_+ = M_{2,n}(\mathbb{R}))\),
\[
p_+ \otimes p_- = \mathcal{L} \oplus \mathcal{H}' \oplus \mathcal{H}^c
\]
where
\[
\mathcal{L} = \mathbb{C} \left( \sum_{j=1}^n X_j \otimes \overline{X}_j \right),
\]
\[
\mathcal{H}' = \left\{ \sum_{i,j=1}^n c_{i,j}X_i \otimes \overline{X}_j : c_{i,j} = -c_{j,i} \right\},
\]
\[
\mathcal{H}^c = \left\{ \sum_{i,j=1}^n c_{i,j}X_i \otimes \overline{X}_j : c_{i,j} = c_{j,i}, \sum_i c_{i,j} = 0 \right\}.
\]
\(\mathcal{H} = \mathcal{L} \oplus \mathcal{H}'\) yields the Hua system. A highest weight vector in \(p_+\) is given by
\[
X_1 + iX_2 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}.
\]
Since \((X_1 + iX_2) \otimes (\overline{X}_1 + i\overline{X}_2) = X_1 \otimes \overline{X}_1 - X_2 \otimes \overline{X}_2 + iX_1 \otimes \overline{X}_2 + iX_2 \otimes \overline{X}_1\) belongs to \(\mathcal{H}^c\), we see that \(\mathcal{H}^c\) is the Cartan component in \(p_+ \otimes p_-\).
4.5. **Type E III.** In this case $K = \text{Spin}(10) \times \mathbb{T}$ and $G$ has Lie algebra $\varepsilon_{6(-14)}$, a certain real form for the complex Lie algebra $\varepsilon_6$. The space $G/K$ has rank 2 and is of non-tube type.

One can identify $\mathfrak{p}_+$ with $\Lambda^\text{ven} (\mathbb{C}^5) \cong \mathbb{C}^{16}$ and $\sigma_+$ is the positive half-spin representation. The contragredient representation $\sigma_+^* \cong \sigma_-$ is equivalent to the negative half-spin representation on $\Lambda^\text{odd} (\mathbb{C}^5) \cong \mathbb{C}^{16}$.

As $\mathfrak{t}_\mathbb{C} = \mathfrak{so}(10, \mathbb{C}) \oplus \mathbb{C}$, the subspace $\mathcal{H} = \varphi^*(\mathfrak{t}_\mathbb{C}^*)$ of $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ has two irreducible components:

$$\mathcal{H} = \mathcal{L} \oplus \mathcal{H}' = \varphi^*(\mathbb{C}^*) \oplus \varphi^*(\mathfrak{so}(10, \mathbb{C})^*).$$

The highest weights for $\sigma_\pm$ are $(1/2)(L_1 + L_2 + L_3 + L_4 \pm L_5)$ in the notation of [FH]. (See [FH], Proposition 20.15.) Thus, the Cartan component $W$ in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ has highest weight $L_1 + L_2 + L_3 + L_4$. A application of the Weyl dimension formula shows that $\text{dim}(W) = 210$. (The Weyl dimension formula for $\text{SO}(2m)$ is Equation (24.41) in [FH].) Since

$$\text{dim}(\mathcal{H}) + \text{dim}(W) = \text{dim}(\mathfrak{t}_\mathbb{C}) + 210 = 46 + 210 = 256 = 16^2 = \text{dim}(\mathfrak{p}_+ \otimes \mathfrak{p}_-),$$

we conclude that $W = \mathcal{H}^c$, the orthogonal complement to $\mathcal{H}$ in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. This proves that $\mathcal{H}^c$ is irreducible and is the Cartan component in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$.

4.6. **Type E VII.** Finally, we consider the exceptional case where $K = E_6 \times \mathbb{T}$ and $G$ has Lie algebra $\varepsilon_{7(-25)}$, a real form of $\varepsilon_7$. The space $G/K$ has rank 3 and is a tube domain.

In this case $\mathfrak{p}_+$ can be identified with an exceptional Jordan algebra $\mathcal{J}$ of dimension 27. The representation of $E_6$ on $\mathcal{J}$ is described in [CS]. We have the decomposition

$$\mathfrak{p}_+ \otimes \mathfrak{p}_- = \mathcal{L} \oplus \mathcal{H}' \oplus \mathcal{H}^c = \varphi^*(\mathbb{C}^*) \oplus \varphi^*(\varepsilon_6^*) \oplus \text{Ker}(\varphi)$$

as usual. It remains to show that $\mathcal{H}^c$ is irreducible and is the Cartan component in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$.

The fundamental weights for the complex simple Lie algebra $\varepsilon_6$ are usually denoted $\omega_1, \ldots, \omega_6$. (See, for example, the tables in [Bou].) The representation $\sigma_+$ has highest weight $\omega_1$ and its contragredient $\sigma_+^* \simeq \sigma_-$ has highest weight $\omega_6$. Hence $\omega_1 + \omega_6$ is the highest weight for the Cartan component $W$ in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. One can use the Weyl dimension formula to show that $\text{dim}(W) = 650$. In fact, this dimension can be obtained from a table in [GS]. We now see that

$$\text{dim}(\mathcal{H}) + \text{dim}(W) = \text{dim}(\mathfrak{t}_\mathbb{C}) + 650 = 79 + 650 = 729 = 27^2 = \text{dim}(\mathfrak{p}_+ \otimes \mathfrak{p}_-).$$

Hence $W = \mathcal{H}^c$, completing the proof for Proposition 4.2.

5. **The main theorem**

The aim of this paper is to characterize functions on $G/K$ which are annihilated by a system of type (1,1) containing the Laplace-Beltrami operator. Proposition 4.2 shows that, apart from the systems $\mathcal{L}$ and $\mathfrak{p}_+ \otimes \mathfrak{p}_-$, we have the Hua system $\mathcal{H} = \mathcal{L} \oplus \mathcal{H}'$.
and the complementary system \( \mathcal{L} \oplus \mathcal{H}^c \). For \( G/K = SU(n, m)/S(U(n) \times U(m)) \), the Hua system can be decomposed: \( \mathcal{H} = \mathcal{L} \oplus \mathcal{H}'_1 \oplus \mathcal{H}'_2 \) and one can consider two further systems of type (1,1): \( \mathcal{L} \oplus \mathcal{H}'_1 \) and \( \mathcal{L} \oplus \mathcal{H}'_2 \). These exhaust the possibilities.

Theorems 5.1 through 5.5 below characterize the functions annihilated by each of these systems. The first two results concern zeros of the Hua system.

**Theorem 5.1 ([JK]).** A function \( f \) on a Hermitian symmetric space of tube type satisfies \( \mathcal{H}(f) = 0 \) if and only if it is the Poisson-Szegö integral of a hyperfunction supported on the Shilov boundary of \( G/K \).

**Theorem 5.2 ([BBDHPT],[B]).** Let \( G/K \) be an irreducible Hermitian symmetric space of non-tube type and let \( f \) be a real-valued function on \( G/K \). Then \( \mathcal{H}(f) = 0 \) if and only if \( f \) is pluriharmonic.

Let us recall that \( f \) defined on \( D \subset \mathbb{C}^n \) is pluriharmonic if it is the real part of a holomorphic function. Equivalently, \( f \) is annihilated by all operators \( \partial_{z_j} \partial_{z_k} \) (\( 1 \leq j, k \leq n \)). That is, the pluriharmonic functions are those annihilated by the system \( p_+ \otimes p_- \). The main result of this paper is the following.

**Main Theorem 5.3.** Let \( G/K \) be an irreducible Hermitian symmetric space of rank \( r \geq 2 \) and let \( f \) be a bounded real-valued function on \( G/K \). Then \( f \) is \( \mathcal{L} \oplus \mathcal{H}^c \)-harmonic if and only if \( f \) is pluriharmonic.

In the statement of Theorem 5.3, we require \( r \geq 2 \), because \( \mathcal{H}^c = 0 \) for the rank one Hermitian symmetric spaces. It seems likely that the boundedness hypothesis on \( f \) can be removed from Theorem 5.3, as in the statement of Theorem 5.2, but we are not able to show this using the methods of this paper. This is one problem for future research.

For \( G/K = S(n, m)/S(U(n) \times U(m)) \) we must also consider the systems \( \mathcal{L} \oplus \mathcal{H}'_1 \) and \( \mathcal{L} \oplus \mathcal{H}'_2 \). As in Section 4.1 we assume that \( 1 \leq n \leq m \) and have \( \mathcal{H}'_1 \cong sl(n, \mathbb{C}) \), \( \mathcal{H}'_2 \cong sl(m, \mathbb{C}) \). The following result is due to N. Berline and M. Vergne.

**Theorem 5.4 ([BV]).** Let \( G/K = SU(n, m)/S(U(n) \times U(m)) \) with \( 1 \leq n \leq m \). A function \( f \) on \( G/K \) satisfies \( \mathcal{L} \oplus \mathcal{H}'_1(f) = 0 \) if and only if it is the Poisson-Szegö integral of a hyperfunction supported on the Shilov boundary.

By symmetry, when \( n = m \) the \( \mathcal{L} \oplus \mathcal{H}'_2 \) system also characterizes Poisson-Szegö integrals. These are tube domains. On the other hand, when \( n < m \) we have a non-tube domain and the following result.

**Theorem 5.5.** Let \( G/K = SU(n, m)/S(U(n) \times U(m)) \) with \( 1 \leq n < m \). A real-valued function \( f \) on \( G/K \) satisfies \( \mathcal{L} \oplus \mathcal{H}'_2(f) = 0 \) if and only if \( f \) is pluriharmonic.

**Proof.** Let \( f \) be \( \mathcal{L} \oplus \mathcal{H}'_2 \)-harmonic. Note that the system \( \mathcal{H}'_2 \) contains a copy of \( \mathcal{H}'_1 \) via the obvious inclusion \( sl(n, \mathbb{C}) \hookrightarrow sl(m, \mathbb{C}) \). Thus Theorem 5.4 implies that \( f \) is the Poisson-Szegö integral of a hyperfunction. But now, Theorem 5.4 also shows that \( f \)
is $L \oplus H$-harmonic. Thus in fact $f$ is annihilated by all of $H \oplus L \oplus H = H$. Finally as $n < m$, $G/K$ is of non-tube type, so Theorem 5.2 shows $f$ is pluriharmonic. □

The rest of this paper concerns the proof of Theorem 5.3. Our strategy can be outlined as follows. Let $G/K$ have rank at least two and $f$ be a bounded real-valued function on $G/K$. First note that if $f$ is pluriharmonic then $L \oplus H$ annihilates it because pluriharmonic functions are annihilated by all of $p_+ \otimes p_-$. Hence we need only show that $L \oplus H$ annihilates $f$ implies $f$ is pluriharmonic. Having the boundedness assumption (which is invariant under biholomorphic mappings), we transfer our problem to the group $S$ acting simply transitively on the Siegel domain $cD$. We show that $f$ is annihilated by some $S$-invariant operators derived from the $L \oplus H$ system. Using two of these operators and a technique from [BBDHPT], we show that $f \circ c^{-1}$ is a Poisson-Szegő integral. In the tube case, Theorem 5.1 now implies that $f$ is Hua-harmonic. So $f$ is in fact annihilated by all of $p_+ \otimes p_- = H^c \oplus L \oplus H^c$, hence pluriharmonic. In the non-tube case, according to [BV], Poisson-Szegő integrals are annihilated by a certain system of third order operators. We will use this fact and methods described in [BDH] and [B] to obtain pluriharmonicity of $f$ in the non-tube case.

6. The $L \oplus H^c$ System and Berline-Vergne Operators

6.1. The $L \oplus H^c$ System. Below we introduce some operators which belong to the $L \oplus H^c$ system. We make extensive use of the notation and results from Section 3.2. As before, $r$ always denotes the rank of $G/K$. We assume throughout that $r \geq 2$, since $H^c = 0$ when $r = 1$. (Most formulas to be derived below are vacuous when $r = 1$.)

