A FAMILY OF FINITE GELFAND PAIRS ASSOCIATED WITH WREATH PRODUCTS

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ABSTRACT. Consider the wreath product $G_n = \Gamma^n \rtimes S_n$ of a finite group Γ with the symmetric group S_n. Letting Δ_n denote the diagonal in Γ^n the direct product $K_n = \Delta_n \times S_n$ forms a subgroup. In case Γ is abelian (G_n, K_n) forms a Gelfand pair with relevance to the study of parking functions. For Γ non-abelian it was suggested by Kürşat Aker and Mahir Bilen Can that (G_n, K_n) must fail to be a Gelfand pair for n sufficiently large. We prove here that this is indeed the case: for Γ non-abelian there is some integer $2 < N(\Gamma) \leq |\Gamma|$ for which (K_n, G_n) is a Gelfand pair for all $n < N(\Gamma)$ but (K_n, G_n) fails to be a Gelfand pair for all $n \geq N(\Gamma)$. For dihedral groups $\Gamma = D_p$ with p an odd prime we prove that $N(\Gamma) = 6$.

1. Introduction

Gelfand pairs are fundamental to the study of harmonic analysis on topological groups. In the context of finite groups the definition is as follows. We denote by $L(G)$ the space of complex-valued functions on a finite group G. This is an algebra under the convolution product

$$f \star g(x) = \sum_{y \in G} f(xy^{-1})g(y).$$

Given a subgroup $K \subset G$, the set

$$L(K \backslash G/K) = \{f \in L(G) \ : \ f(k_1 x k_2) = f(x) \ \forall k_1, k_2 \in K\}$$

of K-bi-invariant functions on G forms a subalgebra of $L(G)$. One calls (G, K) a *Gelfand pair* when $L(K \backslash G/K)$ is commutative. This condition is equivalent to each of the following.

- The left quasi-regular representation $ind_K^G(1_K)$ of G in $L(G/K)$ is multiplicity free.
- For each irreducible representation (π, V) of G the space V^K of K-fixed vectors in V has dimension $\dim(V^K) \leq 1$.

Irreducible representations of G which occur in $L(G/K)$ are called *K-spherical*. These are precisely those admitting non-zero K-fixed vectors. We refer the reader to [CSST08], [Mac95, Chapter VII] or [Ter99] for proofs of these equivalences as well as general background concerning Gelfand pairs and their applications in the finite groups setting.

Given a finite group Γ the symmetric group S_n acts by automorphisms on the cartesian product Γ^n of n copies of Γ via

$$\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

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The resulting semi-direct product

$$G_n := \Gamma^n \rtimes S_n$$

is the *wreath product* of Γ with S_n, sometimes written as $\Gamma \wr S_n$. The representation theory of such wreath products is discussed in [Mac95, Chapter I Appendix B] and, in greater generality, in [CSST06, CSST14].

Let

$$L(\Gamma^n)^{S_n} = \{f \in L(\Gamma^n) \ : \ f(\sigma \cdot \mathbf{x}) = f(\mathbf{x}) \ \forall \sigma \in S_n, \mathbf{x} \in \Gamma^n\}$$

denote the space of complex valued functions on Γ^n invariant under the action of S_n. This is a convolution subalgebra of $L(\Gamma^n)$ and routine calculations show that the map

$$\Phi : L(S_n \backslash G_n / S_n) \to L(\Gamma^n)^{S_n}, \quad \Phi(f)(\mathbf{x}) = n! f(\mathbf{x}, e) \quad (e \in S_n \text{ the identity}) \tag{1.1}$$

is an isomorphism of convolution algebras. In particular (G_n, S_n) is a Gelfand pair if and only if $L(\Gamma^n)^{S_n}$ is commutative. This is certainly the case whenever Γ is abelian. On the other hand for Γ non-abelian choose points $x, y \in \Gamma$ with $xy \neq yx$ and let $\mathbf{x}, \mathbf{y} \in \Gamma^n$ be the points $\mathbf{x} = (x, \dots, x)$, $\mathbf{y} = (y, \dots, y)$. The characteristic functions $\delta_{\mathbf{x}}$, $\delta_{\mathbf{y}}$ for these points belong to $L(\Gamma^n)^{S_n}$ and we have

$$\delta_{\mathbf{x}} \star \delta_{\mathbf{y}} = \delta_{\mathbf{xy}} \neq \delta_{\mathbf{yx}} = \delta_{\mathbf{y}} \star \delta_{\mathbf{x}}.$$

Thus (G_n, S_n) is a Gelfand pair if and only if Γ is abelian.

The *diagonal subgroup* in Γ^n,

$$\Delta_n := \{(x, \dots, x) \ : \ x \in \Gamma\},$$

played a role in the preceding discussion. The S_n-action preserves $\Delta_n \subset \Gamma^n$ and is trivial on Δ_n. Thus the direct product

$$\big(K_n := \Delta_n \times S_n\big) \cong \Gamma \times S_n$$

is a subgroup of $G_n = \Gamma^n \rtimes S_n$ and we consider the pair (G_n, K_n).

Restricting the map Φ given in (1.1) to $L(K_n \backslash G_n / K_n) \subset L(S_n \backslash G_n / S_n)$ produces an algebra isomorphism onto

$$\mathcal{A}_n(\Gamma) := L(\Delta_n \backslash \Gamma^n / \Delta_n) \cap L(\Gamma^n)^{S_n}, \tag{1.2}$$

the algebra of functions $\Gamma^n \to \mathbb{C}$ which are both Δ_n-bi-invariant and S_n-invariant. Thus if either (G_n, S_n) or (Γ^n, Δ_n) is a Gelfand pair then so is (G_n, K_n). It follows in particular that

- (G_n, K_n) is a Gelfand pair for Γ abelian and
- (G_n, K_n) is a Gelfand pair for $n = 2$.

The latter point follows from the well-known fact that $(\Gamma \times \Gamma, \Delta_2)$ is a Gelfand pair [Mac95, §VII-1, Example 9].

For cyclic groups Γ the resulting Gelfand pairs (G_n, K_n) arise in combinatorics in connection with *parking functions* [AC12]. This fact motivates interest in pairs (G_n, K_n) for other finite groups Γ. It is suggested in [AC12] that for Γ non-abelian (G_n, K_n) will fail to be a Gelfand pair for n sufficiently large. The following theorems show that this is indeed the case. These are our main results.

