A FAMILY OF FINITE GELFAND PAIRS
ASSOCIATED WITH WREATH PRODUCTS

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Abstract. Consider the wreath product $G_n = \Gamma^n \rtimes S_n$ of a finite group $\Gamma$ with the symmetric group $S_n$. Letting $\Delta_n$ denote the diagonal in $\Gamma^n$ the direct product $K_n = \Delta_n \times S_n$ forms a subgroup. In case $\Gamma$ is abelian $(G_n, K_n)$ forms a Gelfand pair with relevance to the study of parking functions. For $\Gamma$ non-abelian it was suggested by Kürsat Aker and Mahir Bilen Can that $(G_n, K_n)$ must fail to be a Gelfand pair for $n$ sufficiently large. We prove here that this is indeed the case: for $\Gamma$ non-abelian there is some integer $2 < N(\Gamma) \leq |\Gamma|$ for which $(K_n, G_n)$ is a Gelfand pair for all $n < N(\Gamma)$ but $(K_n, G_n)$ fails to be a Gelfand pair for all $n \geq N(\Gamma)$. For dihedral groups $\Gamma = D_p$ with $p$ an odd prime we prove that $N(\Gamma) = 6$.

1. Introduction

Gelfand pairs are fundamental to the study of harmonic analysis on topological groups. In the context of finite groups the definition is as follows. We denote by $L(G)$ the space of complex-valued functions on a finite group $G$. This is an algebra under the convolution product $f \star g(x) = \sum_{y \in G} f(xy^{-1})g(y)$.

Given a subgroup $K \subset G$, the set $L(K \backslash G/K) = \{ f \in L(G) : f(k_1xk_2) = f(x) \forall k_1, k_2 \in K \}$ of $K$-bi-invariant functions on $G$ forms a subalgebra of $L(G)$. One calls $(G, K)$ a Gelfand pair when $L(K \backslash G/K)$ is commutative. This condition is equivalent to each of the following.

- The left quasi-regular representation $ind^G_K(1_K)$ of $G$ in $L(G/K)$ is multiplicity free.
- For each irreducible representation $(\pi, V)$ of $G$ the space $V^K$ of $K$-fixed vectors in $V$ has dimension $\dim(V^K) \leq 1$.

Irreducible representations of $G$ which occur in $L(G/K)$ are called $K$-spherical. These are precisely those admitting non-zero $K$-fixed vectors. We refer the reader to [CSST08], [Mac95, Chapter VII] or [Ter99] for proofs of these equivalences as well as general background concerning Gelfand pairs and their applications in the finite groups setting.

Given a finite group $\Gamma$ the symmetric group $S_n$ acts by automorphisms on the cartesian product $\Gamma^n$ of $n$ copies of $\Gamma$ via $\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

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The resulting semi-direct product

\[ G_n := \Gamma^n \rtimes S_n \]

is the \textit{wreath product} of \( \Gamma \) with \( S_n \), sometimes written as \( \Gamma / S_n \). The representation theory of such wreath products is discussed in [Mac95, Chapter I Appendix B] and, in greater generality, in [CSST06, CSST14].

Let

\[ L(\Gamma^n)^{S_n} = \{ f \in L(\Gamma^n) : f(\sigma \cdot x) = f(x) \ \forall \sigma \in S_n, x \in \Gamma^n \} \]

denote the space of complex valued functions on \( \Gamma^n \) invariant under the action of \( S_n \). This is a convolution subalgebra of \( L(\Gamma^n) \) and routine calculations show that the map

\[ (1.1) \ \Phi : L(S_n \setminus G_n / S_n) \rightarrow L(\Gamma^n)^{S_n}, \ \Phi(f)(x) = n! f(x, e) \ (e \in S_n \text{ the identity}) \]

is an isomorphism of convolution algebras. In particular \((G_n, S_n)\) is a Gelfand pair if and only if \( L(\Gamma^n)^{S_n} \) is commutative. This is certainly the case whenever \( \Gamma \) is abelian. On the other hand for \( \Gamma \) non-abelian choose points \( x, y \in \Gamma \) with \( xy \neq yx \) and let \( x, y \in \Gamma^n \) be the points \( x = (x, \ldots, x), y = (y, \ldots, y) \). The characteristic functions \( \delta_x, \delta_y \) for these points belong to \( L(\Gamma^n)^{S_n} \) and we have

\[ \delta_x \star \delta_y = \delta_{xy} \neq \delta_y \star \delta_x. \]

Thus \((G_n, S_n)\) is a Gelfand pair if and only if \( \Gamma \) is abelian.

The \textit{diagonal subgroup} in \( \Gamma^n \),

\[ \Delta_n := \{(x, \ldots, x) : x \in \Gamma\}, \]

played a role in the preceding discussion. The \( S_n \)-action preserves \( \Delta_n \subset \Gamma^n \) and is trivial on \( \Delta_n \). Thus the direct product

\[ (K_n := \Delta_n \times S_n) \cong \Gamma \times S_n \]

is a subgroup of \( G_n = \Gamma^n \rtimes S_n \) and we consider the pair \((G_n, K_n)\).

Restricting the map \( \Phi \) given in (1.1) to \( L(K_n \setminus G_n / K_n) \subset L(S_n \setminus G_n / S_n) \) produces an algebra isomorphism onto

\[ (1.2) \ \mathcal{A}_n(\Gamma) := L(\Delta_n \setminus \Gamma^n / \Delta_n) \cap L(\Gamma^n)^{S_n}, \]

the algebra of functions \( \Gamma^n \rightarrow \mathbb{C} \) which are both \( \Delta_n \)-bi-invariant and \( S_n \)-invariant. Thus if either \((G_n, S_n)\) or \((\Gamma^n, \Delta_n)\) is a Gelfand pair then so is \((G_n, K_n)\). It follows in particular that

- \((G_n, K_n)\) is a Gelfand pair for \( \Gamma \) abelian and
- \((G_n, K_n)\) is a Gelfand pair for \( n = 2 \).