For $\alpha \in Q_{ij}$, put

\begin{equation}
\Delta_\alpha = E_\alpha E_{-\alpha} + E_{\bar{\alpha}} E_{-\bar{\alpha}} - E_{\gamma_i} E_{-\gamma_i} - E_{\gamma_j} E_{-\gamma_j},
\end{equation}

regarded as an operator arising from $p_+ \otimes p_-$. Recall that the system $H^c$ is the kernel of the map $\varphi : p_+ \otimes p_- \to k_C$ given by $\varphi(X,Y) = [X,Y]$.

Lemma 6.2. For $\alpha \in Q_{ij}$ the operator $\Delta_\alpha$ belongs to the system $H^c$.

Proof. Recall from (3.10) that we have $\tilde{\alpha} = \gamma_i + \gamma_j - \alpha$, also in $Q_{ij}$. Thus

\begin{equation}
\tilde{H}_\alpha + \tilde{H}_{\bar{\alpha}} - \tilde{H}_{\gamma_i} - \tilde{H}_{\gamma_j} = 0,
\end{equation}

and hence

\begin{equation}
\frac{(\alpha, \alpha)}{2} H_\alpha + \frac{\langle \alpha, \bar{\alpha} \rangle}{2} H_{\bar{\alpha}} - \frac{\langle \gamma_i, \gamma_i \rangle}{2} H_{\gamma_i} - \frac{\langle \gamma_j, \gamma_j \rangle}{2} H_{\gamma_j} = 0.
\end{equation}

But all elements of $Q^+$ have a common length, so $(\alpha, \alpha) = \langle \alpha, \bar{\alpha} \rangle = \langle \gamma_i, \gamma_i \rangle = \langle \gamma_j, \gamma_j \rangle$ and hence $H_\alpha + H_{\bar{\alpha}} - H_{\gamma_i} - H_{\gamma_j} = 0$. 

Now we have
\[
\varphi(\Delta_\alpha) = [E_\alpha, E_{-\alpha}] + [E_{\bar{\alpha}}, E_{-\bar{\alpha}}] - [E_{\gamma_1}, E_{-\gamma_1}] - [E_{\gamma_2}, E_{-\gamma_2}]
\]
\[
= H_\alpha + H_{\bar{\alpha}} - H_{\gamma_1} - H_{\gamma_2}
\]
\[
= 0,
\]
which yields the lemma. \(
\square
\)

**Proposition 6.5.** The operator
\[
\Delta_1 = \sum_{\alpha \in \bigcup_i Q_i} E_\alpha E_{-\alpha} + c \sum E_{\gamma_i} E_{-\gamma_i}
\]
is an element of \(L \oplus H^c\) for some positive constant \(c\).

**Proof.** As \(L\) is the unique \(K\)-invariant operator in \(p_+ \otimes p_- \cong p_+ \otimes (p_-)^*\), we have
\[
L = \sum_{\alpha \in I^+} E_\alpha E_{-\alpha} = \left( \sum_{\alpha \in \bigcup_i Q_i} + \sum_{\alpha \in \bigcup_i \bigcup_j Q_{ij}} + \sum_{\alpha \in \Gamma} \right) E_\alpha E_{-\alpha}.
\]
The operator \(L - \frac{1}{2} \sum_{\alpha \in \bigcup_i \bigcup_j Q_{ij}} \Delta_\alpha\) belongs to \(L \oplus H^c\) in view of Lemma 6.2. Let \(q_2 = |Q_{ij}|\) denote the common cardinality of the sets \(Q_{ij}\). Then we can write
\[
L - \frac{1}{2} \sum_{\alpha \in \bigcup_i \bigcup_j Q_{ij}} \Delta_\alpha = \sum_{\alpha \in \bigcup_i Q_i} E_\alpha E_{-\alpha} + \sum_{k=1}^{r} E_{\gamma_k} E_{-\gamma_k} + \frac{q_2}{2} \sum_{i<j} (E_{\gamma_i} E_{-\gamma_i} + E_{\gamma_j} E_{-\gamma_j})
\]
\[
= \sum_{\alpha \in \bigcup_i Q_i} E_\alpha E_{-\alpha} + \left( 1 + \frac{(r-1)q_2}{2} \right) \sum_{k=1}^{r} E_{\gamma_k} E_{-\gamma_k}.
\]
So \(\Delta_1 = \sum_{\alpha \in \bigcup_i Q_i} E_\alpha E_{-\alpha} + c \sum E_{\gamma_i} E_{-\gamma_i}\) belongs to \(L \oplus H^c\) for \(c = 1 + (r-1)q_2/2\). \(
\square
\)

**Definition 6.6.** \(\tilde{Q}_r = Q_r \cup \bigcup_i Q_{ir}\).

**Lemma 6.7.** If \(\alpha \in Q_{ij} \cup Q_i\) and \(i, j < r\) then there exist \(\beta, \delta \in \tilde{Q}_r\) such that
\[
\Delta_\alpha' = E_\alpha E_{-\alpha} + E_{\gamma_r} E_{-\gamma_r} - E_\beta E_{-\beta} - E_\delta E_{-\delta} \in H^c.
\]

**Proof.** Since \(\alpha \notin \tilde{Q}_r\), we see that \(\alpha \pm \gamma_r\) is not a root, and so \(\alpha\) and \(\gamma_r\) are strongly orthogonal. Let \(\Gamma'\) be a maximal set of strongly orthogonal roots containing \(\alpha\) and \(\gamma_r\). Take any root \(\beta \sim_{\Gamma'} \frac{\alpha + \gamma_r}{2}\) and put \(\delta = \alpha + \gamma_r - \beta\). Then by (3.10) \(\delta\) is a root, and \(\delta \sim_{\Gamma'} \frac{\alpha + \gamma_r}{2}\). Since \((\beta, \gamma_r) \neq 0, (\delta, \gamma_r) \neq 0\) and neither is equal to \(\gamma_r\), we have \(\beta, \delta \in Q_r \cup \bigcup_i Q_{ir}\). Now Lemma 6.2 concludes the proof. \(
\square
\)

**Proposition 6.8.** The operator
\[
\Delta_2 = \sum_{\alpha \in Q_r \cup Q_{ir}} c_\alpha E_\alpha E_{-\alpha} - c_{\gamma_r} E_{\gamma_r} E_{-\gamma_r}
\]
is an element of \(L \oplus H^c\) for some positive constants \(c_\alpha\).
Proof. Take
\[ \mathcal{L} - \frac{1}{c} \Delta_1 - (1 - \frac{1}{c}) \sum_{\alpha \in U_i, q_i} \Delta'_\alpha - \sum_{\alpha \in U_{i<j}, Q_{ij}} \Delta'_\alpha \]
with c as in Proposition 6.5.

6.2. Berline-Vergne operators. In this subsection we restrict our attention to non-tube spaces. In Section 8 we will prove that any bounded function annihilated by the \( \mathcal{L} \oplus \mathcal{H}^c \) system is a Poisson-Szegö integral from the Shilov boundary. This class of functions is characterized by operators due to N. Berline and M. Vergne.

**Theorem 6.9** ([BV]). Let \( G/K \) be a Hermitian symmetric non-tube space and let \( f \in C^\infty(G/K) \). Then \( f \) is a Poisson-Szegö integral if and only if \( f \) is G-harmonic and
\[ \mathcal{B}V(f) = \sum_{\alpha, \beta, \delta \in Q^+} E_\alpha E_{-\beta} E_\delta \tilde{f} [E_{-\alpha}, [E_{\beta}, E_\delta]] = 0. \]

The phrase "\( f \) is G-harmonic" means \( Df = 0 \) for all operators \( D \in \mathcal{D}(G/K) \) with no constant term. (In [BV] and other works, G-harmonic functions are simply referred to as "harmonic".)

Note that \([E_{-\alpha}, [E_{\beta}, E_\delta]]\) in the expression above is an element of \( p_- \). We will compute the projection of \( \mathcal{B}V(f) \) onto the one-dimensional space spanned by \( E_{-\gamma_r} \). So we need to understand the effect of the operators \( E_\alpha E_{-\beta} E_\delta \) on \( \tilde{f} \) when \( \alpha - \beta + \delta = \gamma_r \). We get a non-zero contribution to the sum only when \( \beta - \delta = \alpha - \gamma_r \). We consider the possibilities, assuming throughout that \( f \in C^\infty(G/K) \) is annihilated by the \( \mathcal{L} \oplus \mathcal{H}^c \) system.

To simplify our notation put
\[ U_\alpha = E_\alpha E_{-\alpha} E_{\gamma_r}. \]

Since \([p_+, p_-] \subset \mathfrak{h}_C \) and \( p_+ \) is abelian, we have
\[ E_\alpha E_{-\delta} E_\gamma \tilde{f} = E_\alpha E_\gamma E_{-\delta} \tilde{f} = E_\gamma E_\alpha E_{-\delta} \tilde{f} = E_\gamma E_\delta E_\alpha \tilde{f} \]
for all \( \alpha, \beta, \gamma \) in \( Q^+ \). In particular, \( U_\alpha = E_\alpha E_{-\alpha} E_{\gamma_r} = E_{\gamma_r} E_{-\alpha} E_\alpha \).

**CASE 1.** \( \alpha = \beta = \gamma_r \), then \( \delta = -\alpha + \beta + \gamma_r = \gamma_r \) and
\[ [E_{-\gamma_r}, [E_{\gamma_r}, E_{-\gamma_r}]] = [E_{-\gamma_r}, H_{\gamma_r}] = 2E_{-\gamma_r}. \]
The coefficient of \( E_{-\gamma_r} \) will be \( 2E_{-\gamma_r} E_{-\gamma_r} E_{\gamma_r} \tilde{f} = 2U_{\gamma_r} \tilde{f} \).

**CASE 2.** \( \alpha = \beta \neq \gamma_r \), then again \( \delta = \gamma_r \). \( E_\beta \) does not commute with \( E_{-\gamma_r} \) if and only if \( \alpha = \beta \in Q_r \). Hence by the Jacobi identity
\[ [E_{-\alpha}, [E_\alpha, E_{-\gamma_r}]] = -[E_\alpha, [E_{-\gamma_r}, E_\alpha]] - [E_{-\gamma_r}, [E_{-\alpha}, E_\alpha]] = -[H_\alpha, E_{-\gamma_r}] = E_{-\gamma_r}. \]
The coefficient will be \( E_\alpha E_{-\alpha} E_{\gamma_r} \tilde{f} = U_\alpha \tilde{f} \).
Case 3. \( \alpha \neq \beta = \delta \), then \( \alpha = \gamma_r \) and as before
\[
[E_{-\gamma_r}, [E_\beta, E_\delta]] = E_{-\gamma_r}
\]
for \( \beta \in \widetilde{Q_r} \). The coefficient will be \( E_{\gamma_r} E_\delta E_\beta \bar{f} = U_\beta \bar{f} \).

Case 4. Assume that \( \alpha \neq \beta \) and \( \beta \neq \delta \) and \( \beta - \delta = \alpha - \gamma_r \) is a root.
Then \( (\alpha - \gamma_r)(H_{\gamma_r}) = 1 \), so \( \alpha \) must belong to \( \widetilde{Q_r} \). Thus \( \beta - \delta \sim (\gamma_i + \gamma_r)/2 \) or \( -\gamma_r/2 \). Since \( \beta \in Q^+ \), we must have \( \delta \in \widetilde{Q_r} \) or \( \delta = \gamma_r \). But \( \alpha \neq \beta \), so we conclude that both \( \alpha, \delta \) are in \( \widetilde{Q_r} \).

We further analyze this case in a series of lemmas:

**Lemma 6.12.** Suppose that \( \alpha, \delta \in \widetilde{Q_r}, \beta \in Q^+ \) such that \( \beta - \delta = \alpha - \gamma_r \) is a root. Then there is some \( d = \pm 1 \) such that \( dE_{-\beta}E_\delta - E_{-\alpha}E_{\gamma_r} \) and \( dE_{-\beta}E_\alpha - E_{-\delta}E_{\gamma_r} \) are in \( \mathcal{H}^e \).