Theorem 1.1. *If (G_{n+1}, K_{n+1}) is a Gelfand pair then so is (G_n, K_n).*

Theorem 1.2. *If Γ is non-abelian then $(G_{|\Gamma|}, K_{|\Gamma|})$ fails to be a Gelfand pair.*

Thus for Γ non-abelian there is some integer $2 < N(\Gamma) \leq |\Gamma|$ for which

- (K_n, G_n) is a Gelfand pair for all $n < N(\Gamma)$ but
- (K_n, G_n) fails to be a Gelfand pair for all $n \geq N(\Gamma)$.

Examples 1.3. The authors of [AC12] used the GAP computer algebra system to verify that $N(S_3) = 6$, $N(A_4) = 4$, $N(GL(2, \mathbb{F}_3)) = 3$ and $N(SL(3, \mathbb{F}_2)) = 3$.

We remark that we do not know whether or not $N(\Gamma)$ can be arbitrarily large.

Proofs for Theorems 1.1 and 1.2 are given below in Sections 2 and 4. Section 3 concerns decomposition of the spaces $L(\Gamma^n/\Delta_n)$ and $L(G_n/K_n)$ under the left actions of Γ^n and G_n. Section 5 concerns examples. We show that for primes $p \geq 3$ the dihedral groups D_p have $N(D_p) = 6$. As $D_3 \cong S_3$ this is consistent with [AC12]. In [AM03] the reader will find a different family of Gelfand pairs involving wreath products with dihedral groups.

2. Proof of Theorem 1.1

For $f \in L(\Gamma^{n+1})$ let $f^\circ \in L(\Gamma^n)$ be defined as

$$f^\circ(x_1, \ldots, x_n) = \sum_{\gamma \in \Gamma} f(x_1, \ldots, x_n, \gamma)$$

and consider the map

$$\Psi : L(\Gamma^{n+1}) \to L(\Gamma^n), \quad \Psi(f) = f^\circ.$$

This is an algebra map. That is,

Lemma 2.1. *$(f \star g)^\circ = f^\circ \star g^\circ$ for $f, g \in L(\Gamma^{n+1})$.*

Proof. In fact

$$\begin{aligned}
(f \star g)^\circ(x_1, \ldots, x_n) &= \sum_{\gamma \in \Gamma} f \star g(x_1, \ldots, x_n, \gamma) \\
&= \sum_{\gamma \in \Gamma} \sum_{y_1, \ldots, y_{n+1} \in \Gamma} f(x_1 y_1^{-1}, \ldots, x_n y_n^{-1}, \gamma y_{n+1}^{-1}) g(y_1, \ldots, y_{n+1}) \\
&= \sum_{y_1, \ldots, y_n} \sum_{y_{n+1}} \sum_{\gamma} f(x_1 y_1^{-1}, \ldots, x_n y_n^{-1}, \gamma y_{n+1}^{-1}) g(y_1, \ldots, y_{n+1}) \\
&= \sum_{y_1, \ldots, y_n} \sum_{y_{n+1}} \sum_{\gamma'} f(x_1 y_1^{-1}, \ldots, x_n y_n^{-1}, \gamma') g(y_1, \ldots, y_{n+1}) \\
&= \sum_{y_1, \ldots, y_n} \sum_{y_{n+1}} f^\circ(x_1 y_1^{-1}, \ldots, x_n y_n^{-1}) g(y_1, \ldots, y_{n+1}) \\
&= \sum_{y_1, \ldots, y_n} f^\circ(x_1 y_1^{-1}, \ldots, x_n y_n^{-1}) g^\circ(y_1, \ldots, y_n) \\
&= f^\circ \star g^\circ(x_1, \ldots, x_n).
\end{aligned}$$

□

Lemma 2.2. $\Psi\big(L(\Gamma^{n+1})^{S_{n+1}}\big) \subset L(\Gamma^n)^{S_n}$.

Proof. Say $f \in L(\Gamma^{n+1})^{S_{n+1}}$ and $\sigma \in S_n$. Then

$$\begin{aligned}
f^\circ(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) &= \sum_{\gamma \in \Gamma} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \gamma) = \sum_{\gamma \in \Gamma} f(x_1, \ldots, x_n, \gamma) \\
&= f^\circ(x_1, \ldots, x_n)
\end{aligned}$$

since $(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \gamma)$ is a permutation of $(x_1, \ldots, x_n, \gamma)$. □

For each m the group $\Gamma \times \Gamma$ acts on the set Γ^m via

$$(\gamma_1, \gamma_2) \cdot (x_1, \ldots, x_m) = (\gamma_1 x_1 \gamma_2^{-1}, \ldots, \gamma_1 x_n \gamma_2^{-1})$$

and on $L(\Gamma^m)$ via

$$(\gamma_1, \gamma_2) \cdot f(\mathbf{x}) = f\big((\gamma_1^{-1}, \gamma_2^{-1}) \cdot \mathbf{x}\big).$$

Lemma 2.3. *Ψ is $(\Gamma \times \Gamma)$-equivariant. That is, $\big((\gamma_1, \gamma_2) \cdot f\big)^\circ = (\gamma_1, \gamma_2) \cdot f^\circ$ for $f \in L(\Gamma^{n+1})$, $\gamma_1, \gamma_2 \in \Gamma$.*

Proof. Indeed

$$\begin{aligned}\big((\gamma_1, \gamma_2) \cdot f\big)^\circ(x_1, \ldots, x_n) &= \sum_{\gamma \in \Gamma} f(\gamma_1^{-1} x_1 \gamma_2, \ldots, \gamma_1^{-1} x_n \gamma_2, \gamma_1^{-1} \gamma \gamma_2) \\ &= \sum_{\gamma' \in \Gamma} f(\gamma_1^{-1} x_1 \gamma_2, \ldots, \gamma_1^{-1} x_n \gamma_2, \gamma') \\ &= f^\circ(\gamma_1^{-1} x_1 \gamma_2, \ldots, \gamma_1^{-1} x_n \gamma_2) \\ &= (\gamma_1, \gamma_2) \cdot f^\circ(x_1, \ldots, x_n).\end{aligned}$$

□

Corollary 2.4. $\Psi\big(L(\Delta_{n+1} \backslash \Gamma^{n+1} / \Delta_{n+1}\big) \subset L(\Delta_n \backslash \Gamma^n / \Delta_n)$.