The latter point follows from the well-known fact that \((\Gamma \times \Gamma, \Delta_2)\) is a Gelfand pair [Mac95, §VII-1, Example 9].

For cyclic groups \( \Gamma \) the resulting Gelfand pairs \((G_n, K_n)\) arise in combinatorics in connection with \textit{parking functions} [AC12]. This fact motivates interest in pairs \((G_n, K_n)\) for other finite groups \( \Gamma \). It is suggested in [AC12] that for \( \Gamma \) non-abelian \((G_n, K_n)\) will fail to be a Gelfand pair for \( n \) sufficiently large. The following theorems show that this is indeed the case. These are our main results.

**Theorem 1.1.** If \((G_{n+1}, K_{n+1})\) is a Gelfand pair then so is \((G_n, K_n)\).

**Theorem 1.2.** If \( \Gamma \) is non-abelian then \((G_{|\Gamma|}, K_{|\Gamma|})\) fails to be a Gelfand pair.

Thus for \( \Gamma \) non-abelian there is some integer \( 2 < N(\Gamma) \leq |\Gamma| \) for which
• \((K_n, G_n)\) is a Gelfand pair for all \(n < N(\Gamma)\) but
• \((K_n, G_n)\) fails to be a Gelfand pair for all \(n \geq N(\Gamma)\).

**Examples 1.3.** The authors of [AC12] used the GAP computer algebra system to verify that \(N(S_4) = 6, N(A_4) = 4, N(GL(2, F_3)) = 3\) and \(N(SL(3, F_2)) = 3\).

We remark that we do not know whether or not \(N(\Gamma)\) can be arbitrarily large.

Proofs for Theorems 1.1 and 1.2 are given below in Sections 2 and 4. Section 3 concerns decomposition of the spaces \(L(\Gamma^n/\Delta_n)\) and \(L(G_n/K_n)\) under the left actions of \(\Gamma^n\) and \(G_n\). Section 5 concerns examples. We show that for primes \(p \geq 3\) the dihedral groups \(D_p\) have \(N(D_p) = 6\). As \(D_3 \cong S_3\) this is consistent with [AC12]. In [AM03] the reader will find a different family of Gelfand pairs involving wreath products with dihedral groups.

2. **Proof of Theorem 1.1**

For \(f \in L(\Gamma^{n+1})\) let \(f^o \in L(\Gamma^n)\) be defined as

\[f^o(x_1, \ldots, x_n) = \sum_{\gamma \in \Gamma} f(x_1, \ldots, x_n, \gamma)\]

and consider the map

\[\Psi : L(\Gamma^{n+1}) \to L(\Gamma^n), \quad \Psi(f) = f^o.\]

This is an algebra map. That is,

**Lemma 2.1.** \((f \ast g)^o = f^o \ast g^o\) for \(f, g \in L(\Gamma^{n+1})\).

*Proof.* In fact

\[(f \ast g)^o(x_1, \ldots, x_n) = \sum_{\gamma \in \Gamma} f \ast g(x_1, \ldots, x_n, \gamma)\]

\[= \sum_{\gamma \in \Gamma} \sum_{y_1 \ast \ldots \ast y_{n+1} \in \Gamma} f(x_1 y_1^{-1}, \ldots, x_n y_n^{-1}, y_{n+1}^{-1}) g(y_1, \ldots, y_{n+1})\]

\[= \sum_{y_1, \ldots, y_n, y_{n+1}} f(x_1 y_1^{-1}, \ldots, x_n y_n^{-1}, y_{n+1}^{-1}) g(y_1, \ldots, y_{n+1})\]

\[= \sum_{y_1, \ldots, y_n, y_{n+1}} f^o(x_1 y_1^{-1}, \ldots, x_n y_n^{-1}) g(y_1, \ldots, y_{n+1})\]

\[= f^o \ast g^o(x_1, \ldots, x_n).\]

** Lemma 2.2.** \(\Psi(L(\Gamma^{n+1})S_{n+1}) \subset L(\Gamma^n)S_n\).

*Proof.* Say \(f \in L(\Gamma^{n+1})S_{n+1}\) and \(\sigma \in S_n\). Then

\[f^o(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \sum_{\gamma \in \Gamma} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \gamma) = \sum_{\gamma \in \Gamma} f(x_1, \ldots, x_n, \gamma)\]

\[= f^o(x_1, \ldots, x_n)\]

since \((x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \gamma)\) is a permutation of \((x_1, \ldots, x_n, \gamma)\).
For each $m$ the group $\Gamma \times \Gamma$ acts on the set $\Gamma^m$ via
\[(\gamma_1, \gamma_2) \cdot (x_1, \ldots, x_m) = (\gamma_1 x_1 \gamma_2^{-1}, \ldots, \gamma_1 x_n \gamma_2^{-1})\]
and on $L(\Gamma^m)$ via
\[(\gamma_1, \gamma_2) \cdot f(x) = f((\gamma_1^{-1}, \gamma_2^{-1}) \cdot x)\]

**Lemma 2.3.** $\Psi$ is $(\Gamma \times \Gamma)$-equivariant. That is, $((\gamma_1, \gamma_2) \cdot f)^\circ = (\gamma_1, \gamma_2) \cdot f^\circ$ for $f \in L(\Gamma^{n+1})$, $\gamma_1, \gamma_2 \in \Gamma$.

**Proof.** Indeed
\[\[(\gamma_1, \gamma_2) \cdot f)^\circ (x_1, \ldots, x_n) = \sum_{\gamma \in \Gamma} f(\gamma_1^{-1} x_1 \gamma \gamma_2^{-1} x_2 \gamma \cdots \gamma_1 \gamma_n \gamma_2^{-1} \gamma_2 \gamma_n)\]
\[= \sum_{\gamma' \in \Gamma} f(\gamma_1^{-1} x_1 \gamma_2^{-1} x_2 \gamma \cdots \gamma_1 \gamma_n \gamma_2^{-1} \gamma_2 \gamma_n)\]
\[= f^\circ(\gamma_1^{-1} x_1 \gamma_2^{-1} x_2 \gamma \cdots \gamma_1 \gamma_n \gamma_2^{-1} \gamma_2 \gamma_n)\]
\[= (\gamma_1, \gamma_2) \cdot f^\circ (x_1, \ldots, x_n)\]

**Corollary 2.4.** $\Psi(L(\Delta_{n+1}^m) \cap \Gamma^{m+1}/ \Delta_{n+1}^m) \subset L(\Delta_n^m \cap \Gamma \cap \Delta_n)$.