**Proof.** Since \( \beta - \delta = \alpha - \gamma_r \) is a root, there is some \( d = \pm 1 \) such that
\[
[E_\beta, E_{-\delta}] = d[E_\alpha, E_{-\gamma_r}].
\]
By applying the conjugation \( \tau \), we also have
\[
[E_{-\beta}, E_\delta] = d[E_{-\alpha}, E_{\gamma_r}].
\]
From (6.13), and the fact that \( \mathcal{p}_- \) is abelian, we obtain:
\[
[E_{-\alpha}, [E_\beta, E_{-\delta}]] = d[E_{-\alpha}, [E_\alpha, E_{-\gamma_r}] = d[E_{-\gamma_r}, [E_\alpha, E_{-\alpha}] + d[E_\alpha, [E_{-\alpha}, E_{-\gamma_r}]] = d[E_{-\gamma_r}, H_\alpha] = d\gamma_r(H_\alpha)E_{-\gamma_r} = dE_{-\gamma_r}.
\]
On the other hand,
\[
[E_{-\alpha}, [E_\beta, E_{-\delta}]] = [E_\beta, [E_{-\alpha}, E_{-\delta}]] + [E_{-\delta}, [E_\beta, E_{-\alpha}]] = [E_{-\delta}, [E_\beta, E_{-\alpha}]]
\]
Thus we obtain:
\[
dH_{\gamma_r} = d[E_{\gamma_r}, E_{-\gamma_r}] = [E_{\gamma_r}, [E_{-\delta}, [E_\beta, E_{-\alpha}]]) = [E_{-\delta}, [E_{\gamma_r}, [E_\beta, E_{-\alpha}]]] + [[E_\beta, E_{-\alpha}], [E_{-\delta}, E_{\gamma_r}]] = \epsilon_1 H_\delta + \epsilon_2 H_{\beta-\alpha}.
\]
We compare this last equation to
\[
\bar{H}_{\gamma_r} = \bar{H}_\delta - \bar{H}_{\beta-\alpha}.
\]
If \( \beta - \alpha \) and \( \delta \) are not linearly independent, then \( \delta - \gamma_r = \beta - \alpha = c\delta \) for some \( c \neq 0 \), so \( \delta \) is a multiple of \( \gamma_r \), namely \( \delta = \gamma_r/2 \). Then our last calculation implies
that $\delta H_{\gamma_r} = (\epsilon_1 + \epsilon_2)H_{-\gamma_r}$, which is impossible. Thus we conclude that $\epsilon_2 \neq 0$, and therefore $\beta - \alpha = \delta - \gamma_r$ is a root.

So in fact we have $H_{\gamma_r} = H_{\delta} - H_{\beta - \alpha}$, and therefore $\epsilon_1 = d$, $\epsilon_2 = -d$. Now

$$[[E_{\beta}, E_{-\alpha}], [E_{-\delta}, E_{\gamma_r}]] = -dH_{\beta - \alpha},$$

and

$$[E_{-\delta}, E_{\gamma_r}] = \rho E_{\gamma_r - \delta} = \rho E_{\alpha - \beta}$$

with $\rho = \pm 1$. Thus

$$[E_{\beta}, E_{-\alpha}] = -\rho dE_{\beta - \alpha}$$

and hence, by applying the conjugation $\tau$,

$$(6.15) \quad [E_{-\beta}, E_{\alpha}] = \rho dE_{\alpha - \beta} = d[E_{-\delta}, E_{\gamma_r}].$$

Combining (6.14) and (6.15), we see that both $dE_{-\beta}E_{\delta} - E_{-\alpha}E_{-\gamma_r}$ and $dE_{-\beta}E_{\alpha} - E_{-\delta}E_{\gamma_r}$ are in $\ker(\varphi) = \mathcal{H}^c$.

**Lemma 6.16.** Suppose that $\alpha, \delta \in \widetilde{Q_r}$, $\beta \in Q^+$ and $\beta - \delta = \alpha - \gamma_r$ is a root. Then $U_{\alpha \tilde{f}} = U_{\delta \tilde{f}}$.

**Proof.** By Lemma 6.12,

$$U_{\alpha \tilde{f}} = E_{\alpha}E_{-\alpha}E_{\gamma_r} \tilde{f} = dE_{\alpha}E_{-\beta}E_{\delta} \tilde{f} = dE_{\delta}E_{-\beta}E_{\alpha} \tilde{f} = E_{\delta}E_{-\delta}E_{\gamma_r} \tilde{f} = U_{\delta \tilde{f}}.$$

\[\square\]

**Lemma 6.17.** For any $\alpha \in Q_{ir}$, $U_{\alpha \tilde{f}} = U_{\alpha \tilde{f}}$.

**Proof.** Take $\beta = \gamma_\iota$, $\delta = \bar{\alpha}$ in Lemma 6.16. \[\square\]

**Lemma 6.18.** For any $\alpha \in Q_{ir}$ and $\delta \in Q_r$, $U_{\alpha \tilde{f}} = U_{\delta \tilde{f}}$.

**Proof.** In light of lemmas 6.16 and 6.17 we need to show that, given $\alpha \in Q_{ir}$ and $\delta \in Q_r$, there is a $\beta \in Q_i$ such that

$\beta - \delta = \alpha - \gamma_r \in \Delta$ or $\beta - \delta = \bar{\alpha} - \gamma_r \in \Delta$.

If $\delta - \alpha \in \Delta$, then $\delta - \alpha \sim -\gamma_\iota/2$, and hence it is an element of $C_i$. Then $\beta = \delta - \alpha + \gamma_\iota \in Q_i$, so $\beta - \delta = \bar{\alpha} - \gamma_r \in \Delta$.

If $\delta - \alpha \not\in \Delta$, then $\alpha$ and $\delta$ are strongly orthogonal, and we can find a maximal set $\Gamma'$ of strongly orthogonal roots containing this pair. Since $\gamma_{\iota}(H_\alpha) = \alpha(H_{\gamma_r}) = 1$ and $\gamma_{\iota}(H_\delta) = \delta(H_{\gamma_r}) = 1$, we must have $\gamma_{\iota} \sim_{\Gamma'} (\alpha + \delta)/2$. Then $\beta = \alpha + \gamma - \gamma_r$ is a root, and $\beta - \delta = \alpha - \gamma_r \in \Delta$. \[\square\]

Thus we have shown that $U_{\alpha \tilde{f}}$ has a common value in the sets $Q_{ir}, Q_r$, and hence:

**Proposition 6.19.** $U_{\alpha \tilde{f}}$ has the same value for every $\alpha$ in $\widetilde{Q_r}$.

Combining all four cases, we now have the following consequence of Theorem 6.9:
Theorem 6.20. Let $G/K$ be a Hermitian symmetric non-tube space and let $f$ be a Poisson-Szegő integral annihilated by the $\mathcal{L} \oplus \mathcal{H}^c$ system. Then

$$BV(\tilde{f}) = (E_{\gamma_r}E_{-\gamma_r} + c \sum_{\alpha \in \tilde{Q}_r} E_\alpha E_{-\alpha} E_{\gamma_r}) \tilde{f} = 0,$$

with $c$ greater than 1.

7. The $\mathcal{L} \oplus \mathcal{H}^c$ System on Siegel Domains

Our proof of Theorem 5.3 in Section 8 uses the operators $\mathcal{L}$, $\Delta_1$, $\Delta_2$, and $BV$ together with some Fourier analysis on the group $S$. Therefore, we are going to transfer these operators to $S$ and make them act on corresponding functions there.

The scheme is as follows. Given a function $f$ on $D$ and a function $\tilde{f}(g) = f(g \cdot o)$ on $G$, we choose coordinates $z = \sum z_\alpha E_\alpha$ in $p_+$ to express $\mathcal{L}$, $\Delta_1$, $\Delta_2$, $BV$ in partial derivatives (see (7.10)).

Let $\tilde{U}$ be one of operators $\mathcal{L}$, $\Delta_1$, $\Delta_2$, $BV$ and $U$ the corresponding right-hand side operator in (7.10). Since $\tilde{U}$ is left-invariant on $G$ we have

$$U(f \circ g)(o) = \tilde{U}\tilde{f}(g).$$

Next we need to have (7.10), but on $^cD$ for the function

$$(7.1) \quad ^c f(x) = f(c^{-1} \cdot x).$$

For that we compute the differential of the Cayley transform $c$, we choose convenient coordinates (7.11) in $p_+$ and we write $\tilde{U}\tilde{f}(e)$ as in (7.12) i.e.

$$\tilde{U}\tilde{f}(e) = (^cU^c f)(c \cdot o),$$

there $^cU$ denotes the corresponding operator on the right-hand side of (7.12).

Finally, we extend $^cU$ by $S$-invariance

$$((^cU^c \tilde{f})(s) = ^c U(^c f \cdot s)(c \cdot o),$$

where

$$(7.2) \quad ^c \tilde{f}(s) = ^c f(s \cdot (c \cdot o)).$$

Then, by invariance, $\tilde{U}\tilde{f} = 0$ implies $^cU^c \tilde{f} = 0$. Then $\mathcal{L}$, $\Delta_1$, $\Delta_2$, $BV$ are written in Theorem 7.24 in terms of some basic building blocks $\Delta(W, Z)$, as explained in (7.13).

7.1. Operators on the Domain $D$. All our operators are sums of elements having the form $E_\alpha E_{-\alpha} E_{\gamma_r}$ and $E_\alpha E_{\gamma_r}$. In $D \subset p_+$ let us introduce complex coordinates $\{z_\alpha\}_{\alpha \in Q^+}$ (we shall denote $z_\alpha = x_\alpha + iy_\alpha$) corresponding to the root space decomposition $z = \sum z_\alpha E_\alpha$. Then we prove:
Let the algebra spanned by $E_j$ be denoted by $E$. By (7.7):

$$E_jE_j = (X_j^2 + Y_j^2)\tilde{f},$$

$$E_jE_k\tilde{f} = E_kE_j\tilde{f} = (X_jX_k + X_kX_j + i(Y_jX_k + Y_kX_j))\tilde{f}.$$

To prove the above theorem we need to understand better the action of the group $G$ on the domain $\mathcal{D}$. It is known ([K]) that for the case of $\mathcal{D} \cong SU(n, m)/SU(n) \times SU(m)$ and $\mathfrak{p}_+ = M_{n,m}$:

$$g \cdot 0 = BD^{-1}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(n, m)$. Using this formula we shall be able to compute the action of some elements of $G$ on $\mathcal{D} \cong G/K$ in the general case.