Proof. As $L(\Delta_m \backslash \Gamma^m / \Delta_m) = L(\Gamma^m)^{\Gamma \times \Gamma}$ for all m this follows from Lemma 2.3. □

Recalling that $\mathcal{A}_m(\Gamma) = L(\Delta_m \backslash \Gamma^m / \Delta_m) \cap L(\Gamma^m)^{S_m}$ (see Equation 1.2) Corollary 2.4 together with Lemma 2.2 give the following.

Corollary 2.5. $\Psi\big(\mathcal{A}_{n+1}(\Gamma)\big) \subset \mathcal{A}_n(\Gamma)$.

We wish to show that in fact

Lemma 2.6. $\Psi\big(\mathcal{A}_{n+1}(\Gamma)\big) = \mathcal{A}_n(\Gamma)$. *That is, $\Psi : \mathcal{A}_{n+1}(\Gamma) \to \mathcal{A}_n(\Gamma)$ is surjective.*

Working towards a proof for this we introduce, for each m, the projection map

$$P_m : L(\Gamma^m)^{S_m} \to \mathcal{A}_m(\Gamma), \quad P_m(f) = \sum_{\gamma_1, \gamma_2 \in \Gamma} (\gamma_1, \gamma_2) \cdot f.$$

As Ψ is $(\Gamma \times \Gamma)$-equivariant (Lemma 2.3) the diagram

$$\begin{array}{ccc} L(\Gamma^{n+1})^{S_{n+1}} & \xrightarrow{\Psi} & L(\Gamma)^{S_n} \\ \big\downarrow P_{n+1} & & \big\downarrow P_n \\ \mathcal{A}_{n+1}(\Gamma) & \xrightarrow{\Psi} & \mathcal{A}_n(\Gamma) \end{array}$$

commutes. As P_n is surjective we see that to prove Lemma 2.6 it suffices to show that $\Psi : L(\Gamma^{n+1})^{S_{n+1}} \to L(\Gamma^n)^{S_n}$ is surjective. For this we require Lemma 2.7 below.

List the elements of Γ as

$$\Gamma = \{\gamma_1, \ldots, \gamma_r\}$$

say where $r = |\Gamma|$. For $\mathbf{x} \in \Gamma^m$ we let

$$[\mathbf{x}] := S_m \cdot \mathbf{x}$$

denote the S_m-orbit though $\mathbf{x}$. This contains a unique point in which any γ_1's appear first followed by any γ_2's etcetera in order. So

$$[\mathbf{x}] = \langle k_1, k_2, \ldots, k_r \rangle := \big[\underbrace{\gamma_1, \ldots, \gamma_1}_{k_1}, \underbrace{\gamma_2, \ldots, \gamma_2}_{k_2}, \ldots, \underbrace{\gamma_r, \ldots, \gamma_r}_{k_r}\big]$$

for some non-negative integers $k_1, \dots, k_r$ with $k_1 + \cdots + k_r = m$. The characteristic functions

$$\delta_{\langle k_1,k_2,\dots,k_r\rangle}(x) = \begin{cases} 1 & \text{if } x \in \langle k_1, k_2, \dots, k_r\rangle \\ 0 & \text{otherwise} \end{cases}$$

give a basis for $L(\Gamma^m)^{S_m}$.

Lemma 2.7. *For integers $k_1, \dots, k_r \geq 0$ with $k_1 + \cdots + k_r = n+1$ we have*

$$\delta^\circ_{\langle k_1,k_2,\dots,k_r\rangle} = \sum_{\substack{1\leq j\leq r \\ k_j \neq 0}} \delta_{\langle k_1,\dots,k_j-1,\dots,k_r\rangle}.$$

Proof. For $\mathbf{x} = (x_1, \dots, x_n) \in \Gamma^n$ we have

$$\delta^\circ_{\langle k_1,k_2,\dots,k_r\rangle}(\mathbf{x}) = \sum_{j=1}^{r} \delta_{\langle k_1,k_2,\dots,k_r\rangle}(x_1, \dots, x_n, \gamma_j).$$

Here $\delta_{\langle k_1,k_2,\dots,k_r\rangle}(x_1, \dots, x_n, \gamma_j) = 1$ if and only if $k_j > 0$ and $\mathbf{x}$ is a permutation of

$$\big(\underbrace{\gamma_1, \dots, \gamma_1}_{k_1}, \dots, \underbrace{\gamma_j, \dots, \gamma_j}_{k_j-1}, \dots, \underbrace{\gamma_r, \dots, \gamma_r}_{k_r}\big).$$

Thus $\delta_{\langle k_1,k_2,\dots,k_r\rangle}(x_1, \dots, x_r, \gamma_j) = 1$ if and only if $k_j > 0$ and $\delta_{\langle k_1,\dots,k_j-1,\dots,k_r\rangle}(\mathbf{x}) = 1$. This completes the proof. □

Proof of Lemma 2.6. As explained above it suffices to show that $\Psi : L(\Gamma^{n+1})^{S_{n+1}} \to L(\Gamma^n)^{S_n}$ is surjective. For this we will verify that $\delta_{\langle k_1,\dots,k_r\rangle} \in \Psi\big(L(\Gamma^{n+1})^{S_{n+1}}\big)$ for all $k_1, \dots, k_r \geq 0$ with $k_1 + \cdots + k_r = n$. We do this by reverse induction on $k_1 \in \{0, \dots, n\}$.

First suppose that $k_1 = n$ so that $\langle k_1, \dots, k_r\rangle = \langle n, 0, \dots, 0\rangle$. As $\delta^\circ_{\langle n+1,0,\dots,0\rangle} = \delta_{\langle n,0,\dots,0\rangle}$ this shows that $\delta_{\langle k_1,\dots,k_r\rangle} \in \Psi\big(L(\Gamma^{n+1})^{S_{n+1}}\big)$ when $k_1 = n$.