**Proof.** As $L(\Delta_m^m \cap \Gamma^m) = L(\Gamma^m)^{\Gamma \times \Gamma}$ for all $m$ this follows from Lemma 2.3. □

Recalling that $A_n(\Gamma) = L(\Delta_m^m \cap \Gamma^m) \cap L(\Gamma^m)^{\Gamma \times \Gamma}$ (see Equation 1.2) Corollary 2.4 together with Lemma 2.2 give the following.

**Corollary 2.5.** $\Psi(A_n(\Gamma)) \subset A_n(\Gamma)$.

We wish to show that in fact

**Lemma 2.6.** $\Psi(A_n+1(\Gamma)) = A_n(\Gamma)$. That is, $\Psi: A_{n+1}(\Gamma) \to A_n(\Gamma)$ is surjective.

Working towards a proof for this we introduce, for each $m$, the projection map
\[P_m : L(\Gamma)^{S_m} \to A_m(\Gamma), \quad P_m(f) = \sum_{\gamma_1, \gamma_2 \in \Gamma} (\gamma_1, \gamma_2) \cdot f.\]

As $\Psi$ is $(\Gamma \times \Gamma)$-equivariant (Lemma 2.3) the diagram
\[
\begin{array}{ccc}
L(\Gamma^{n+1})^{S_n+1} & \xrightarrow{\Psi} & L(\Gamma)^{S_n} \\
\downarrow P_{n+1} & & \downarrow P_n \\
A_{n+1}(\Gamma) & \xrightarrow{\Psi} & A_n(\Gamma)
\end{array}
\]
commutes. As $P_n$ is surjective we see that to prove Lemma 2.6 it suffices to show that $\Psi : L(\Gamma^{n+1})^{S_n+1} \to L(\Gamma)^{S_n}$ is surjective. For this we require Lemma 2.7 below.

List the elements of $\Gamma$ as
\[\Gamma = \{\gamma_1, \ldots, \gamma_r\}\]
say where $r = |\Gamma|$. For $x \in \Gamma^m$ we let
\[|x| := S_m \cdot x\]
denote the $S_m$-orbit though $x$. This contains a unique point in which any $\gamma_1$'s appear first followed by any $\gamma_2$'s etcetera in order. So
\[|x| = \langle k_1, k_2, \ldots, k_r \rangle := \left[\begin{array}{c}
\gamma_1, k_1, \gamma_2, k_2, \ldots, \gamma_r, k_r
\end{array}\right]_{k_1, k_2, \ldots, k_r}
\]
for some non-negative integers $k_1, \ldots, k_r$ with $k_1 + \cdots + k_r = m$. The characteristic functions
\[
\delta_{(k_1, k_2, \ldots, k_r)}(x) = \begin{cases} 
1 & \text{if } x \in (k_1, k_2, \ldots, k_r) \\
0 & \text{otherwise}
\end{cases}
\]
give a basis for $L((\Gamma^m)S_m)$.

**Lemma 2.7.** For integers $k_1, \ldots, k_r \geq 0$ with $k_1 + \cdots + k_r = n + 1$ we have
\[
\delta_{(k_1, k_2, \ldots, k_r)}^\circ = \sum_{1 \leq j \leq r \text{ \ and \ } k_j \neq 0} \delta_{(k_1, k_2, \ldots, k_r)\langle k_j \rangle}.
\]

**Proof.** For $x = (x_1, \ldots, x_n) \in \Gamma^n$ we have
\[
\delta_{(k_1, k_2, \ldots, k_r)}^\circ(x) = \sum_{j=1}^r \delta_{(k_1, k_2, \ldots, k_r)}(x_1, \ldots, x_n, \gamma_j).
\]
Here $\delta_{(k_1, k_2, \ldots, k_r)}(x_1, \ldots, x_n, \gamma_j) = 1$ if and only if $k_j > 0$ and $x$ is a permutation of
\[
(\gamma_1, \ldots, \underbrace{\gamma_{j-1}}_{k_j-1}, \underbrace{\gamma_j, \ldots, \gamma_r}_{k_r}).
\]
Thus $\delta_{(k_1, k_2, \ldots, k_r)}(x_1, \ldots, x_n, \gamma_j) = 1$ if and only if $k_j > 0$ and $\delta_{(k_1, \ldots, k_j-1, \ldots, k_r)}(x) = 1$. This completes the proof. \hfill \Box

**Proof of Lemma 2.6.** As explained above it suffices to show that $\Psi : L((\Gamma^{n+1})^S_{n+1}) \rightarrow L(\Gamma^n)^{S_n}$ is surjective. For this we will verify that $\delta_{(k_1, \ldots, k_r)} \in \Psi(L((\Gamma^{n+1})^S_{n+1}))$ for all $k_1, \ldots, k_r \geq 0$ with $k_1 + \cdots + k_r = n$. We do this by reverse induction on $k_1 \in \{0, \ldots, n\}$.

First suppose that $k_1 = n$ so that $(k_1, \ldots, k_r) = (n, 0, \ldots, 0)$. As $\delta_{(n, 0, \ldots, 0)}^\circ = \delta_{(n, 0, \ldots, 0)}$ this shows that $\delta_{(k_1, \ldots, k_r)} \in \Psi(L((\Gamma^{n+1})^S_{n+1}))$ when $k_1 = n$.