**Lemma 7.8 ([H1],[KW]).** Let $\alpha \in Q^+$, then

$$\exp tX_\alpha \cdot o = \tanh(t)E_\alpha, \quad \exp rX_\alpha \cdot tE_\alpha = \frac{t \cosh(r) + \sinh(r)}{\cosh(r) + t \sinh(r)}E_\alpha,$$

$$\exp tY_\alpha \cdot o = i \tanh(t)E_\alpha, \quad \exp rY_\alpha \cdot tE_\alpha = \frac{t \cosh(r) + i \sinh(r)}{\cosh(r) - i \sinh(r)}E_\alpha.$$

**Proof.** The algebra spanned by $E_\alpha, E_\bar{\alpha}, H_\alpha$ is isomorphic with $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(1, 1)^\mathbb{C}$. By $j^{-1}$ we shall denote the isomorphism:

$$j^{-1}(E_\alpha) = E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad j^{-1}(E_{-\alpha}) = \bar{E} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad j^{-1}(H_\alpha) = H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which extends to an isomorphism $j$ of $\exp Lie(E_\alpha, E_\bar{\alpha}, H_\alpha)$ and $SL(2, \mathbb{C})$. Moreover, $j$ commutes with taking the $P^+$ component in both $G^\mathbb{C}$ and $SL(2, \mathbb{C})$:

$$p_+ \circ j = j \circ \bar{p}_+.$$

Therefore, by (7.7):

$$\exp tX_\alpha \cdot o = p_+ (\exp tX_\alpha) = p_+ (j(\exp tX)) = j \circ \bar{p}_+ (\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix})$$

$$= j \circ \log \begin{pmatrix} 1 & \tanh(t) \\ 0 & 1 \end{pmatrix} = \tanh(t)E_\alpha,$$

$$\exp rX_\alpha \cdot tE_\alpha = p_+ (\exp rX_\alpha \exp tE) = j \circ \bar{p}_+ (\begin{pmatrix} \cosh(r) & t \cosh(r) + \sinh(r) \\ \sinh(r) & \cosh(r) + t \sinh(r) \end{pmatrix})$$

$$= j \circ \log \begin{pmatrix} 1 & \frac{t \cosh(r) + \sinh(r)}{\cosh(r) + t \sinh(r)} \\ 0 & 1 \end{pmatrix} = \frac{t \cosh(r) + \sinh(r)}{\cosh(r) + t \sinh(r)}E_\alpha.$$
Using Lemma 7.8 we conclude:

\[ X_\alpha \tilde{f}(e) = \partial_{x_\alpha}^2 f(o), \quad Y_\alpha \tilde{f}(e) = \partial_{y_\alpha}^2 f(o), \]
\[ X_\alpha f(o) - 2 \partial_{x_\alpha} f(o), \quad Y_\alpha f(o) - 2 \partial_{y_\alpha} f(o), \]
\[ X_\alpha Y_\alpha \tilde{f}(e) = \partial_{x_\alpha} \partial_{y_\alpha}^2 f(o) + 2 \partial_{x_\alpha} \partial_{y_\alpha} f(o), \quad Y_\alpha X_\alpha \tilde{f}(e) = \partial_{y_\alpha} \partial_{x_\alpha}^2 f(o) + 2 \partial_{y_\alpha} \partial_{x_\alpha} f(o), \]

which by (7.6) proves (7.4) and (7.5) for \( \alpha = \gamma_r \). To cope with the general version of equation (7.5) we need a stronger version of Lemma 7.8:

**Lemma 7.9.** If \( \alpha \in Q_r \cup \bigcup_i Q_{ir} \) then

\[
\exp sX_{\gamma_r} \cdot tE_\alpha = \frac{t}{\cosh(s)} E_\alpha + \tanh(s) E_{\gamma_r},
\]
\[
\exp sY_{\gamma_r} \cdot tE_\alpha = \frac{t}{\cosh(s)} E_\alpha + \tanh(s) (iE_{\gamma_r}).
\]

**Proof.** Put \( \beta = \alpha - \gamma_r \), then by the Restricted Roots Theorem \( \beta \) is a root and we may assume

\[
E_\beta = [E_\alpha, E_{-\gamma}], \\
E_{-\beta} = [E_{\gamma_r}, E_{-\alpha}], \\
H_\beta = [E_\beta, E_{-\beta}] = H_\alpha - H_{\gamma_r}.
\]

The vectors \( E_\alpha, E_\beta, E_{\gamma_r}, E_{-\alpha}, E_{-\beta}, E_{-\gamma_r}, H_\alpha, H_\beta \) span a subalgebra of \( g_\mathbb{C} \) isomorphic with \( sl(3, \mathbb{C}) = su(2, 1)_\mathbb{C} \):

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Applying (7.7) we obtain

\[
\exp sX_{\gamma_r} \cdot tE_\alpha = p_+ \circ j(\exp sX_3 \exp tE_2) = j \circ \tilde{p}_+ \begin{pmatrix}
1 & 0 & t \\
0 & \cosh(s) & \sinh(s) \\
0 & \sinh(s) & \cosh(s)
\end{pmatrix},
\]
\[
\exp sY_{\gamma_r} \cdot tE_\alpha = j \circ \tilde{p}_+ \begin{pmatrix}
1 & 0 & t \\
0 & \cosh(s) & i \sinh(s) \\
0 & -i \sinh(s) & \cosh(s)
\end{pmatrix}.
\]
\[
\frac{t}{\cosh(s)} E_\alpha + \tanh(s)(iE_{\gamma}).
\]

By Lemma 7.9:

\[
\begin{align*}
X_\gamma X_\alpha^2 \tilde{f}(e) &= \partial_{x_\gamma} \partial_{x_\alpha}^2 f(o), \\
X_\gamma Y_\alpha^2 \tilde{f}(e) &= \partial_{x_\gamma} \partial_{y_\alpha}^2 f(o), \\
Y_\gamma X_\alpha^2 \tilde{f}(e) &= \partial_{y_\gamma} \partial_{x_\alpha}^2 f(o), \\
Y_\gamma Y_\alpha^2 \tilde{f}(e) &= \partial_{y_\gamma} \partial_{y_\alpha}^2 f(o),
\end{align*}
\]

which finishes the proof of Theorem 7.3.

Having this result it is easy to write our operators on the domain at the point \(o\):

\[
\begin{align*}
\mathcal{L}\tilde{f}(e) &= \left( \sum_{\alpha \in Q^+} \partial_{z_\alpha} \partial_{z_\alpha} \right) f(o), \\
\Delta_1\tilde{f}(e) &= \left( \sum_{\alpha \in \mathcal{U} \cup Q_i} \partial_{z_\alpha} \partial_{z_\alpha} + c_0 \sum_{\gamma \in \Gamma} \partial_{z_\gamma} \partial_{z_\gamma} \right) f(o), \\
\Delta_2\tilde{f}(e) &= \left( \sum_{\alpha \in \mathcal{Q}_i \cup \mathcal{Q}_{ij}} c_\alpha \partial_{z_\alpha} \partial_{z_\alpha} - c_{\gamma_i} \partial_{z_\gamma} \partial_{z_\gamma} \right) f(o), \\
BV\tilde{f}(e) &= \left( c_1 \sum_{\alpha \in \mathcal{Q}_j \cup \mathcal{Q}_{ij}} \partial_{z_\alpha} \partial_{z_\alpha} \partial_{z_\gamma} + \partial_{z_\gamma} \partial_{z_\gamma} \partial_{z_\gamma} \right) f(o).
\end{align*}
\] (7.10)

7.2. The Cayley transform. In order to write down the above operators on \(S\), we compute the differential of \(c\). We shall use the formula given by [S], Lemma II.5.3, which says that the Jacobian of the mapping \(z \mapsto c \cdot z\) at the point \(o\) is given by

\[
\text{Jac}(o \mapsto c \cdot o) = \text{Ad} c_K|_{p^+},
\]

where \(c_K\) denotes the \(K^C\) component of \(c\) in the decomposition \(P^+ K^C P^-\). By [KW], Lemma 3.5:

\[
c_K = \exp \left( - \sum_{1 \leq i \leq r} \log \cosh \left( \frac{\pi}{4} i \cdot H_{\gamma_i} \right) \right) = \prod_{1 \leq i \leq r} \exp \left( - \log \frac{\sqrt{2}}{2} \cdot H_{\gamma_i} \right).
\]

Hence the vectors \(E_\alpha\) are eigenvectors of \(\text{Ad}(c_K)\):

\[
\text{Ad} c_K(E_\alpha) = \begin{cases} 
2E_\alpha, & \text{for } \alpha \in \Gamma, \\
2E_\alpha, & \text{for } \alpha \in Q_{ij}, \\
\sqrt{2}E_\alpha, & \text{for } \alpha \in Q_j.
\end{cases}
\]

Therefore, to have the operators on \(cD\) it is enough to multiply all terms by appropriate constants. But it will be more convenient for us to write the result using the basis (3.16):

\[
v = u + it = \sum v_j X_j + \sum_{\alpha \in \mathcal{Q}_i} v_\alpha X_\alpha + \sum_{\alpha \in \mathcal{Q}_{ij}} v^k_\alpha X^k_\alpha.
\]
We assume below that $G/K$ is of non-tube type (for tube spaces using similar methods we get the same values of $^c \mathcal{L}$ and $^c \Delta_1$). Then it is known that $\alpha \neq \tilde{\alpha}$ for $\alpha \in \mathcal{Q}_{ij}$.

By (3.16):

$$x_\alpha = \frac{1}{\sqrt{2}} (u_\alpha^1 - t_\alpha^2), \quad y_\alpha = \frac{1}{\sqrt{2}} (u_\alpha^2 + t_\alpha^1),$$

(7.11)

$$x_{\tilde{\alpha}} = -\frac{\varepsilon_\alpha}{\sqrt{2}} (u_\alpha^1 + t_\alpha^2), \quad y_{\tilde{\alpha}} = \frac{\varepsilon_\alpha}{\sqrt{2}} (u_\alpha^2 - t_\alpha^1),$$

therefore

$$\partial_{z_{\alpha}} \partial_{\bar{z}_{\alpha}} {^c f(c \cdot o)} = (\partial_{x_{\alpha}}^2 + \partial_{y_{\alpha}}^2) {^c f(c \cdot o)}$$

$$= \left( \frac{1}{2} (\partial_{u_{\alpha}^1} - \partial_{t_{\alpha}^2})^2 + \frac{1}{2} (\partial_{u_{\alpha}^2} + \partial_{t_{\alpha}^1})^2 \right) {^c f(c \cdot o)},$$

$$\partial_{\bar{z}_{\alpha}} \partial_{z_{\alpha}} {^c f(c \cdot o)} = \left( \frac{1}{2} (\partial_{u_{\alpha}^1} + \partial_{t_{\alpha}^2})^2 + \frac{1}{2} (\partial_{u_{\alpha}^2} - \partial_{t_{\alpha}^1})^2 \right) {^c f(c \cdot o)},$$

$$(\partial_{z_{\alpha}} \partial_{\bar{z}_{\alpha}} + \partial_{\bar{z}_{\alpha}} \partial_{z_{\alpha}}) {^c f(c \cdot o)} = (\partial_{w_{\alpha}} \partial_{\bar{w}_{\alpha}} + \partial_{\bar{w}_{\alpha}} \partial_{w_{\alpha}}) {^c f(c \cdot o)}.$$

Finally for any Hermitian symmetric space $G/K$:

$$\mathcal{L} \tilde{f}(c) = 4 \left( \sum_{\alpha \in \mathcal{Q}_i} \partial_{w_{\alpha}} \partial_{\bar{w}_{\alpha}} + \sum_{\alpha \in \mathcal{Q}_{ij}} \partial_{v_{\alpha}} \partial_{w_{\alpha}} + \sum_{j} \partial_{v_{\alpha}} \partial_{\bar{v}_{\alpha}} \right) {^c f(c \cdot o)},$$

(7.12)

$$\Delta_1 \tilde{f}(c) = 4 \left( \sum_{\alpha \in \mathcal{Q}_i} \partial_{w_{\alpha}} \partial_{\bar{w}_{\alpha}} + c_0 \sum_{j} \partial_{v_{\alpha}} \partial_{\bar{v}_{\alpha}} \right) {^c f(c \cdot o)},$$

and for $G/K$ being of non-tube type:

$$BV \tilde{f}(c) = 8 \left( c_1 \left( \sum_{\alpha \in \mathcal{Q}_i} \partial_{w_{\alpha}} \partial_{\bar{w}_{\alpha}} \partial_{v_{\alpha}} \partial_{v_{\alpha}} + \sum_{\alpha \in \mathcal{Q}_{ij}} \partial_{v_{\alpha}} \partial_{w_{\alpha}} \partial_{v_{\alpha}} \partial_{v_{\alpha}} \right) + \partial_{v_{\alpha}} \partial_{v_{\alpha}} \partial_{v_{\alpha}} \partial_{v_{\alpha}} \right) {^c f(c \cdot o)},$$

$$\Delta_2 \tilde{f}(c) = 4 \left( \sum_{\alpha \in \mathcal{Q}_i} c_\alpha \partial_{w_{\alpha}} \partial_{\bar{w}_{\alpha}} + \sum_{\alpha \in \mathcal{Q}_{ij}} (c_\alpha \partial_{w_{\alpha}} \partial_{\bar{w}_{\alpha}} + c_\alpha \partial_{v_{\alpha}} \partial_{v_{\alpha}}) - c_{\gamma} \partial_{v_{\alpha}} \partial_{v_{\alpha}} \right) {^c f(c \cdot o)},$$

where in the last operator it is more convenient not to change coordinates for $\alpha \in Q_{ir}$.