Next suppose that $0 \leq k_1 \leq n-1$ and assume inductively that $\delta_{\langle k'_1,\dots,k'_r\rangle} \in \Psi\big(L(\Gamma^{n+1})^{S_{n+1}}\big)$ for all $k'_1, \dots, k'_r \geq 0$ with $k'_1 + \cdots + k'_r = n$ and $k'_1 = k_1 + 1$. Lemma 2.7 shows that

$$\delta^\circ_{\langle k_1+1,k_2,\dots,k_r\rangle} = \delta_{\langle k_1,k_2,\dots,k_r\rangle} + \sum_{\substack{2\leq j\leq r \\ k_j \neq 0}} \delta_{\langle k_1+1,\dots,k_j-1,\dots,k_r\rangle}.$$

By inductive hypothesis all terms in the sum belong to $\Psi\big(L(\Gamma^{n+1})^{S_{n+1}}\big)$ and hence so does $\delta_{\langle k_1,k_2,\dots,k_r\rangle}$ as desired. □

Proof of Theorem 1.1. Suppose that (G_{n+1}, K_{n+1}) is a Gelfand pair. Equivalently the algebra $\mathcal{A}_{n+1}(\Gamma)$ commutes under convolution. Given $f, g \in \mathcal{A}_n(\Gamma)$ Lemma 2.6 ensure that there exist functions $F, G \in \mathcal{A}_{n+1}(\Gamma)$ with $F^\circ = f$ and $G^\circ = g$. Applying Lemma 2.1 now yields

$$f \star g = F^\circ \star G^\circ = (F \star G)^\circ = (G \star F)^\circ = G^\circ \star F^\circ = g \star f.$$

Thus $\mathcal{A}_n(\Gamma)$ is commutative and hence (G_n, K_n) is a Gelfand pair. □

3. Decomposition of $L(\Gamma^n/\Delta_n)$ and $L(G_n/K_n)$

3.1. Decomposition of $L(\Gamma^n/\Delta_n)$. One checks easily that the map

$$\Gamma^n/\Delta_n \to \Gamma^{n-1}, \quad (x_1,\dots,x_n)\Delta_n \mapsto (x_1x_n^{-1},\dots,x_{n-1}x_n^{-1})$$

is a well-defined bijection. Using this to identify Γ^n/Δ_n with Γ^{n-1} the left quasi-regular representation of Γ^n on $L(\Gamma^n/\Delta_n)$ is realized on $L(\Gamma^{n-1})$ as $\rho_n : \Gamma^n \to GL\big(L(\Gamma^{n-1})\big)$ where

$$\rho_n(\gamma_1,\dots,\gamma_{n-1},\gamma')f(y_1,\dots,y_{n-1}) = f(\gamma_1^{-1}y_1\gamma',\dots,\gamma_{n-1}^{-1}y_{n-1}\gamma'). \tag{3.1}$$

This is the restriction of the left-right regular representation of $\Gamma^{n-1}\times\Gamma^{n-1}$ to $\Gamma^{n-1}\times\Delta_{n-1}$ upon identification of the diagonal subgroup $\Delta_{n-1}\subset\Gamma^{n-1}$ with Γ. The Peter-Weyl Theorem shows that $L(\Gamma^{n-1})$ decomposes under $\Gamma^{n-1}\times\Gamma^{n-1}$ as

$$L(\Gamma^{n-1}) \simeq \sum_{\pi\in\widehat{\Gamma^{n-1}}} \pi\widehat{\otimes}\pi^*.$$

Here $\widehat{\Gamma^{n-1}}$ denotes the set of irreducible representations of Γ^{n-1} modulo equivalence, π^* is the dual (or contragredient) representation for π and $\pi\widehat{\otimes}\pi^*$ is an exterior tensor product representation for the product group $\Gamma^{n-1}\times\Gamma^{n-1}$. The irreducible representations for the product group Γ^{n-1} are themselves exterior tensor products $\pi_1\widehat{\otimes}\cdots\widehat{\otimes}\pi_{n-1}$ of irreducible representations $\pi_j\in\widehat{\Gamma}$. The restriction of $\pi_1\widehat{\otimes}\cdots\widehat{\otimes}\pi_{n-1}$ to $\Delta_{n-1}\cong\Gamma$ is the interior tensor product representation $\pi_1\otimes\cdots\otimes\pi_{n-1}$. For $\pi_n\in\widehat{\Gamma}$ let $m(\pi_1,\dots,\pi_{n-1}|\pi_n)$ denote the multiplicity of π_n in $\pi_1\otimes\cdots\otimes\pi_{n-1}$ so that

$$\pi_1\otimes\cdots\otimes\pi_{n-1} \simeq \sum_{\pi_n\in\widehat{\Gamma}} m(\pi_1,\dots,\pi_{n-1}|\pi_n)\pi_n.$$

Now (3.1) yields the decomposition

$$L(\Gamma^n/\Delta_n) \simeq \sum_{\pi_1,\dots,\pi_{n-1}}\sum_{\pi_n} m(\pi_1,\dots,\pi_{n-1}|\pi_n)\pi_1\widehat{\otimes}\cdots\widehat{\otimes}\pi_{n-1}\widehat{\otimes}\pi_n^* \tag{3.2}$$

for $L(\Gamma^n/\Delta_n)$ as a Γ^n-module. As (Γ^n,Δ_n) is a Gelfand pair if and only if $L(\Gamma^n/\Delta_n)$ is multiplicity free this proves the following.

Proposition 3.1. *(Γ^n,Δ_n) is a Gelfand pair if and only if the interior tensor product representation $\pi_1\otimes\cdots\otimes\pi_{n-1}$ is multiplicity free for all irreducible representations $\pi_1,\dots,\pi_{n-1}\in\widehat{\Gamma}$.*

In particular, taking $n=2$ we recover the well-known fact that $(\Gamma\times\Gamma,\Delta_2)$ is a Gelfand pair.

3.2. Decomposition of $L(G_n/K_n)$. The irreducible representations for the wreath product $G_n=\Gamma^n\rtimes S_n$ are constructed via the Mackey machine as follows. Let π be an irreducible representation of Γ^n. We have say $\pi=\pi_1\widehat{\otimes}\cdots\widehat{\otimes}\pi_n$, the exterior tensor product of irreducible representations $(\pi_j,V_j)\in\widehat{\Gamma}$. The stabilizer of π in S_n, namely

$$S_\pi = \{\sigma\in S_n \;:\; \pi_{\sigma(j)}=\pi_j \text{ for } j=1,\dots,n\},$$

acts on $V_1\otimes\cdots\otimes V_n$ via the intertwining representation

$$\omega : S_\pi \to GL(V_1\otimes\cdots\otimes V_n), \quad \omega(\sigma)(v_1\otimes\cdots\otimes v_n) = v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(n)}.$$

Given any $\rho \in \widehat{S_\pi}$ the induced representation

$$R_{\pi,\rho} = ind_{\Gamma^n \rtimes S_\pi}^{G_n}\big((\pi \circ \omega)\widehat{\otimes}\rho\big)$$

is irreducible and every irreducible representation of G_n is of this form.