Next suppose that $0 \leq k_1 \leq n - 1$ and assume inductively that $\delta_{(k_1', \ldots, k_r')} \in \Psi(L((\Gamma^{n+1})^S_{n+1}))$ for all $k_1', \ldots, k_r' \geq 0$ with $k_1' + \cdots + k_r' = n$ and $k_1' = k_1 + 1$. Lemma 2.7 shows that
\[
\delta_{(k_1+1, k_2, \ldots, k_r)} = \delta_{(k_1, k_2, \ldots, k_r)} + \sum_{2 \leq j \leq r \text{ \ and \ } k_j \neq 0} \delta_{(k_1+1, k_2, \ldots, k_j-1, \ldots, k_r)}.
\]
By inductive hypothesis all terms in the sum belong to $\Psi(L((\Gamma^{n+1})^S_{n+1}))$ and hence so does $\delta_{(k_1, k_2, \ldots, k_r)}$ as desired. \hfill \Box

**Proof of Theorem 1.1.** Suppose that $(G_{n+1}, K_{n+1})$ is a Gelfand pair. Equivalently the algebra $A_{n+1}(\Gamma)$ commutes under convolution. Given $f, g \in A_n(\Gamma)$ Lemma 2.6 ensure that there exist functions $F, G \in A_{n+1}(\Gamma)$ with $F^\circ = f$ and $G^\circ = g$. Applying Lemma 2.1 now yields
\[
f \ast g = F^\circ \ast G^\circ = (F \ast G)^\circ = (G \ast F)^\circ = G^\circ \ast F^\circ = g \ast f.
\]
Thus $A_n(\Gamma)$ is commutative and hence $(G_n, K_n)$ is a Gelfand pair. \hfill \Box
3. Decomposition of $L(\Gamma^n/\Delta_n)$ and $L(G_n/K_n)$

3.1. Decomposition of $L(\Gamma^n/\Delta_n)$. One checks easily that the map

$$\Gamma^n/\Delta_n \to \Gamma^{n-1}, \quad (x_1,\ldots,x_n)\Delta_n \mapsto (x_1x_n^{-1}x_2,\ldots,x_{n-1}x_n^{-1})$$

is a well-defined bijection. Using this to identify $\Gamma^n/\Delta_n$ with $\Gamma^{n-1}$, the left quasiregular representation of $\Gamma^n$ on $L(\Gamma^n/\Delta_n)$ is realized on $L(\Gamma^{n-1})$ as $\rho_n : \Gamma^n \to GL(L(\Gamma^{n-1}))$ where

$$(3.1) \quad \rho_n(\gamma_1,\ldots,\gamma_{n-1},\gamma)f(y_1,\ldots,y_{n-1}) = f(\gamma_1^{-1}y_1\gamma',\ldots,\gamma_{n-1}^{-1}y_{n-1}\gamma').$$

This is the restriction of the left-right regular representation of $\Gamma^{n-1} \times \Gamma^{n-1}$ to $\Gamma^{n-1} \times \Delta_{n-1}$ upon identification of the diagonal subgroup $\Delta_{n-1} \subset \Gamma^{n-1}$ with $\Gamma$. The Peter-Weyl Theorem shows that $L(\Gamma^{n-1})$ decomposes under $\Gamma^{n-1} \times \Gamma^{n-1}$ as

$$L(\Gamma^{n-1}) \simeq \sum_{\pi \in \Gamma^{n-1}} \pi \hat{\otimes} \pi^*.$$

Here $\Gamma^{n-1}$ denotes the set of irreducible representations of $\Gamma^{n-1}$ modulo equivalence, $\pi^*$ is the dual (or contragredient) representation for $\pi$ and $\pi \hat{\otimes} \pi^*$ is an exterior tensor product representation for the product group $\Gamma^{n-1} \times \Gamma^{n-1}$. The irreducible representations for the product group $\Gamma^{n-1}$ are themselves exterior tensor products $\pi_1 \hat{\otimes} \cdots \hat{\otimes} \pi_{n-1}$ of irreducible representations $\pi_j \in \tilde{\Gamma}$. The restriction of $\pi_1 \hat{\otimes} \cdots \hat{\otimes} \pi_{n-1}$ to $\Delta_{n-1} \cong \Gamma$ is the interior tensor product representation $\pi_1 \otimes \cdots \otimes \pi_{n-1}$. For $\pi_n \in \tilde{\Gamma}$ let $m(\pi_1,\ldots,\pi_{n-1}|\pi_n)$ denote the multiplicity of $\pi_n$ in $\pi_1 \otimes \cdots \otimes \pi_{n-1}$ so that

$$\pi_1 \otimes \cdots \otimes \pi_{n-1} \simeq \sum_{\pi_n \in \Gamma} m(\pi_1,\ldots,\pi_{n-1}|\pi_n)\pi_n.$$ 

Now (3.1) yields the decomposition

$$(3.2) \quad L(\Gamma^n/\Delta_n) \simeq \sum_{\pi_1,\ldots,\pi_{n-1}} \sum_{\pi_n} m(\pi_1,\ldots,\pi_{n-1}|\pi_n)\pi_1 \hat{\otimes} \cdots \hat{\otimes} \pi_{n-1} \hat{\otimes} \pi_n^*$$

for $L(\Gamma^n/\Delta_n)$ as a $\Gamma^n$-module. As $(\Gamma^n, \Delta_n)$ is a Gelfand pair if and only if $L(\Gamma^n/\Delta_n)$ is multiplicity free this proves the following.

**Proposition 3.1.** $(\Gamma^n, \Delta_n)$ is a Gelfand pair if and only if the interior tensor product representation $\pi_1 \otimes \cdots \otimes \pi_{n-1}$ is multiplicity free for all irreducible representations $\pi_1,\ldots,\pi_{n-1} \in \tilde{\Gamma}$.

In particular, taking $n = 2$ we recover the well-known fact that $(\Gamma \times \Gamma, \Delta_2)$ is a Gelfand pair.