### 7.3. $S$-invariant operators on Siegel domains

Now we extend the above operators by $S$-invariance. Let $W$ and $Z$ be $S$-invariant holomorphic vector fields which agree with $\partial_w$ and $\partial_z$ at $o$. We have to find $S$-invariant operators on $^c \mathcal{D}$, $\Delta(W, Z)$ and $D(W, Z)$, such that:

$$\Delta(W, Z) {^c \tilde{f}(c)} = \partial_w \partial_z {^c f(c \cdot o)},$$

$$D(W, Z) {^c \tilde{f}(c)} = \partial_w \partial_{\bar{w}} \partial_z {^c f(c \cdot o)}$$

for any function $^c f$ on $^c \mathcal{D}$.

The operators $\Delta(W, Z)$ are explicitly computed in [DHMP]:

(7.13)

$$\Delta(W, Z) = W \bar{Z} - \nabla_W \bar{Z} = \bar{Z} W - \nabla_{\bar{Z}} W$$
where $\nabla$ denotes the Riemannian connection on $S$. $\Delta(Z, Z)$ is a real, second order, elliptic degenerate operator which annihilates holomorphic (consequently pluriharmonic) functions. Moreover any left-invariant operator with the above properties is a linear combination of $\Delta(W, Z)$'s. To simplify our notation we shall denote $\Delta(Z, Z)$ by $\Delta_Z$. In view of (3.18)

$$
\Delta_Z^j \tilde{f}(e) = \partial_{v_j} \partial_{v_j}^c f(c \cdot o),
$$

$$
\Delta_Z^k \tilde{f}(e) = \partial_{v_k} \partial_{v_k}^c f(c \cdot o),
$$

$$
\Delta_Z^\alpha \tilde{f}(e) = \partial_{v_\alpha} \partial_{v_\alpha}^c f(c \cdot o),
$$

and for $\alpha \in \overline{Q}_{ir}$

$$
\Delta_W^{\alpha} \tilde{f}(e) = \partial_{z_\alpha} \partial_{z_\alpha}^c f(c \cdot o),
$$

$$
\Delta_W^{\tilde{\alpha}} \tilde{f}(e) = \partial_{\tilde{z}_\alpha} \partial_{\tilde{z}_\alpha}^c f(c \cdot o),
$$

where

$$
W_\alpha = \frac{1}{\sqrt{2}} \left[ (X^1_\alpha - Y^2_\alpha) - i(X^2_\alpha + Y^1_\alpha) \right],
$$

$$
W_{\tilde{\alpha}} = -\frac{\varepsilon_{\alpha}}{\sqrt{2}} \left[ (X^1_\alpha + Y^2_\alpha) + i(X^2_\alpha - Y^1_\alpha) \right].
$$

(7.14)

It remains to calculate the operators $D(W, Z_r)$.

**Theorem 7.15.** Let $W$ be one of the holomorphic vector fields $Z_{\alpha}, Z_{\beta}^k, Z_r$, for $\alpha \in Q_r, \beta \in \bigcup_i \overline{Q}_{ir}$ then

$$
D(W, Z_r) = W \Delta(W, Z_r) - i c_W \Delta_{Z_r}
$$

with a positive constant $c_W$.

**Proof.** We begin the proof with some general observations. Given any coordinates $\{w_i\}$ in $\mathcal{D}$, take $W_n$ to be the $S$-invariant holomorphic vector field such that:

$$
W_n g(c \cdot o) = \partial_{w_n} g(c \cdot o),
$$

(7.16)

then $W_n$ can be written in the form

$$
W_n g(z) = \sum_i h^n_i(z) \partial_{w_i} g(z)
$$

and of course by (7.16)

$$
h^n_i(c \cdot o) = \delta_{in}.$$

Let us write the building blocks $\Delta(W_p, W_q)$ in these coordinates.

$$\Delta(W_p, W_q) = |W_p W_q - \nabla W_p W_q|$$

$$= (\sum_i h_i^p(z) \partial_{w_i}) (\sum_j h_j^q(z) \partial_{w_j}) - \sum_i h_i^p(z) \nabla \partial_{w_i} (\sum_j h_j^q(z) \partial_{w_j})$$

$$= \sum_{i,j} h_i^p(z) h_j^q(z) \partial_{\overline{w}_i} \partial_{w_j} + \sum_{i,j} h_i^p(z) \partial_{\overline{w}_i} h_j^q(z) \partial_{w_j}$$

$$- \sum_{i,j} h_i^p(z) \partial_{w_i} h_j^q(z) \partial_{w_j} - \sum_{i,j} h_i^p(z) h_j^q(z) \nabla \partial_{\overline{w}_i} \partial_{w_j}$$

$$= \sum_{i,j} h_i^p(z) h_j^q(z) \partial_{\overline{w}_i} \partial_{w_j},$$

Since

$$W_n \Delta(W_n, W_k) g(c \cdot o) = W_n (\sum_{i,j} h_i^n(z) h_j^k(z) \partial_{\overline{w}_i} \partial_{w_j}) g(c \cdot o)$$

$$= (\sum_{i,j} h_i^n(z) h_j^k(z) \partial_{w_i} \partial_{\overline{w}_j} + \sum_{i,j} \partial_{w_i} (h_i^n h_j^k)(z) \partial_{\overline{w}_j}) g(c \cdot o)$$

$$= \partial_{w_i} \partial_{\overline{w}_j} g(c \cdot o) + \sum_{i,j} \partial_{w_i} (h_i^n h_j^k)(c \cdot o) \partial_{\overline{w}_j} g(c \cdot o),$$

we see that

$$D(W_n, W_k) = W_n \Delta(W_n, W_k) - \sum_{i,j} \partial_{w_i} (h_i^n h_j^k)(c \cdot o) \Delta(W_i, W_j).$$

Thus we need to compute:

$$\partial_{w_i} (h_i^n h_j^k)(c \cdot o) = (\partial_{w_i} h_i^n)(c \cdot o) h_j^k(c \cdot o) + h_i^n(c \cdot o) (\partial_{w_i} h_j^k)(c \cdot o)$$

and to do it, we are going to use the action of the $W_i$'s on $cD$.

Taking $g(z) = z_j$ we get

$$W_k g(z) = \sum_i h_i^k(z) \partial_{w_i} g(z) = h_j^k(z)$$

For $z = s \cdot (c \cdot o)$:

$$h_j^k(z) = W_k g(s) = \frac{1}{2} (X_k - iY_k) g(s)$$

$$= \frac{1}{2} \left( \partial_t g(s \exp t X_k)|_{t=0} - i \partial_t g(s \exp t Y_k)|_{t=0} \right)$$
To simplify the notation we write \( g(s) \) instead of \( g(s \cdot o) \) identifying the function on \( S \) with that on \( cD \). Furthermore,

\[
\partial w_n h_j^k (c \cdot o) = W_n h_j^k (c \cdot o) = \frac{1}{2} (X_n - iY_n) h_j^k (c \cdot o)
\]

\[
= \frac{1}{2} \left( \partial_s h_j^k (\exp s X_n)|_{s=0} - i \partial_s h_j^k (\exp s Y_n)|_{s=0} \right)
\]

(7.19)

\[
= \frac{1}{4} \partial_s \partial_t \left( \left( g(\exp s X_n \exp t X_k) - g(\exp s Y_n \exp t Y_k) \right)
\]

\[
- i \left( g(\exp s X_n \exp t Y_k) + g(\exp s Y_n \exp t X_k) \right) \right) \bigg|_{t=0}^{s=0}
\]

and

\[
\partial w_n h_i^m (c \cdot o) = \partial w_n h_i^m (c \cdot o)
\]

(7.20)

\[
= \frac{1}{4} \partial_s \partial_t \left( \left( \overline{g}(\exp s X_n \exp t X_n) - \overline{g}(\exp s Y_n \exp t Y_n) \right)
\]

\[
+ i \left( \overline{g}(\exp s X_n \exp t Y_n) - \overline{g}(\exp s Y_n \exp t X_n) \right) \right) \bigg|_{t=0}^{s=0}
\]

From now we shall use the standard notation (3.13) of Siegel domains. To prove the theorem we need to consider \( W_n = X - iY \in \{ Z_\alpha, Z_\alpha^k, Z_r \} \) and \( W_k = Z_r = X_r - iY_r \). Let us also recall the action of the group \( S \) on the domain \( cD \):

(7.21) \((\zeta, x)s \cdot (w, z) = (\zeta + \sigma(s)w, x + sz + 2i\Phi(\sigma(s)w, \zeta) + i\Phi(\zeta, \zeta)), \)

where \( \sigma \) is a representation of \( N_0 \tilde{A} \):

\[
\sigma : N_0 \tilde{A} \ni s \mapsto \sigma(s) \in GL(Z).
\]

Observe that (7.21) implies:

\[
\partial_s \partial_t g(\exp s X \exp t X_r)|_{t=0}^{s=0} = 0,
\]

\[
\partial_s \partial_t g(\exp s X \exp t H_r)|_{t=0}^{s=0} = 0,
\]

which simplifies (7.19) and (7.20). Furthermore

(7.22) \( \partial_s \partial_t g(\exp s Y \exp t X_r)|_{t=0}^{s=0} \) and \( \partial_s \partial_t g(\exp s Y \exp t H_r)|_{t=0}^{s=0} \)

are nonzero only for \( Y = H_r \). Thus for \( W_n \in \{ Z_\alpha, Z_\alpha^k \} \) the formula (7.18) reduces to:

(7.18') \( \partial w_n (h_i^m(z) h_j^k(z)) (c \cdot o) = (\partial w_n \overline{h_i^m})(c \cdot o). \)

while for \( W_n = Z_r \) (8.12) becomes

(7.18'') \( \partial z_r (h_i^m(z) h_j^k(z)) (c \cdot o) = (\partial z_r \overline{h_i^m} + \partial z_r h_j^k)(c \cdot o). \)
Now we are going to calculate (7.18') and (7.18'').