Example 3.2. Suppose we have $n = 3$, $\pi = \pi_1 \widehat{\otimes} \pi_1 \widehat{\otimes} \pi_2$. The stabilizer of π in S_3 is $S_\pi \cong S_2 \times S_1$. Letting $\rho_\circ$ denote the trivial representation of S_π the induced representation $R_{\pi,\rho_\circ}$ is

$$R_{\pi,\rho_\circ} = (\pi_1 \widehat{\otimes} \pi_1 \widehat{\otimes} \pi_2) \oplus (\pi_1 \widehat{\otimes} \pi_2 \widehat{\otimes} \pi_1) \oplus (\pi_2 \widehat{\otimes} \pi_1 \widehat{\otimes} \pi_1)$$

as a representation of Γ^3 with S_3 permuting the factors.

Recall that $(G_n = \Gamma^n \rtimes S_n, K_n = \Delta_n \times S_n)$ is a Gelfand pair if and only if the space R^{K_n} of K_n-fixed vectors in R has $\dim(R^{K_n}) \leq 1$ for every irreducible representation $R \in \widehat{G_n}$. Abusing terminology we call $\dim(R^{K_n})$ the "number of K_n-fixed vectors in R." Letting $K_\pi = \Delta_n \times S_\pi$ for given $\pi \in \widehat{\Gamma^n}$ one has

$$R_{\pi,\rho}|_{K_n} = ind_{K_\pi}^{K_n}\big((\pi \circ \omega)\widehat{\otimes}\rho\big).$$

An application of Frobenius reciprocity now yields the following.

Lemma 3.3. *The number of K_n-fixed vectors in $R_{\pi,\rho}$ is equal to the number of K_π-fixed vectors in $(\pi \circ \omega)\widehat{\otimes}\rho$.*

Thus in order for $R_{\pi,\rho}$ to be K_n-spherical there must be K_π-fixed vectors in $(\pi \circ \omega)\widehat{\otimes}\rho$ and hence Δ_n-fixed vectors in $\pi = \pi_1 \widehat{\otimes} \cdots \widehat{\otimes} \pi_n$. Now (3.2) yields the following necessary condition for $R_{\pi,\rho}$ to be spherical.

Lemma 3.4. *If $R_{\pi,\rho}$ is K_n-spherical then π_n^* occurs in the (internal) tensor product $\pi_1 \otimes \ldots \otimes \pi_{n-1}$.*

We will make use of this criterion in connection with examples in Section 5.

4. Proof of Theorem 1.2

Among the irreducible representations for $G_n = \Gamma^n \rtimes S_n$ discussed above are the following. Given an irreducible representation (π, V) of Γ one obtains an irreducible representation $\widetilde{\pi}$ of G_n in the n'th tensor power $W = \otimes^n V$ via

$$\widetilde{\pi}(\mathbf{x}, \sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \pi(x_1)v_{\sigma^{-1}(1)} \otimes \pi(x_2)v_{\sigma^{-1}(2)} \cdots \otimes \pi(x_n)v_{\sigma^{-1}(n)}$$

on decomposable tensors.[1] Observe that the space of $\widetilde{\pi}(S_n)$-invariant vectors in W is $W^{S_n} = S^n(V)$, the n'th symmetric power of V. The action of the diagonal subgroup Δ_n on W via $\widetilde{\pi}$ preserves W^{S_n} as Δ_n and S_n commute in G_n. Moreover the representation $(\widetilde{\pi}|_{\Delta_n}, W^{S_n} = S^n(V))$ of Δ_n coincides with $(S^n(\pi), S^n(V))$, the n'th symmetric power of the representation (π, V), under the obvious isomorphism $\Delta_n \cong \Gamma$. It follows that the space W^{K_n} of $(K_n = \Delta_n \times S_n)$-invariant vectors in W is precisely

$$W^{K_n} = S^n(V)^\Gamma,$$

the space of Γ-invariant vectors in the n'th symmetric power of (π, V).

[1] Here $\widetilde{\pi} = R_{\pi \widehat{\otimes} \cdots \widehat{\otimes} \pi, 1_{S_n}}$ in the notation from the previous section.

Proof of Theorem 1.2. Let Γ be a finite non-abelian group of order $r = |\Gamma|$ and (π, V) an irreducible representation of Γ with $d = \dim(V) > 1$. Such a representation exists as Γ is non-abelian. A fundamental result in Invariant Theory asserts that the algebra $S(V)^\Gamma$ of Γ-invariants in $S(V)$ contains d algebraically independent homogeneous elements. In fact Proposition 3.4 in [Sta79] shows that one can construct d such elements with degrees all dividing r. Suitable powers of these yield d linearly independent vectors in $S^r(V)^\Gamma$. Letting $(\widetilde{\pi}, W)$ be the irreducible representation of $G_r = \Gamma^r \rtimes S_r$ described above we now have that the space W^{K_r} of $(K_r = \Delta_r \times S_r)$-fixed vectors in W has dimension at least d. Thus (G_r, K_r) fails to be a Gelfand pair as was to be shown. □

Remark 4.1. The result that $S(V)^\Gamma$ contains d algebraically independent homogeneous elements does not require that (π, V) be irreducible. Most of the literature on Invariant Theory, including [Sta79], concerns invariants in polynomial rings $\mathbb{C}[V]$ rather than symmetric algebras $S(V)$. To pass to this context one can replace (π, V) above by its dual representation (π^*, V^*) and note that $S(V^*)$ is canonically isomorphic to $\mathbb{C}[V]$ as a Γ-module.

5. Wreath products with dihedral groups

In this section we take $\Gamma = D_p$, where p is any odd prime. We show that $(\Gamma^n \rtimes S_n, \Delta_n \times S_n)$ is a Gelfand pair for $n \leq 5$ and not a Gelfand pair for $n \geq 6$. The strategy is to first identify representations of Γ^n which occur in $L(\Gamma^n/\Delta_n)$ by finding the tensor product representations which have Δ_n-fixed vectors.