3.2. Decomposition of $L(G_n/K_n)$. The irreducible representations for the wreath product $G_n = \Gamma^n \rtimes S_n$ are constructed via the Mackey machine as follows. Let $\pi$ be an irreducible representation of $\Gamma^n$. We have say $\pi = \pi_1 \hat{\otimes} \cdots \hat{\otimes} \pi_n$, the exterior tensor product of irreducible representations $(\pi_j, V_j) \in \tilde{\Gamma}$. The stabilizer of $\pi$ in $S_n$, namely

$$S_\pi = \{\sigma \in S_n : \pi_\sigma(j) = \pi_j \text{ for } j = 1,\ldots,n\},$$

acts on $V_1 \otimes \cdots \otimes V_n$ via the intertwining representation

$$\omega : S_\pi \to GL(V_1 \otimes \cdots \otimes V_n), \quad \omega(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(n)}.$$
Given any \( \rho \in \hat{S}_n \) the induced representation

\[ R_{\pi, \rho} = ind_{\Gamma \times S_n}^{\Gamma_n}((\pi \circ \omega) \hat{\otimes} \rho) \]

is irreducible and every irreducible representation of \( G_n \) is of this form.

**Example 3.2.** Suppose we have \( n = 3 \), \( \pi = \pi_1 \otimes \pi_2 \otimes \pi_3 \). The stabilizer of \( \pi \) in \( S_3 \) is \( S_3 \cong S_2 \times S_1 \). Letting \( \rho_0 \) denote the trivial representation of \( S_n \) the induced representation \( R_{\pi, \rho_0} \) is

\[ R_{\pi, \rho_0} = (\pi_1 \otimes \pi_1 \otimes \pi_2) \oplus (\pi_1 \otimes \pi_2 \otimes \pi_1) \oplus (\pi_2 \otimes \pi_1 \otimes \pi_1) \]

as a representation of \( \Gamma^3 \) with \( S_3 \) permuting the factors.

Recall that \( (G_n = \Gamma^n \rtimes S_n, K_n = \Delta_n \rtimes S_n) \) is a Gelfand pair if and only if the space \( R^{K_n} \) of \( K_n \)-fixed vectors in \( R \) has \( \dim(R^{K_n}) \leq 1 \) for every irreducible representation \( R \in \hat{G}_n \). Abusing terminology we call \( \dim(R^{K_n}) \) the “number of \( K_n \)-fixed vectors in \( R \).” Letting \( K_\pi = \Delta_n \rtimes S_\pi \) for given \( \pi \in \hat{\Gamma^n} \) one has

\[ R_{\pi, \rho}|_{K_\pi} = ind_{K_\pi}^{\hat{K}_\pi}((\pi \circ \omega) \hat{\otimes} \rho). \]

An application of Frobenius reciprocity now yields the following.

**Lemma 3.3.** The number of \( K_n \)-fixed vectors in \( R_{\pi, \rho} \) is equal to the number of \( K_\pi \)-fixed vectors in \( (\pi \circ \omega) \hat{\otimes} \rho \).

Thus in order for \( R_{\pi, \rho} \) to be \( K_n \)-spherical there must be \( K_\pi \)-fixed vectors in \( (\pi \circ \omega) \hat{\otimes} \rho \) and hence \( \Delta_n \)-fixed vectors in \( \pi = \pi_1 \otimes \cdots \otimes \pi_n \). Now (3.2) yields the following necessary condition for \( R_{\pi, \rho} \) to be spherical.

**Lemma 3.4.** If \( R_{\pi, \rho} \) is \( K_n \)-spherical then \( \pi_\pi^* \) occurs in the (internal) tensor product \( \pi_1 \otimes \cdots \otimes \pi_n \).

We will make use of this criterion in connection with examples in Section 5.

### 4. Proof of Theorem 1.2

Among the irreducible representations for \( G_n = \Gamma^n \rtimes S_n \) discussed above are the following. Given an irreducible representation \( (\pi, V) \) of \( \Gamma \) one obtains an irreducible representation \( \pi \) of \( G_n \) in the \( n \)th tensor power \( W = \otimes^n V \) via

\[ \pi(x, \sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \pi(x_1)v_{\sigma^{-1}(1)} \otimes \pi(x_2)v_{\sigma^{-1}(2)} \cdots \otimes \pi(x_n)v_{\sigma^{-1}(n)} \]

on decomposable tensors.\(^1\) Observe that the space of \( \pi(S_n) \)-invariant vectors in \( W \) is \( W^{S_n} = S^n(V) \), the \( n \)th symmetric power of \( V \). The action of the diagonal subgroup \( \Delta_n \) on \( V \) via \( \pi \) preserves \( W^{S_n} \) as \( \Delta_n \) and \( S_n \) commute in \( G_n \). Moreover the representation \( (\pi|_{\Delta_n}, W^{S_n} = S^n(V)) \) of \( \Delta_n \) coincides with \( (S^n(\pi), S^n(V)) \), the \( n \)th symmetric power of the representation \( (\pi, V) \), under the obvious isomorphism \( \Delta_n \cong \Gamma \). It follows that the space \( W^{K_n} \) of \( (K_n = \Delta_n \rtimes S_n) \)-invariant vectors in \( W \) is precisely

\[ W^{K_n} = S^n(V)^\Gamma, \]

the space of \( \Gamma \)-invariant vectors in the \( n \)th symmetric power of \( (\pi, V) \).

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\(^1\)Here \( \pi = R_{\pi \otimes \cdots \otimes \pi} \) in the notation from the previous section.
The strategy is to first identify representations of $\Gamma$ isomorphic to $C(\Gamma)$. Let $(\pi, V)$ be the irreducible representation of $G_r = \Gamma^r \rtimes S_r$ described above we now have that the space $W^{K_r}$ of $(K_r = \Delta_r \times S_r)$-fixed vectors in $W$ has dimension at least $d$. Thus $(G_r, K_r)$ fails to be a Gelfand pair as was to be shown.

Remark 4.1. The result that $S(V)^\Gamma$ contains $d$ algebraically independent homogeneous elements does not require that $(\pi, V)$ be irreducible. Most of the literature on Invariant Theory, including [Sta79], concerns invariants in polynomial rings $C[V]$ rather than symmetric algebras $S(V)$. To pass to this context one can replace $(\pi, V)$ above by its dual representation $(\pi^*, V^*)$ and note that $S(V^*)$ is canonically isomorphic to $C[V]$ as a $\Gamma$-module.