**Case I.** $W_n = Z_r$. Then $\exp tX_r \in V_{rr}$, $\exp tH_r \in a_r$. Notice that in (7.19) the values $\partial_s \partial_t g$ are nonzero only if $g((\zeta, z)) = z_r$. Then

$$\partial_s \partial_t g(\exp sH_r \exp tX_r \cdot (c \cdot o))|_{t=0} = \partial_s \partial_t g(\exp sH_r \cdot \exp tX_r + i \exp sH_r)|_{t=0} = 1,$$

$$\partial_s \partial_t g(\exp sH_r \exp tH_r \cdot (c \cdot o))|_{t=0} = \partial_s \partial_t g(i \exp(s + t)H_r)|_{t=0} = i,$$

and we get

$$\partial_s h_r^*(o) = -2i.$$

Similarly using (7.20) we obtain:

$$\partial_s h_r^*(o) = -2i.$$

Hence by (7.18'')

$$\partial_s (h_r^*(z)h_r^*(z))(c \cdot o) = -4i.$$

**Case II.** $W_n = Z_k^\alpha$ for $\alpha \in Q_{jr}$. Let $x_t = \exp tX_k^\alpha$ then $x_t \in V_{jr}$. To describe the action of $y_t = \exp tY_k^\alpha = \exp(2x_t \Box c_j)$ we need a Jordan algebra lemma ([FK], Lemma VI.3.1):

**Lemma 7.23.** Take $z \in V$ and decompose $z = z_1 + z_2 + z_0$ into its Peirce decomposition with respect to $c_j$. Then for $a = \exp(2x_t \Box c_j)z$, we have:

- $a_1 = z_1$,
- $a_2 = 2L(x_t)z_1 + z_2$,
- $a_0 = 2L(e - c_j)L(x_t)^2 z_1 + 2L(e - c_j)L(x_t)z_2 + z_0$. $\square$

We shall also use a key fact: $V_{ij} \cdot V_{ij} \subset V_{ii} + V_{jj}$. We have:

$$y_s(x_t) = x_t + 2L(e - c_j)L(x_s)(x_t) = x_t + C stX_r, \ C \neq 0.$$

Therefore,

$$\partial_s \partial_t g(\exp sY \exp tX)|_{t=0} = C$$

for $g(\zeta, z) = z_r$ and it is zero for other variables. We have

$$y_t \cdot e = e + 2L(x_t)c_j + 2L(e - c_j)L(x_t)^2 c_j$$

$$= e + x_t + L(e - c_j)L(x_t)x_t$$

To calculate

$$\partial_s \partial_t g(\exp sY \exp tY)|_{t=0}$$

we need to know only the term with $st$ in $y_s \cdot (y_t \cdot e)$. Therefore it is enough to calculate

$$y_s \cdot (e + x_t) = e + x_s + L(e - c_j)L(x_s)x_s + x_t + 2L(e - c_j)L(x_s)x_t.$$

Only the last term matters, it is equal $iC stX_r$ and so

$$\partial_s \partial_t g(\exp sY \exp tY)|_{t=0} = -iC.$$
Finally the only nonzero term in (7.18') is
\[ \partial_z^k h^\alpha_{\nu,k}(0) = -2iC. \]

**Case III.** \( W_n = Z_\alpha \) for \( \alpha \in Q_r \). Then \( \exp tX = \exp tX_\alpha = tX_\alpha, \Phi(X_\alpha, X_\alpha) = CX_r \) and for \( g((\zeta, z)) = z_r \nabla \frac{\partial_s^s}{\partial t^t} \exp sX \exp tX) |_{t=0} = \partial_s^s \partial_t^t 2sti\Phi(x, x) = -2iC. \)

By analogy, for \( \exp tY = \exp tY_\alpha = itX_\alpha \):
\[ \partial_s^s \partial_t^t \exp sY \exp tY |_{t=0} = -2iC, \]
\[ \partial_s^s \partial_t^t \exp sX \exp tY |_{t=0} = -2C, \]
\[ \partial_s^s \partial_t^t \exp sY \exp tX |_{t=0} = 2C. \]

Thus
\[ \partial_{z_\alpha} h^\alpha_{\nu}(0) = -8iC. \]

\[ \square \]

Now we are ready to transfer operators \( L, \Delta_1, \Delta_2, BV \) to \( S \):

**Theorem 7.24.** The images of operators \( L, \Delta_1, \Delta_2 \) and \( BV \) are respectively:
\[
\begin{align*}
\mathcal{L}^c &= \sum_{\alpha \in Q_r} \Delta Z_\alpha + \sum_i \Delta Z_i + \sum_{\alpha \in i Q_i j} \Delta Z^k_{\alpha}, \\
\Delta_1^c &= \sum_{\alpha \in Q_r} \Delta Z_\alpha + c_0 \sum_i \Delta Z_i, \\
\Delta_2^c &= \sum_{\alpha \in Q_r} c_\alpha \Delta Z_\alpha + \sum_{\alpha \in i Q_i j} (c_\alpha \Delta W_\alpha + c_{\bar{\alpha}} \Delta W_{\bar{\alpha}}) - c_r \Delta Z_r, \\
BV^c &= A \left( \sum_{\alpha \in Q_r} Z_\alpha \Delta(Z_\alpha, Z_r) + \sum_{\alpha \in i Q_i j} Z^k_{\alpha} \Delta(Z^k_\alpha, Z_r) \right) + Z_r \Delta Z_r - iB \Delta Z_r,
\end{align*}
\]

where \( W_\alpha, W_{\bar{\alpha}} \) are as in (7.14), \( c_0, c_\alpha, c_r, B \) are strictly positive constants and \( A > 1 \).

One can simplify the operator \( BV^c \). For that we use the following formula for the Bergmann connection
\[ \nabla_W Z = \pi_Q([W, Z]), \]
(see e.g. [BBDHPT]) where \( W, Z \) are holomorphic vectors fields and \( \pi_Q \) denotes the projection onto the space of holomorphic vectors fields. Then for \( Z \in \{ Z_\alpha, Z^k_\alpha \} \) we have
\[
\begin{align*}
\nabla_Z Z_r &= \pi_Q [X + iY, X_r - iH_r] = \frac{1}{2} \pi_Q (iX - Y) = \frac{1}{2} \pi_Q (i(X - iY)) = 0.
\end{align*}
\]
which by (7.13) implies
\[ \Delta(Z, Z_r) = ZZ_r. \]
Furthermore applying
\[ Z\overline{Z} = (X - iY)(X + iY) = X^2 + Y^2 + i[X, Y] = X^2 + Y^2 - iX_r, \]
we obtain
\[ ^cBV = A \left( \sum_{\alpha \in Q_r} ((\mathcal{X}_\alpha)^2 + (\mathcal{Y}_\alpha)^2) + \sum_{\alpha \in \mathcal{U}_i \setminus Q_r} ((\mathcal{X}_\alpha^k)^2 + (\mathcal{Y}_\alpha^k)^2) \right) Z_r \]
\[ - kAiX_rZ_r + Z_r \Delta_{Z_r} - iB\Delta_{Z_r} \]
for \( k = q_1 + (r - 1)q_2 \). (Recall that \( q_1 \) denotes the cardinality of the sets \( Q_i \) and \( q_2 \) the cardinality of the \( Q_{ij} \)'s.)

Similarly, we rewrite \(^c\Delta_2\). If \( \alpha \in Q_{jr} \) then
\[ [Y_{\alpha}^{\varepsilon_1}, X_{\alpha}^{\varepsilon_2}] = X_r \]
when \( \varepsilon_1 = \varepsilon_2 \) and \([Y_{\alpha}^{\varepsilon_1}, X_{\alpha}^{\varepsilon_2}] = 0 \) when \( \varepsilon_1 \neq \varepsilon_2 \). Now using (7.13) and (7.26) we calculate
\[ \Delta_{Z_r} = X_r^2 + H_r^2 - H_r \]
\[ \Delta_{W_\alpha} = \frac{1}{2}(X_\alpha^1 - Y_\alpha^2)^2 + \frac{1}{2}(X_\alpha^2 + Y_\alpha^1)^2 - H_r \]
\[ \Delta_{W_{\overline{\alpha}}} = \frac{1}{2}(X_\alpha^1 + Y_\alpha^2)^2 + \frac{1}{2}(X_\alpha^2 - Y_\alpha^1)^2 - H_r \]
\[ \Delta_{Z_{\overline{\alpha}}} = X_{\alpha}^2 + Y_{\alpha}^2 - H_r \]
and so
\[ ^c\Delta_2 = 4\left( - c_{Y_r}(X_r^2 + H_r^2 - H_r) + \sum_{\alpha \in \mathcal{U}_i \setminus Q_r} c_{\alpha}\left( \frac{1}{2}(X_\alpha^1 - Y_\alpha^2)^2 + \frac{1}{2}(X_\alpha^2 + Y_\alpha^1)^2 - H_r \right) \right) \]
\[ + \sum_{\alpha \in \mathcal{U}_i \setminus Q_r} c_{\overline{\alpha}}\left( \frac{1}{2}(X_\alpha^1 + Y_\alpha^2)^2 + \frac{1}{2}(X_\alpha^2 - Y_\alpha^1)^2 - H_r \right) + \sum_{\alpha \in Q_r} c_{\alpha}(X_{\alpha}^2 + Y_{\alpha}^2 - H_r) \].

8. Proof of the main theorem

We turn now to the proof of Theorem 5.3. Our proof will depend on whether or not \( G/K \) is of tube type. The first step, given by Proposition 8.1, applies to both cases.
8.1. Poisson-Szegő integrals. We will show that bounded \( \mathcal{L} \oplus \mathcal{H}^c \)-harmonic functions are Poisson-Szegő integrals. In fact, we have the following.

**Proposition 8.1.** Let \( F \) be a bounded function on \( S \) annihilated by \( ^c \mathcal{L} \) and \( ^c \Delta_1 \). Then \( F \) is the Poisson-Szegő integral of a function \( f \in L^\infty(N(\Phi)) \).

**Proof.** The first part of this proof closely follows that for Proposition 3.3 in [BBDHPT]. Since \( ^c \mathcal{L} F = 0 \), \( F \) can be written as the Poisson integral of a function \( \tilde{f} \) over \( N = N(\Phi)N_0 \). As in [BBDHPT] we can assume that \( \tilde{f} \) is continuous. We will prove below that \( y \mapsto \tilde{f}(y) \) is constant on \( N_0 \). It then follows that \( F \) is the Poisson-Szegő integral of \( f = \frac{\tilde{f}|_{N(\Phi)}}{N(\Phi)} \) over \( N(\Phi) \), completing the proof.

Let \( H = \sum_j H_j \) and set \( F_t(g) = F(exp(tH)g) \) for \( g \in S \). By Formula (37) in [BBDHPT], \( F_H = \lim_{t \to -\infty} F_t \) exists and is given by

\[
F_H(wya) = \int_{N_0} \tilde{f}(yaua^{-1}) \left( \int_{N(\Phi)} \tilde{P}_L(vu) dv \right) du
\]

where \( w, v \in N(\Phi), y, u \in N_0, a \in \tilde{A} \), and \( \tilde{P}_L \) denotes the Poisson kernel for \( ^c \mathcal{L} \) on \( N = N(\Phi)N_0 \).

The operators \( ^c \mathcal{L} \) and \( ^c \Delta_1 \) act from the right and annihilate \( F \). So \( ^c \mathcal{L} F_t = 0 = ^c \Delta_1 F_t \) and hence

\[
^c \mathcal{L} F_H = 0, \quad ^c \Delta_1 F_H = 0.
\]

Moreover, as in Lemma 3.4 in [BBDHPT], one can use (8.2) to show that \( t \mapsto F_H(wyaexp(tH)) \) is constant and hence

\[
HF_H = 0.
\]

Equation (8.2) shows that \( F_H(wya) = F_H(ya) \), independent of \( w \in N(\Phi) \). Substituting the expressions

\[
\Delta z_i = X_i^2 + H_i^2 - H_i, \quad \Delta z_\alpha = X_\alpha^2 + Y_\alpha^2 - H_i \quad (\alpha \in Q_i)
\]

into the formula for \( ^c \Delta_1 \) in Proposition 7.24 and noting that \( X_i, X_\alpha, Y_\alpha \) belong to the Lie algebra \( \mathcal{Z} \oplus V \) for \( N(\Phi) \), we see that

\[
^c \Delta_1 F_H(ya) = \left( c_0 \sum H_i^2 - (c_0 + q_1) \sum H_i \right) F_H(ya)
\]

where \( q_1 \) is the common cardinality of the sets \( Q_i \). Now using (8.4) and (8.5), we conclude that \( F_H(ya) \) is annihilated by the Laplace-Beltrami operator \( \sum H_i^2 \) for the
abelian group $\tilde{A}$. As $F_H$ is bounded, $F_H(ya)$ can not depend on $a \in \tilde{A}$. We have shown $F_H(wya) = F_H(y)$ and (8.3) now yields

$$0 = \left( \sum H_i^2 + \sum (Y^k)^2 - \sum a_iH_i \right) F_H(y) = \left( \sum (Y^k)^2 \right) F_H(y).$$

So $F_H(y)$ is annihilated by a left-invariant second order elliptic differential operator on the nilpotent group $N_0$. Thus $F_H(y)$ is a constant function. Now from (8.2) we conclude that $\tilde{f}(y)$ is also constant on $N_0$.