5.1. **The case $n = 3$.** We begin by reviewing the representation theory of D_p. The conjugacy classes are the identity element $\{I\}$, pairs of rotations $\{R, R^{-1}\}$, and the set S of all reflections. There are two one-dimensional irreducible representations, the trivial representation θ_1 and the determinant θ_2. In addition, there are $m = (p-1)/2$ two-dimensional representations π_j with characters χ_j. In each of these representations, any rotation has eigenvalues λ and λ^{-1}, where λ is a pth root of unity.

The character table is:

	(1)	(2)	(2)	...	(m)
	I	R_1	R_2	...	S
θ_1	1	1	1	...	1
θ_2	1	1	1	...	-1
χ_j	2	μ_j	μ_{2j}	...	0

The second row of this table lists the conjugacy classes, while the row above indicates the number of elements in each conjugacy class. In the last row, π_j is a two-dimension representation, with $\mu_j = \lambda^j + \lambda^{-j}$ for $j = 1, \ldots, m = (p-1)/2$.

Since all of the characters are real-valued, by Lemma 3.4 we consider representations $\pi_i \otimes \pi_j \otimes \pi_k$, where π_k occurs in $\pi_i \otimes \pi_j$. Equivalently, we want $\pi_i \widehat{\otimes} \pi_j \widehat{\otimes} \pi_k$ to contain a Δ_3-fixed vector. Let us consider some examples.

The representations $\pi_1 \otimes \pi_1$ and $\pi_1 \otimes \pi_2$ have characters

	I	R_1	R_2	...	S
χ_1^2	4	μ_1^2	μ_2^2	...	0
$\chi_1\chi_2$	4	$\mu_1\mu_2$	$\mu_2\mu_4$	...	0

A calculation shows that

$$\mu_i\mu_j = \mu_{i+j} + \mu_{i-j},$$

where we understand that the subscripts are taken mod p, that $\mu_{-k} = \mu_k$, and that $\mu_0 = 2$. Thus in our examples, we get the characters

	I	R_1	R_2	$\dots$	S
χ_1^2	4	$\mu_2 + 2$	$\mu_4 + 2$	$\dots$	0
$\chi_1\chi_2$	4	$\mu_1 + \mu_3$	$\mu_2 + \mu_6$	$\dots$	0

and hence $\pi_1 \otimes \pi_1 \simeq \pi_2 \oplus \theta_1 \oplus \theta_2$ and $\pi_1 \otimes \pi_2 \simeq \pi_1 \oplus \pi_3$. This tells us that the triples $\pi_1\widehat{\otimes}\pi_1\widehat{\otimes}\pi_2$, $\pi_1\widehat{\otimes}\pi_1\widehat{\otimes}\theta_1$, $\pi_1\widehat{\otimes}\pi_1\widehat{\otimes}\theta_2$ and $\pi_1\widehat{\otimes}\pi_2\widehat{\otimes}\pi_3$ are all Δ_3-spherical representations of Γ^3. In each case, there is a single Δ_3-fixed vector, and hence each representation occurs in $L(\Gamma^3/\Delta_3)$ with multiplicity one.

The spherical representations of Γ^3 are:

- $\pi_i\widehat{\otimes}\pi_j\widehat{\otimes}\pi_{i+j}$ for $i \neq j$,
- $\pi_i\widehat{\otimes}\pi_i\widehat{\otimes}\pi_{2i}$
- $\pi_i\widehat{\otimes}\pi_i\widehat{\otimes}\theta_1$
- $\pi_i\widehat{\otimes}\pi_i\widehat{\otimes}\theta_2$
- $\theta_1\widehat{\otimes}\theta_1\widehat{\otimes}\theta_1$
- $\theta_1\widehat{\otimes}\theta_2\widehat{\otimes}\theta_2$

and all permutations of the tensor product factors. Each of these representations has a single Δ_3-fixed vector, hence it occurs in $L(\Gamma^3/\Delta_3)$ with multiplicity one. Note that for small values of p, not all cases may occur. Now we have:

Proposition 5.1. *For $\Gamma = D_p$, (Γ^3, Δ_3) and $(\Gamma^3 \rtimes S_3, \Delta_3 \times S_3)$ are Gelfand pairs.*

5.2. **The case $n = 4$.** We use identities of the following type to find the spherical representations:

$$\chi_i\chi_j\chi_k = \sum \chi_{i\pm j\pm k};$$

$$\chi_j^3 = \chi_j(\chi_{2j} + \theta_1 + \theta_2) = \chi_j + \chi_{3j} + 2\chi_j = 3\chi_j + \chi_{3j};$$

$$\chi_j^2\chi_k = (\chi_{2j} + \theta_1 + \theta_2)\chi_k = \chi_{2j-k} + \chi_{2j+k} + 2\chi_k.$$

Representations of Γ^4 which occur in $L(\Gamma^4/\Delta_4)$:

- $\pi_i\widehat{\otimes}\pi_j\widehat{\otimes}\pi_k\widehat{\otimes}\pi_{i\pm j\pm k}$ for i, j, k distinct.
- $\pi_j^{(4)}$ with multiplicity three. (Here $\pi^{(k)}$ denotes $\pi\widehat{\otimes}\cdots\widehat{\otimes}\pi$.)
- $\pi_j^{(3)}\widehat{\otimes}\pi_{3j}$,
- $\pi_j^{(2)}\widehat{\otimes}\pi_k\widehat{\otimes}\pi_{2j\pm k}$
- $\pi_j^{(2)}\widehat{\otimes}\pi_k^{(2)}$ with multiplicity two.
- $\theta_i\widehat{\otimes}\pi_j^{(2)}\widehat{\otimes}\pi_{2j}$.
- $\theta_i\widehat{\otimes}\theta_j\widehat{\otimes}\pi_k^{(2)}$ for all choices of i, j, k.
- $\theta_i\widehat{\otimes}\pi_j\widehat{\otimes}\pi_k\widehat{\otimes}\pi_{j\pm k}$ for $j \neq k$.
- $\theta_i^{(4)}$.
- $\theta_1^{(2)}\widehat{\otimes}\theta_2^{(2)}$.

Since there are two cases of multiplicity, (Γ^4, Δ_4) is not a Gelfand pair for $\Gamma = D_p$.