5. Wreath products with dihedral groups

In this section we take $\Gamma = D_p$, where $p$ is any odd prime. We show that $(\Gamma^n \times S_n, \Delta_n \times S_n)$ is a Gelfand pair for $n \leq 5$ and not a Gelfand pair for $n \geq 6$. The strategy is to first identify representations of $\Gamma^n$ which occur in $L(\Gamma^n/\Delta_n)$ by finding the tensor product representations which have $\Delta_n$-fixed vectors.

5.1. The case $n = 3$. We begin by reviewing the representation theory of $D_p$. The conjugacy classes are the identity element $\{I\}$, pairs of rotations $\{R, R^{-1}\}$, and the set $S$ of all reflections. There are two one-dimensional irreducible representations, the trivial representation $\theta_1$ and the determinant $\theta_2$. In addition, there are $m = (p - 1)/2$ two-dimensional representations $\pi_j$ with characters $\chi_j$. In each of these representations, any rotation has eigenvalues $\lambda$ and $\lambda^{-1}$, where $\lambda$ is a $p$th root of unity.

The character table is:

$$
\begin{array}{cccccc}
\theta_1 & \theta_2 & \chi_j & \lambda & \lambda^{-1} \\
(1) & (2) & (3) & (4) & (5) & (6) \\
I & R_1 & R_2 & S & 1 & 1 \\
\theta_1 & 1 & 1 & 1 & 1 & 0 \\
\theta_2 & 1 & 1 & 1 & 1 & 0 \\
\chi_j & 2 & \mu_j & \mu_{2j} & 0 & 0 \\
\end{array}
$$

The second row of this table lists the conjugacy classes, while the row above indicates the number of elements in each conjugacy class. In the last row, $\pi_j$ is a two-dimension representation, with $\mu_j = \lambda^j + \lambda^{-j}$ for $j = 1, \ldots, m = (p - 1)/2$.

Since all of the characters are real-valued, by Lemma 3.4 we consider representations $\pi_i \otimes \pi_j \otimes \pi_k$, where $\pi_k$ occurs in $\pi_i \otimes \pi_j$. Equivalently, we want $\pi_i \otimes \pi_j \otimes \pi_k$ to contain a $\Delta_3$-fixed vector. Let us consider some examples.

The representations $\pi_1 \otimes \pi_1$ and $\pi_1 \otimes \pi_2$ have characters

$$
\begin{array}{cccccc}
\chi_1^2 & \chi_2^2 & \chi_1 \chi_2 & \lambda_1 \lambda_2 & \lambda_1 \lambda_2 \lambda_3 & \lambda_1 \lambda_2 \lambda_3 \\
I & R_1 & R_2 & S & 1 & 1 \\
\chi_1^2 & 4 & \mu_1^2 & \mu_2^2 & 0 & 0 \\
\chi_1 \chi_2 & 4 & \mu_1 \mu_2 & \mu_2 \mu_3 & 0 & 0 \\
\end{array}
$$
A calculation shows that
\[ \mu_i \mu_j = \mu_{i+j} + \mu_{i-j}, \]
where we understand that the subscripts are taken mod \( p \), that \( \mu_{-k} = \mu_k \), and that \( \mu_0 = 2 \). Thus in our examples, we get the characters
\[
\begin{array}{cccccc}
I & R_1 & R_2 & \ldots & S \\
\chi^2_1 & 4 & \mu_2 + 2 & \mu_4 + 2 & \ldots & 0 \\
\chi_{1\chi_2} & 4 & \mu_1 + \mu_3 & \mu_2 + \mu_6 & \ldots & 0 \\
\end{array}
\]
and hence \( \pi_1 \otimes \pi_1 \simeq \pi_2 \oplus \theta_1 \oplus \theta_2 \) and \( \pi_1 \otimes \pi_2 \simeq \pi_1 \oplus \pi_3 \). This tells us that the triples \( \pi_1 \otimes \pi_1 \otimes \pi_2 \), \( \pi_1 \otimes \pi_1 \otimes \theta_1 \), \( \pi_1 \otimes \pi_1 \otimes \theta_2 \) and \( \pi_1 \otimes \pi_2 \otimes \pi_3 \) are all \( \Delta_3 \)-spherical representations of \( \Gamma^3 \). In each case, there is a single \( \Delta_3 \)-fixed vector, and hence each representation occurs in \( L(\Gamma^3/\Delta_3) \) with multiplicity one.

The spherical representations of \( \Gamma^3 \) are:
- \( \pi_i \otimes \pi_j \otimes \pi_{i+j} \) for \( i \neq j \),
- \( \pi_i \otimes \pi_j \otimes \theta_1 \)
- \( \pi_i \otimes \pi_j \otimes \theta_2 \)
- \( \theta_1 \otimes \theta_1 \otimes \theta_1 \)
- \( \theta_1 \otimes \theta_2 \otimes \theta_2 \)
and all permutations of the tensor product factors. Each of these representations has a single \( \Delta_3 \)-fixed vector, hence it occurs in \( L(\Gamma^3/\Delta_3) \) with multiplicity one. Note that for small values of \( p \), not all cases may occur. Now we have:

**Proposition 5.1.** For \( \Gamma = D_p \), \( (\Gamma^3, \Delta_3) \) and \( (\Gamma^3 \times S_3, \Delta_3 \times S_3) \) are Gelfand pairs.