8.2. **Proof of Theorem 5.3 for tube domains.** Let $G/K$ be of tube type and $F$ be a real-valued bounded $L \oplus \mathcal{H}^c$-harmonic function on $G/K$. Proposition 8.1 shows that $F$ is a Poisson-Szegő integral. The Johnson-Koranyi Theorem 5.1 implies that $F$ is $\mathcal{H}$-harmonic. So $F$ is in fact annihilated by all of $\mathcal{H}' \oplus L \oplus \mathcal{H}^c = p_+ \otimes p_-$. This means $F$ is pluriharmonic.

8.3. **Proof of Theorem 5.3 in the non-tube case.** The Hua system does not annihilate the Poisson-Szegő kernel on a non-tube domain, so Poisson-Szegő integrals need not be Hua harmonic. Thus we require a different proof for Theorem 5.3 in the non-tube case.

For the rest of this section we assume $G/K$ to be of non-tube type. As above, $F$ denotes a real-valued bounded $L \oplus \mathcal{H}^c$-harmonic function on $G/K$ and $cF$, $c\tilde{F}$ are as in (7.1) and (7.2). We note the following consequence of Theorem 6.20 and Proposition 8.1.

**Corollary 8.6.** $cBV(c\tilde{F}) = 0$.

Our proof will make use of the operators $cBV$, $c\Delta_2$ and a characterization of bounded pluriharmonic functions on Siegel domains from [BDH]. It is shown in [BDH] that $F$ is pluriharmonic if and only if $c\tilde{F}$ is annihilated by three operators. Each of these is a linear combination of building-blocks $\Delta_{Z_\alpha}$, $\Delta_{Z_i}$ and $\Delta_{Z_h^\alpha}$. Thus we have:

**Theorem 8.7 ([BDH]).** If a real bounded function $c\tilde{F}$ is annihilated by the operators

$$\Delta_{Z_\alpha} \ (\alpha \in \cup_i Q_i), \ \Delta_{Z_i} \ (1 \leq i \leq r) \ \text{and} \ \Delta_{Z_h^\alpha} \ (\alpha \in \cup_{ij} Q_{ij})$$

then it is pluriharmonic.

Consider the decomposition

$$s = s^{r-1} \oplus s_r,$$
where

\[ s^{-1} = \left( \bigoplus_{j<r} Z_j \right) \oplus \left( \bigoplus_{1 \leq i < j < r} V_{ij} \right) \oplus \left( \bigoplus_{1 \leq i < j < r} n_{ij} \right) \oplus \left( \bigoplus_{1 \leq j < r} \mathbb{R} H_j \right), \]

\[ (8.9) \]

\[ s_r = Z_r \oplus \left( \bigoplus_{1 \leq j < r} V_{jr} \right) \oplus \left( \bigoplus_{1 \leq j < r} n_{jr} \right) \oplus V_{rr} \oplus \mathbb{R} H_r. \]

\[ (8.10) \]

\( s^{-1} \) is a subalgebra of \( s \) and \( s_r \) is an ideal. Therefore we can decompose the group as a semidirect product of a subgroup \( S^{-1} \) and normal subgroup \( S_r \),

\[ (8.11) \]

\[ S = S^{-1} S_r. \]

Let

\[ (8.12) \]

\[ h_r = Z_r \oplus \left( \bigoplus_{j<r} V_{jr} \right) \oplus \left( \bigoplus_{j<r} n_{jr} \right) \oplus V_{rr}, \quad \tilde{a}_r = \mathbb{R} H_r, \]

so that \( s_r = h_r \oplus \tilde{a}_r \). Using the bracket relations, one sees that \( h_r \) is a Heisenberg Lie algebra of dimension \( 2k + 1 \), where, as in formula (7.25), \( k = q_1 + (r - 1)q_2 \). Here

\[ Z_r \text{ has basis } \{ \mathcal{X}_\alpha, \mathcal{Y}_\alpha : \alpha \in Q_r \}, \]

\[ \bigoplus_{j<r} V_{jr} \text{ has basis } \{ X^1_\alpha, X^2_\alpha : \alpha \in \bigcup_{j<r} Q_{jr} \}, \]

\[ \bigoplus_{j<r} n_{jr} \text{ has basis } \{ Y^1_\alpha, Y^2_\alpha : \alpha \in \bigcup_{j<r} Q_{jr} \} \]

and \( V_{rr} = \mathbb{R} X_r \) is the center of \( h_r \). The basis

\[ \mathcal{X}_\alpha, \mathcal{Y}_\alpha, X^k_\alpha, Y^k_\alpha, X_r, H_r, \quad \alpha \in Q_r \cup \bigcup Q_{jr} \]

for \( s_r \) is orthonormal with respect to the Riemannian form \( g_r \) on \( S_r \). The complex structure \( \mathcal{J} \) on \( s \) restricts to yield a complex structure on \( s_r \):

\[ \mathcal{J}(X_r) = H_r, \quad \mathcal{J}(H_r) = -X_r, \]

\[ \mathcal{J}(X^k_\alpha) = Y^k_\alpha, \quad \mathcal{J}(Y^k_\alpha) = -X^k_\alpha, \]

\[ \mathcal{J}(\mathcal{X}_\alpha) = \mathcal{Y}_\alpha, \quad \mathcal{J}(\mathcal{Y}_\alpha) = -\mathcal{X}_\alpha. \]

The operators \( \tilde{\Delta}_Z, \tilde{\Delta}_a, \tilde{\Delta}_{2k} \) defined on \( S_r \) as in (7.13), act from the right and make perfect sense on both \( S_r \) and \( S \). Thus for any smooth function \( g \) on \( S \) we may write:

\[ \tilde{\Delta}_Z(g_{s'}) (s_r) = (\Delta Z, g)(s's_r), \]

\[ \tilde{\Delta}_a(g_{s'}) (s_r) = (\Delta a, g)(s's_r), \]

\[ \tilde{\Delta}_{2k}(g_{s'}) (s_r) = (\Delta_{2k} g)(s's_r). \]

where \( s' \in S^{-1}, s_r \in S_r \) and \( g_{s'}(s_r) = g(s's_r) \). Likewise, Equations (7.25) and (7.27) show that \( ^cBV \) and \( ^c\Delta_2 \) can be applied to functions on \( S_r \). As \( ^cBV \) and \( ^c\Delta_2 \) annihilate \( ^c\tilde{F} \) we have

\[ ^cBV^c(\tilde{F}_{s'}) = ^c\Delta_2^c(\tilde{F}_{s'}) = 0. \]

\[ (8.13) \]
Theorem 8.18, proved below, shows that (8.13) implies \( \tilde{c}\tilde{F}_{s'} \) is a pluriharmonic function on \( S_r \). Hence
\[
\Delta_{Z_r}(\tilde{c}\tilde{F}_{s'}) = \Delta_{Z^\alpha_r}(\tilde{c}\tilde{F}_{s'}) = \Delta_{Z^\beta_r}(\tilde{c}\tilde{F}_{s'}) = 0 \quad \text{for all } \alpha \in Q_r, \beta \in \bigcup_i \overline{Q}_i,
\]
and by (8.12)
\[
(8.14) \quad \Delta_{Z_r}(\tilde{c}\tilde{F}) = \Delta_{Z^\alpha_r}(\tilde{c}\tilde{F}) = \Delta_{Z^\beta_r}(\tilde{c}\tilde{F}) = 0 \quad \text{for all } \alpha \in Q_r, \beta \in \bigcup_i \overline{Q}_i.
\]
Thus the function \( cF \) on \( cD \) satisfies
\[
(8.15) \quad \partial_{v_\alpha} \partial_{\bar{v}_\alpha} cF(c \cdot o) = \partial_{v_\beta} \partial_{\bar{v}_\beta} cF(c \cdot o) = 0, \quad \text{for } \alpha \in \{\gamma_r\} \cup Q_r, \beta \in \bigcup_i \overline{Q}_i,
\]
where, as before, \( v_\alpha, \bar{v}_\beta \) denote coordinates on \( cD \) with respect to basis (3.18). Notice that our considerations do not depend on the order of the roots in \( \Gamma \), hence the role of all \( \gamma_j \)'s is equivalent as far as the conclusion is made on the domain \( cD \) not on the group \( S_r \). In particular, we may interchange \( \gamma_r \) with \( \gamma_j \) and so obtain (8.15) for \( \alpha \in \{\gamma_j\} \cup Q_j, \beta \in \bigcup_i \overline{Q}_{ij} \). But now we may apply \( S \)-invariance (with respect to the group \( S \) defined by the original ordering) and conclude that all building blocks \( \Delta_{Z_j}, \Delta_{Z_{\alpha}}, \Delta_{Z_{\beta}} \) annihilate \( \tilde{c}\tilde{F} \). This shows that \( F \) is pluriharmonic, in view of Theorem 8.7.

8.4. Rank one analysis. The group \( S_r \) in (8.10) can be identified with the semidirect product
\[
S_0 = \mathbb{H}^k A_0
\]
of the Heisenberg group \( \mathbb{H}^k = \mathbb{C}^k \times \mathbb{R} \) with \( A_0 = \mathbb{R}^+ \) acting via dilations. The solvable group \( S_0 \) arises in connection with the rank one Hermitian symmetric space \( G/K = SU(1,k)/S(U(1) \times U(k)) \), whose bounded realization \( D_0 \) is biholomorphically equivalent to the unit ball in \( \mathbb{C}^{k+1} \). \( S_0 \) is identified with the classical Siegel domain
\[
cD = \{ (z, z_{k+1}) \in \mathbb{C}^k \times \mathbb{C} : \Im z_{k+1} > \frac{1}{4} |z|^2 \}.
\]
We write points \( s \in S_0 \) as
\[
s = \left( (\zeta, u), a \right) = \left( (\zeta_1, \ldots, \zeta_k), u \right), \quad \left( \zeta_j = x_j + iy_j \in \mathbb{C}, u \in \mathbb{R}, a \in \mathbb{R}^+ \right)
\]
and denote by \( X_j, Y_j, T \) the left-invariant fields on \( \mathbb{H}^k \), which at \( e \) agree respectively with \( \partial_{x_j}, \partial_{y_j}, \partial_a \). Then the operators \( \tilde{X}_j, \tilde{Y}_j, \tilde{T}, \tilde{H} \) given by
\[
(8.16) \quad \tilde{X}_j = \sqrt{a} X_j, \quad \tilde{Y}_j = \sqrt{a} Y_j, \quad \tilde{T} = aT, \quad \tilde{H} = a\partial_a,
\]
are left-invariant on the group $S_0$, and form a basis of the Lie algebra $\mathfrak{s}_0$ of $S_0$. The elements $X_r$ and $H_r$ of $\mathfrak{s}_r$ correspond to $\tilde{T}$ and $H$ respectively. $X_r^k$'s and $X_\alpha$'s in $\mathfrak{s}_r$ correspond to the $\tilde{X}_j$'s in $\mathfrak{s}_0$ and the $Y_r^k$, $Y_\alpha$'s correspond to the $\tilde{Y}_j$'s. Let $\ell = (r-1)q_2$ and order the bases for $\mathfrak{s}_r$ and $\mathfrak{s}_0$ so that the $X_k^\alpha$'s and $Y_k^\alpha$'s ($\alpha \in \bigcup_j \mathcal{Q}_{jr}$) correspond with $\tilde{X}_j$, $\tilde{Y}_j$ with $\ell < j \leq k$.