5.3. **Characters for $\Gamma^4 \rtimes S_4$.** Let $\pi = \pi_i \widehat{\otimes} \pi_j \widehat{\otimes} \pi_k \widehat{\otimes} \pi_l$ be a representation of Γ^4, S_π the stabilizer of π in S_4, and ω the intertwining representation of S_π. Let χ_π be the character for $\pi \circ \omega$. The technique for computing χ_π is illustrated by the following example:

Suppose that $\pi = \pi_1^{(4)}$, so that $S_\pi = S_4$. Then

$$\begin{aligned}
\chi_\pi(\delta, (1234)) &= \chi_1(\delta^4), \\
\chi_\pi(\delta, (123)) &= \chi_\pi(\delta^3)\chi_\pi(\delta), \\
\chi_\pi(\delta, (12)) &= \chi_\pi(\delta^2)\chi_\pi(\delta)^2, \\
\chi_\pi(\delta, (12)(34)) &= \chi_\pi(\delta^2)\chi_\pi(\delta^2).
\end{aligned}$$

where we identify $\delta \in \Gamma$ with an element of Δ_4. The two cases of multiplicity are $\pi_j^{(4)}$ and $\pi_j^{(2)} \widehat{\otimes} \pi_k^{(2)}$, with stabilizers S_4 and $S_2 \times S_2$ respectively. Since π_1 is determined by an arbitrary p th root of unity, we can take $j = 1$ in both cases.

We need to find the spherical representations of the form $R_{\pi,\rho}$ for $\rho \in \widehat{S_\pi}$. That is, in view of Lemma 3.3, we seek representations which contain $(K_\pi = \Delta_4 \times S_\pi)$-fixed vectors. The number of such vectors is:

$$\frac{1}{|\Delta_4 \times S_\pi|} \sum_{\delta,\sigma} \chi_\pi(\delta, \sigma)\chi_\rho(\sigma) = \frac{1}{|S_\pi|} \sum_{\sigma} \left(\frac{1}{|\Delta_4|} \sum_{\delta} \chi_\pi(\delta, \sigma) \right) \chi_\rho(\sigma) \tag{5.1}$$

For each fixed $\sigma \in S_\pi$, define the function

$$m_{\pi,\sigma}(\delta) = \chi_\pi(\delta, \sigma).$$

This is a class function on $\Delta = \Delta_4$, which can be expressed as a linear combination of irreducible characters. So the sum

$$M_\pi(\sigma) = \frac{1}{|\Delta|} \sum_{\delta} \chi_\pi(\delta, \sigma) = \langle m_{\pi,\sigma}, 1 \rangle_\Delta$$

is the coefficient of the trivial character θ_1 in $m_{\pi,\sigma}$. Moreover, the function M_π is a class function on S_π, so the sum (5.1) is the coefficient of $\chi_\rho = \overline{\chi_\rho}$ in M_π.

For $\pi = \pi_1^{(4)}$, a straightforward calculation shows that:

$$\begin{aligned}
\chi_\pi(\delta, e) &= \chi_1(\delta)^4 = (\chi_4 + 4\chi_2 + 3\theta_1 + 3\theta_2)(\delta), \\
\chi_\pi(\delta, (12)) &= \chi_1(\delta^2)\chi_1(\delta)^2 = (\chi_4 + 2\chi_2 + \theta_1 + \theta_2)(\delta) \\
\chi_\pi(\delta, (123)) &= \chi_1(\delta^3)\chi_1(\delta) = (\chi_4 + \chi_2)(\delta) \\
\chi_\pi(\delta, (1234)) &= \chi_1(\delta^4) = (\chi_2 + \theta_1 - \theta_2)(\delta) \\
\chi_\pi(\delta, (12)(34)) &= \chi_1(\delta^2)\chi_1(\delta^2) = (\chi_4 + 3\theta_1 - 2\theta_2)(\delta)
\end{aligned}$$

We take the coefficient of θ_1 to obtain:

$$M_\pi(e) = 3,\ M_\pi((12)) = 1,\ M_\pi((123)) = 0,\ M_\pi((1234)) = 1,\ M_\pi((12)(34)) = 3.$$

Consulting the character table of S_4, we find that M_π is the sum of the trivial character and a two-dimensional character, with corresponding representations $\rho_\circ$ and ρ_2.

This tells us that $(\pi_1^{(4)} \circ \omega) \widehat{\otimes} \rho_\circ$ and $(\pi_1^{(4)} \circ \omega) \widehat{\otimes} \rho_2$ are K_π-spherical representations, each occurring with multiplicity one. Together, these spaces account for the occurrence of $\pi_1^{(4)}$ with multiplicity three.

For $\pi = \pi_1^{(2)} \widehat{\otimes} \pi_k^{(2)}$ and $S_\pi \cong S_2 \times S_2$, we obtain

$$M_\pi(e) = 2,\ M_\pi((12)) = 0,\ M_\pi((34)) = 0,\ M_\pi((12)(34)) = 2.$$

Thus M_π is the sum of the trivial character $\rho_\circ$ and the sign character ρ_s. This tells us that $(\pi_1^{(2)}\widehat{\otimes}\pi_k^{(2)}\circ\omega)\widehat{\otimes}\rho_\circ$ and $(\pi_1^{(2)}\widehat{\otimes}\pi_k^{(2)}\circ\omega)\widehat{\otimes}\rho_s$ are spherical representations, each occurring with multiplicity one. Together, these spaces account for the occurrence of $\pi_1^{(2)}\widehat{\otimes}\pi_k^{(2)}$ with multiplicity two. Thus we see that $(\Gamma^4 \rtimes S_4, \Delta \times S_4)$ is a Gelfand pair.

5.4. **The case** $n = 5$**.** Using similar techniques to those above, we can find the spherical representations of Γ^5. We list just those with multiplicity:

- $\pi_j^{(4)}\widehat{\otimes}\,\theta_i$ with multiplicity 3.
- $\pi_j^{(2)}\widehat{\otimes}\pi_k^{(2)}\widehat{\otimes}\,\theta_i$ with multiplicity 2.
- $\pi_j^{(4)}\widehat{\otimes}\pi_{2j}$ with multiplicity 4.
- $\pi_j^{(3)}\widehat{\otimes}\pi_k\widehat{\otimes}\pi_{j\pm k}$ with multiplicity 3.
- $\pi_j^{(2)}\widehat{\otimes}\pi_k^{(2)}\widehat{\otimes}\pi_{2k}$ with multiplicity 2.
- $\pi_j^{(2)}\widehat{\otimes}\pi_k\widehat{\otimes}\pi_l\widehat{\otimes}\pi_{k\pm l}$ with multiplicity 2.