### 5.2. The case \( n = 4 \)

We use identities of the following type to find the spherical representations:
\[
\chi_i \chi_j \chi_k = \sum \chi_{i+j \pm k};
\]
\[
\chi^3_j = \chi_j (\chi_{2j} + \theta_1 + \theta_2) = \chi_j + \chi_{3j} + 2\chi_j = 3\chi_j + \chi_{3j};
\]
\[
\chi^2_j \chi_k = (\chi_{2j} + \theta_1 + \theta_2)\chi_k = \chi_{2j-k} + \chi_{2j+k} + 2\chi_k.
\]
Representations of \( \Gamma^4 \) which occur in \( L(\Gamma^4/\Delta_4) \):
- \( \pi_i \otimes \pi_j \otimes \pi_k \otimes \pi_{i+j \pm k} \) for \( i, j, k \) distinct.
- \( \pi_j^{(4)} \) with multiplicity three. (Here \( \pi^{(k)} \) denotes \( \pi \otimes \cdots \otimes \pi \).)
- \( \pi_j^{(3)} \otimes \pi_{3j} \)
- \( \pi_j^{(2)} \otimes \pi_k \otimes \pi_{2j \pm k} \)
- \( \pi_j^{(2)} \otimes \theta_k^{(2)} \) with multiplicity two.
- \( \theta_j^{(3)} \otimes \pi_j^{(2)} \otimes \pi_{2j} \)
- \( \theta_j^{(3)} \otimes \theta_j^{(2)} \otimes \pi_k \) for all choices of \( i, j, k \).
- \( \theta_i^{(3)} \otimes \pi_j \otimes \pi_k \otimes \pi_{j \pm k} \) for \( j \neq k \).
- \( \theta_i^{(4)} \).
- \( \theta_j^{(2)} \otimes \theta_k^{(2)} \).
Since there are two cases of multiplicity, \( (\Gamma^4, \Delta_4) \) is not a Gelfand pair for \( \Gamma = D_p \).
5.3. Characters for $\Gamma^4 \times S_4$. Let $\pi = \pi_1 \otimes \pi_2 \otimes \pi_k \otimes \pi_l$ be a representation of $\Gamma^4$, $S_\pi$ the stabilizer of $\pi$ in $S_4$, and $\omega$ the intertwining representation of $S_\pi$. Let $\chi_\pi$ be the character for $\pi \circ \omega$. The technique for computing $\chi_\pi$ is illustrated by the following example:

Suppose that $\pi = \pi_1^{(4)}$, so that $S_\pi = S_4$. Then

$$\chi_\pi(\delta, (1234)) = \chi_1(\delta^4),$$
$$\chi_\pi(\delta, (123)) = \chi_\pi(\delta^3)\chi_\pi(\delta),$$
$$\chi_\pi(\delta, (12)) = \chi_\pi(\delta^2)\chi_\pi(\delta)^2,$$
$$\chi_\pi(\delta, (12)(34)) = \chi_\pi(\delta^2)\chi_\pi(\delta^2).$$

where we identify $\delta \in \Gamma$ with an element of $\Delta_4$. The two cases of multiplicity are $\pi_j^{(4)}$ and $\pi_j^{(2)} \otimes \pi_k^{(2)}$, with stabilizers $S_4$ and $S_2 \times S_2$ respectively. Since $\pi_1$ is determined by an arbitrary $p$th root of unity, we can take $j = 1$ in both cases.

We need to find the spherical representations of the form $R_{\pi,\rho}$ for $\rho \in \widehat{S_\pi}$. That is, in view of Lemma 3.3, we seek representations which contain $(K_\pi = \Delta_4 \times S_\pi)$-fixed vectors. The number of such vectors is:

$$(5.1) \frac{1}{|\Delta_4 \times S_\pi|} \sum_{\delta,\sigma} \chi_\pi(\delta, \sigma)\chi_\rho(\sigma) = \frac{1}{|S_\pi|} \sum_{\sigma} \left( \frac{1}{|\Delta_4|} \sum_{\delta} \chi_\pi(\delta, \sigma) \right) \chi_\rho(\sigma)$$

For each fixed $\sigma \in S_\pi$, define the function

$$m_{\pi,\sigma}(\delta) = \chi_\pi(\delta, \sigma).$$

This is a class function on $\Delta = \Delta_4$, which can be expressed as a linear combination of irreducible characters. So the sum

$$M_\pi(\sigma) = \frac{1}{|\Delta|} \sum_{\delta} \chi_\pi(\delta, \sigma) = \langle m_{\pi,\sigma}, 1 \rangle_\Delta$$

is the coefficient of the trivial character $\theta_1$ in $m_{\pi,\sigma}$. Moreover, the function $M_\pi$ is a class function on $S_\pi$, so the sum (5.1) is the coefficient of $\chi_\rho = \chi_\sigma$ in $M_\pi$.

For $\pi = \pi_1^{(4)}$, a straightforward calculation shows that:

$$\chi_\pi(\delta, e) = \chi_1(\delta)^4 = (\chi_4 + 4\chi_2 + 3\theta_1 + 3\theta_2)(\delta),$$
$$\chi_\pi(\delta, (12)) = \chi_1(\delta^2)\chi_1(\delta)^2 = (\chi_4 + 2\chi_2 + \theta_1 + \theta_2)(\delta)$$
$$\chi_\pi(\delta, (123)) = \chi_1(\delta^3)\chi_1(\delta) = (\chi_4 + \chi_2)(\delta)$$
$$\chi_\pi(\delta, (1234)) = \chi_1(\delta^4) = (\chi_2 + \theta_1 - \theta_2)(\delta)$$
$$\chi_\pi(\delta, (12)(34)) = \chi_1(\delta^2)\chi_1(\delta^2) = (\chi_4 + 3\theta_1 - 2\theta_2)(\delta)$$

We take the coefficient of $\theta_1$ to obtain:

$$M_\pi(e) = 3, \quad M_\pi((12)) = 1, \quad M_\pi((123)) = 0, \quad M_\pi((1234)) = 1, \quad M_\pi((12)(34)) = 3.$$ 

Consulting the character table of $S_4$, we find that $M_\pi$ is the sum of the trivial character and a two-dimensional character, with corresponding representations $\rho_o$ and $\rho_2$.

This tells us that $(\pi_1^{(4)} \circ \omega) \otimes \rho_o$ and $(\pi_1^{(4)} \circ \omega) \otimes \rho_2$ are $K_\pi$-spherical representations, each occurring with multiplicity one. Together, these spaces account for the occurrence of $\pi_1^{(4)}$ with multiplicity three.