Recall that the operators $^cBV$ and $^c\Delta_2$ on $S$ are now regarded as living on the subgroup $S_r$. Identifying $S_r$ with $S_0$ and working with equations (7.25) and (7.27) we obtain the following expressions in the notation of (8.16).

\begin{align*}
^cBV &= AaL_BZ - ikA\tilde{T}Z + Z\Delta_Z - iB\Delta_Z, \\
^c\Delta_2 &= a \sum_{0 \leq j < \ell} \left( b_{2j+1}((\mathcal{X}_{2j+1} - \mathcal{Y}_{2j+2})^2 + (\mathcal{X}_{2j+2} + \mathcal{Y}_{2j+1})^2) \\
&\quad + b_{2j+2}((\mathcal{X}_{2j+1} + \mathcal{Y}_{2j+2})^2 + (\mathcal{X}_{2j+2} - \mathcal{Y}_{2j+1})^2) \right) \\
&\quad + a \sum_{2l<j \leq k} b_j((\mathcal{X}_j)^2 + (\mathcal{Y}_j)^2) - CH - c_r(\tilde{T}^2 + H^2),
\end{align*}

(8.17)

where

\[ Z = \tilde{T} - iH, \quad \Delta_Z = \tilde{T} + H^2 - H, \quad L_B = \sum_{1 \leq j \leq n} ((\mathcal{X}_j)^2 + (\mathcal{Y}_j)^2). \]

To complete the proof of Theorem 5.3 in the non-tube case, it remains to establish the following result.

**Theorem 8.18.** Let $F$ be a real bounded function on $S_0$ annihilated by two operators: $^cBV$ and $^c\Delta_2$. Then $F$ is pluriharmonic.

Observe that $F$ is a real function while $^cBV$ is a complex operator. Therefore $F$ is annihilated by two real operators:

\begin{align*}
U &= AaL_B\tilde{T} - kA\tilde{T}H + \tilde{T}(\tilde{T}^2 + H^2 - H), \\
V &= AaL_BH + kA\tilde{T}^2 + H(\tilde{T}^2 + H^2 - H) + B(\tilde{T}^2 + H^2 - H),
\end{align*}

(8.19)

$A > 1, B > 0$.

Let $\phi$ be a Schwartz function on $\mathbb{R}$ whose Fourier transform satisfies

\[ \hat{\phi}(\lambda) = \begin{cases} 
1 & \text{for } |\lambda| \leq 1, \\
0 & \text{for } |\lambda| \geq 2.
\end{cases} \]

By [BDH] (Lemma 4.4) there exists a sequence $\{k_n\}_{n=1,2,\ldots}$ of natural numbers tending to infinity such that for

\[ \phi_n(x) = \frac{1}{k_n}\phi\left(\frac{x}{k_n}\right), \]

By [BDH] (Lemma 4.4) there exists a sequence $\{k_n\}_{n=1,2,\ldots}$ of natural numbers tending to infinity such that for

\[ \phi_n(x) = \frac{1}{k_n}\phi\left(\frac{x}{k_n}\right), \]
convolving in the central direction in $\mathbb{H}^k$ the limit

$$\lim_{n \to \infty} \phi_n \ast R \mathcal{F}(\zeta, x) a$$

exists for every $\zeta \in C^k$, $a \in A_0$ and does not depend on the central variable $x$. Denote this limit by $G(\zeta, a)$, then $G$ is annihilated by the same operators as $F$: $U, V, c\Delta_2$.

**Lemma 8.20.** The function $G(\zeta, a)$ is constant.

**Proof.** Let $\widetilde{G}(\zeta, a) = HG(\zeta, a) = (a\partial_a)G(\zeta, a)$, then by Harnack’s inequality the function $\widetilde{G}$ is bounded and by (8.19)

$$VG = (AaL_B + H^2 + (B - 1)H - B)\widetilde{G}(\zeta, a) = 0.$$

This equation implies that $\widetilde{G} = 0$. Indeed, let $\mu_t$ be the semigroup of measures with infinitesimal generator $AaL_B + H^2 + (B - 1)H$. Then $AaL_B + H^2 + (B - 1)H - B$ generates the semigroup $e^{-Bt}\mu_t$ and so $\widetilde{G} \ast e^{-Bt}\mu_t = \widetilde{G}$. Now letting $t \to \infty$ we see that $\widetilde{G}$ vanishes, $G$ does not depend on $a$ and so it is a function on $\mathbb{R}^{2k} = C^k$ annihilated by the elliptic operator (compare (8.17)):

$$\left(\sum_{0 \leq j < l} (b_{2j+1}((\partial_{x_{2j+1}} - \partial_{y_{2j+1}})^2 + (\partial_{x_{2j+2}} + \partial_{y_{2j+1}})^2)ight.$$

$$+ b_{2j+2}((\partial_{x_{2j+1}} + \partial_{y_{2j+1}})^2 + (\partial_{x_{2j+2}} - \partial_{y_{2j+1}})^2) + \sum_{2l < j \leq k} b_j(\eta_j^2 + \xi_j^2)\right)G(x, y) = 0,$$

Taking the Fourier transform of both sides (in the distribution sense) we get

$$\left(\sum_{0 \leq j < l} (b_{2j+1}((\eta_{2j+1} - \xi_{2j+1})^2 + (\eta_{2j+2} + \xi_{2j+1})^2)ight.$$

$$+ b_{2j+2}((\eta_{2j+1} + \xi_{2j+1})^2 + (\eta_{2j+2} - \xi_{2j+1})^2) + \sum_{2l < j \leq k} b_j(\eta_j^2 + \xi_j^2)\right)\hat{G}(\eta, \xi) = 0,$$

which implies

$$\text{supp} \hat{G} \subset \{0\}.$$  

Boundedness of $G$ forces it to be constant. □

We remark that the only use of the operator $\Delta_2$ appears above in the proof of Lemma 8.20. In summary, we have proved the following.

**Theorem 8.21.** Assume that $F$ satisfies the assumptions of Theorem 8.18. Denote by $F(\zeta, \lambda, a)$ the distributional partial Fourier transform of $F$ along the center of $\mathbb{H}^k$. 

Let

\[ \eta_n(x) = k_n \phi(k_n x) - k_n \phi(k_n^{-1} x), \]
\[ F_n(\zeta, x, a) = \eta_n \ast_R F(\zeta, x, a). \]

Then

\[ \text{supp} F_n(\cdot, \cdot, \cdot) \subset \mathbb{C}^n \times \{ \lambda \in \mathbb{R} : k_n^{-1} \leq |\lambda| \leq 2k_n \} \times \mathbb{R}^+, \]

\( F_n \) is annihilated by \( U \) and \( V \) and there is a constant \( c \) such that the sequence \( F_n \) tends to \( F + c \).

As a corollary, we see that to justify Theorem 8.18 it is enough to justify it for functions satisfying

(8.22) \[ \text{supp} F(\cdot, \cdot, \cdot) \subset \mathbb{C}^n \times \{ \lambda \in \mathbb{R} : \varepsilon \leq |\lambda| \leq \varepsilon^{-1} \} \times \mathbb{R}^+, \]

Writing the operator \( U \) on the partial Fourier transform side and using the above assumptions one can easily prove that such \( F \) is annihilated by

(8.23) \[ U_1 = AaL_B - kAH + \tilde{T}^2 + H^2 - H, \]

which is a second order elliptic operator. Therefore by [DH], [R] the function \( F \) may be written as

\[ F((\zeta, x)a) = f \ast_{\mathbb{H}^k} P_a(\zeta, x) \]

for the Poisson kernel \( P_a \) determined by \( U \) and \( f \in L^\infty(\mathbb{H}^k) \). Furthermore

\[ \lim_{a \to 0} F((\zeta, x)a) = f(\zeta, x) \]

in the \(*\)weak sense.

**Proposition 8.24.** The boundary value \( f \) of \( F \) satisfies the following differential equation:

\[ (L_B^2 + k^2 \partial_x^2) f(\zeta, x) = 0 \]

**Proof.** In the proof we shall use the following simple lemma

**Lemma 8.25.** Let \( h \in C^1(0, \infty) \). Assume that

\[ ah'(a) - \gamma h(a) = v(a) \]

for some \( \gamma > 0 \) and

\[ \lim_{a \to 0} v(a) = v_0. \]

Then

\[ \lim_{a \to 0} h(a) = -\frac{v_0}{\gamma}. \]
Let \( F(a)(\zeta, x) = F((\zeta, x)a) \) and let \( \phi \) be a test function. The lemma is applied to \( h(a) = (\partial^p_a F(a), \phi) \), for \( p = 1, 2, 3 \). Observe that by (8.23)

\[
(8.26) \quad a\partial^2_a F(a) - kA\partial_a F(a) = -(AL_B + a\partial^2_x)F(a).
\]

Applying Lemma 8.25 we get

\[
\lim_{a \to 0} \partial_a F(a) = \frac{1}{k} L_B f.
\]

Differentiating (8.26) along \( a \) we obtain:

\[
(8.27) \quad a\partial^3_a F(a) - (kA - 1)\partial^2_a F(a) = -(AL_B\partial_a + \partial^2_a + a\partial^2_x)F(a)
\]

and using again Lemma 8.25 we prove

\[
\lim_{a \to 0} \partial^2_a F(a) = \frac{1}{kA - 1} \left( \frac{A}{k} L^2_B + \partial^2_x \right) f.
\]

Repeating this procedure one can also show that:

\[
\lim_{a \to 0} \partial^3_a F(a) \text{ exists.}
\]

Applying the operator \( V \) we have:

\[
0 = \lim_{a \to 0} \left( AL_B \partial_a + kA\partial^2_x + a\partial_a (\partial^2_x + \partial^2_a) + B'(\partial^2_a + \partial^2_x) \right) F(a) \quad (B' = B + 2)
\]

\[
= \left( \frac{A}{k} L^2_B + kA\partial^2_x + B'\partial^2_a + \frac{B'}{kA - 1} \left( \frac{A}{k} L^2_B + \partial^2_x \right) \right) f
\]

\[
= \frac{1}{k(kA - 1)} \left( (A(kA - 1) + B'A) L^2_B 
\right.
\]

\[
+ (k^2 A(kA - 1) + B'k(kA - 1) + B'k)\partial^2_x f
\]

\[
= \frac{1}{k(kA - 1)} \left( A(kA - 1 + B') L^2_B + k^2 A(kA - 1 + B') \partial^2_x f \right)
\]

\[
= C(L^2_B + k^2 \partial^2_x) f.
\]

\[\square\]

**Lemma 8.28.** There is an \( \varepsilon' \) such that for every \( \zeta \in \mathbb{C}^n \)

\[
\text{supp} \hat{f}_\zeta(\lambda) \subset \{ \varepsilon' < |\lambda| < \varepsilon'^{-1} \}.
\]

**Proof.** Let \( \varepsilon \) be as in 8.22. Taking \( \phi \in \mathcal{S}(\mathbb{R}) \) such that:

\[
\hat{\phi}(\lambda) = \begin{cases} 
1 & \text{for } \varepsilon < |\lambda| < \varepsilon^{-1}, \\
0 & \text{for } |\lambda| \in (0, \varepsilon) \cup (\varepsilon, \infty),
\end{cases}
\]

with \( \varepsilon \) as in (8.22), we obtain \( F = \phi *_R F \). Thus

\[
(f - \phi *_R F) *_{\mathbb{R}^k} P_s = F(s) - \phi *_R F(s) = 0,
\]

but the mapping \( L^\infty(\mathbb{R}^k) \ni g \mapsto g *_{\mathbb{R}^k} P_s \in L^\infty(S_0) \) is injective ([DH]), therefore \( f = \phi *_R f \), establishing the lemma with \( \varepsilon' = \varepsilon/2 \).  \( \square \)
Now we are in the situation described in [B] (see also [BBDHJ]), where the following was proved:

**Theorem 8.29 ([B]).** Let \( f \in L^\infty(\mathbb{H}^k) \) be a boundary value of a real bounded function \( F \) on \( S_0 \) i.e.:

\[
F((\zeta, x)) = f \ast_{\mathbb{H}^k} P_\alpha(\zeta, x).
\]

Assume:
- \( (L_B^2 + k^2 \partial_x^2) f(\zeta, x) = 0, \)
- \( \text{supp} \hat{f}_\zeta(\lambda) \subset \{ \epsilon' < |\lambda| < \epsilon'^{-1} \} \) for every \( \zeta \in \mathbb{C}^k. \)

Then \( F \) is pluriharmonic.

**REFERENCES**


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