Since $\chi_j^4\,\theta_i = \chi_j^4$ and $\chi_j^2\chi_k^2\,\theta_i = \chi_j^2\chi_k^2$, the first two cases are handled in the previous section. For the other cases, we take $j = 1$.

For $\pi = \pi_1^{(4)}\widehat{\otimes}\pi_2$ and $S_\pi \cong S_4 \times S_1$, we obtain

$$M_\pi(e) = 4,\ M_\pi((12)) = 2,\ M_\pi((123)) = 1,\ M_\pi((12)(34)) = 0,\ M_\pi((1234)) = 0.$$

Thus M_π is the character of the standard 4-dimensional representation of S_4, which is the sum of the trivial representation $\rho_\circ$ and a 3-dimensional irreducible representation ρ_3. This tells us that $(\pi_1^{(4)}\widehat{\otimes}\pi_2 \circ \omega)\widehat{\otimes}\rho_\circ$ and $(\pi_1^{(4)}\widehat{\otimes}\pi_2 \circ \omega)\widehat{\otimes}\rho_3$ are spherical representations, each occurring with multiplicity one. Together, these spaces account for the occurrence of $\pi_1^{(4)}\widehat{\otimes}\pi_2$ with multiplicity four.

For $\pi = \pi_1^{(3)}\widehat{\otimes}\pi_k\widehat{\otimes}\pi_{k+1}$ and $S_\pi \cong S_3$, we obtain

$$M_\pi(e) = 3,\ M_\pi((12)) = 1,\ M_\pi((123)) = 0.$$

Thus M_π is the character of the standard 3-dimensional representation of S_3, which is the sum of the trivial representation $\rho_\circ$ and a 2-dimensional irreducible representation ρ_2. This tells us that $(\pi_1^{(3)}\widehat{\otimes}\pi_k\widehat{\otimes}\pi_{k+1} \circ \omega)\widehat{\otimes}\rho_\circ$ and $(\pi_1^{(4)}\widehat{\otimes}\pi_k\widehat{\otimes}\pi_{k+1} \circ \omega)\widehat{\otimes}\rho_2$ are spherical representations, each occurring with multiplicity one. Together, these spaces account for the occurrence of $\pi_1^{(4)}\widehat{\otimes}\pi_k\widehat{\otimes}\pi_{k+1}$ with multiplicity three.

For $\pi = \pi_1^{(2)}\widehat{\otimes}\pi_k^{(2)}\widehat{\otimes}\pi_2$ and $S_\pi \cong S_2 \times S_2$, we obtain

$$M_\pi(e) = 2,\ M_\pi((12)) = 2,\ M_\pi((34)) = 0, M_\pi((12)(34)) = 0.$$

Thus M_π is the sum of two 1-dimensional irreducible representations of $S_2 \times S_2$, one of which is the trivial representation.

For $\pi = \pi_1^{(2)}\widehat{\otimes}\pi_k\widehat{\otimes}\pi_l\widehat{\otimes}\pi_{k\pm l}$ and $S_\pi \cong S_2$, we obtain

$$M_\pi(e) = 2,\ M_\pi((12)) = 0.$$

Thus M_π is the sum of the two 1-dimensional irreducible representations of S_2. We conclude that $(\Gamma^5 \rtimes S_5, \Delta_5 \times S_5)$ is a Gelfand pair.

5.5. **The Case** $n = 6$**.** In this case, $(D_p^6 \rtimes S_6, \Delta_6 \times S_6)$ fails to be a Gelfand pair. This is consistent with the GAP-generated result for $p = 3$ in [AC12].

Let $\pi = \pi_1^{(4)} \widehat{\otimes} \pi_2^{(2)}$, with $S_\pi \cong S_4 \times S_2$. Then we obtain

$$M_\pi(e) = 7,\ M_\pi((12)) = 3,\ M_\pi((123)) = 1,\ M_\pi((1234)) = 1,\ M_\pi((12)(34)) = 3,$$
$$M_\pi((56)) = 1,\ M_\pi((12)(56)) = 1,\ M_\pi((123)(56)) = 1,\ M_\pi((1234)(56)) = 3,$$
$$M_\pi((12)(34)(56)) = 5.$$

One can see that $\langle M_\pi, \rho_\circ \rangle = 2$, and hence that $R_{\pi,\rho_\circ} = Ind_{\Gamma^6 \rtimes S_\pi}^{G_6}\big((\pi \circ \omega) \widehat{\otimes} \rho_\circ\big)$ has multiplicity 2 in $L(G_6/K_6)$. For $p = 3$, we have $\pi_1 = \pi_2$, and the result holds for $\pi = \pi_1^{(6)}$.

References

[AC12] Kürşat Aker and Mahir Bilen Can. From parking functions to Gelfand pairs. *Proc. Amer. Math. Soc.*, 140(4):1113–1124, 2012.

[AM03] Hiroki Akazawa and Hiroshi Mizukawa. Orthogonal polynomials arising from the wreath products of a dihedral group with a symmetric group. *J. Combin. Theory Ser. A*, 104(2):371–380, 2003.

[CSST06] Tullio Ceccherini-Silberstein, Fabio Scarabotti, and Filippo Tolli. Trees, wreath products and finite Gelfand pairs. *Adv. Math.*, 206(2):503–537, 2006.

[CSST08] Tullio Ceccherini-Silberstein, Fabio Scarabotti, and Filippo Tolli. *Harmonic analysis on finite groups*, volume 108 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008. Representation theory, Gelfand pairs and Markov chains.

[CSST14] Tullio Ceccherini-Silberstein, Fabio Scarabotti, and Filippo Tolli. *Representation theory and harmonic analysis of wreath products of finite groups*, volume 410 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2014.

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995.

[Sta79] Richard P. Stanley. Invariants of finite groups and their applications to combinatorics. *Bull. Amer. Math. Soc. (N.S.)*, 1(3):475–511, 1979.

[Ter99] Audrey Terras. *Fourier analysis on finite groups and applications*, volume 43 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1999.

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