For $\pi = \pi_1^{(2)} \otimes \pi_k^{(2)}$ and $S_\pi \cong S_2 \times S_2$, we obtain

$$M_\pi(e) = 2, \quad M_\pi((12)) = 0, \quad M_\pi((34)) = 0, \quad M_\pi((12)(34)) = 2.$$
Thus $M_\pi$ is the sum of the trivial character $\rho_0$ and the sign character $\rho_s$. This tells us that $(\pi_1^{(2)} \otimes \pi_2^{(2)} \circ \omega) \otimes \rho_s$ and $(\pi_1^{(2)} \otimes \pi_2^{(2)} \circ \omega) \otimes \rho_s$ are spherical representations, each occurring with multiplicity one. Together, these spaces account for the occurrence of $\pi_1^{(2)} \otimes \pi_k^{(2)}$ with multiplicity two. Thus we see that $(\Gamma^4 \rtimes S_4, \Delta \times S_4)$ is a Gelfand pair.

5.4. The case $n = 5$. Using similar techniques to those above, we can find the spherical representations of $\Gamma^5$. We list just those with multiplicity:

- $\pi_j^{(4)} \otimes \theta_1$, with multiplicity 3.
- $\pi_j^{(2)} \otimes \pi_k^{(2)} \otimes \theta_1$, with multiplicity 2.
- $\pi_j^{(4)} \otimes \pi_2 \otimes \theta_1$, with multiplicity 4.
- $\pi_j^{(3)} \otimes \pi_k \otimes \pi_{j \pm k}$ with multiplicity 3.
- $\pi_j^{(2)} \otimes \pi_k^{(2)} \otimes \pi_{2k}$ with multiplicity 2.
- $\pi_j^{(2)} \otimes \pi_k \otimes \pi_{k \pm l}$ with multiplicity 2.

Since $\chi_j^1 \theta_1 = \chi_j^1$ and $\chi_j^2 \chi_k^2 \theta_1 = \chi_j^2 \chi_k^2$, the first two cases are handled in the previous section. For the other cases, we take $j = 1$.

For $\pi = \pi_1^{(4)} \otimes \pi_2$ and $S_\pi \cong S_4 \times S_1$, we obtain

$$M_\pi(e) = 4, \ M_\pi((12)) = 2, \ M_\pi((123)) = 1, \ M_\pi((12)(34)) = 0, \ M_\pi((1234)) = 0.$$ 

Thus $M_\pi$ is the character of the standard 4-dimensional representation of $S_4$, which is the sum of the trivial representation $\rho_0$ and a 3-dimensional irreducible representation $\rho_3$. This tells us that $(\pi_1^{(4)} \otimes \pi_2 \circ \omega) \otimes \rho_0$ and $(\pi_1^{(4)} \otimes \pi_2 \circ \omega) \otimes \rho_3$ are spherical representations, each occurring with multiplicity one. Together, these spaces account for the occurrence of $\pi_1^{(4)} \otimes \pi_2$ with multiplicity four.

For $\pi = \pi_1^{(3)} \otimes \pi_k \otimes \pi_{k+1}$ and $S_\pi \cong S_4$, we obtain

$$M_\pi(e) = 3, \ M_\pi((12)) = 1, \ M_\pi((123)) = 0.$$ 

Thus $M_\pi$ is the character of the standard 3-dimensional representation of $S_3$, which is the sum of the trivial representation $\rho_0$ and a 2-dimensional irreducible representation $\rho_2$. This tells us that $(\pi_1^{(3)} \otimes \pi_k \otimes \pi_{k+1} \circ \omega) \otimes \rho_0$ and $(\pi_1^{(4)} \otimes \pi_k \otimes \pi_{k+1} \circ \omega) \otimes \rho_2$ are spherical representations, each occurring with multiplicity one. Together, these spaces account for the occurrence of $\pi_1^{(4)} \otimes \pi_k \otimes \pi_{k+1}$ with multiplicity three.

For $\pi = \pi_1^{(2)} \otimes \pi_k^{(2)} \otimes \pi_2$ and $S_\pi \cong S_2 \times S_2$, we obtain

$$M_\pi(e) = 2, \ M_\pi((12)) = 2, \ M_\pi((34)) = 0, \ M_\pi((12)(34)) = 0.$$ 

Thus $M_\pi$ is the sum of two 1-dimensional irreducible representations of $S_2 \times S_2$, one of which is the trivial representation.

For $\pi = \pi_1^{(2)} \otimes \pi_k \otimes \pi_1 \otimes \pi_{k \pm l}$ and $S_\pi \cong S_2$, we obtain

$$M_\pi(e) = 2, \ M_\pi((12)) = 0.$$ 

Thus $M_\pi$ is the sum of the two 1-dimensional irreducible representations of $S_2$. We conclude that $(\Gamma^5 \rtimes S_5, \Delta_5 \times S_5)$ is a Gelfand pair.
5.5. The Case $n = 6$. In this case, $(D_p^6 \rtimes S_6, \Delta_6 \times S_6)$ fails to be a Gelfand pair.
This is consistent with the GAP-generated result for $p = 3$ in [AC12].

Let $\pi = \pi_1^{(4)} \otimes \pi_2^{(2)}$, with $S_\pi \cong S_4 \times S_2$. Then we obtain

\[
M_\pi(e) = 7, \ M_\pi((12)) = 3, \ M_\pi((123)) = 1, \ M_\pi((1234)) = 1, \ M_\pi((12)(34)) = 3,
\]
\[
M_\pi((56)) = 1, \ M_\pi((12)(56)) = 1, \ M_\pi((123)(56)) = 1, \ M_\pi((1234)(56)) = 3,
\]
\[
M_\pi((12)(34)(56)) = 5.
\]

One can see that $\langle M_\pi, \rho_\circ \rangle = 2$, and hence that $R_{\pi, \rho_\circ} = \text{Ind}_{\Gamma_6 \times S_\pi}^{G_6} (\pi \circ \omega \otimes \rho_\circ)$ has multiplicity 2 in $L(G_6/K_6)$. For $p = 3$, we have $\pi_1 = \pi_2$, and the result holds for $\pi = \pi_1^{(6)}$.

References


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