SPACES OF BOUNDED SPHERICAL FUNCTIONS ON HEISENBERG GROUPS: PART II

CHAL BENSON AND GAIL RATCLIFF

Abstract. Consider a linear multiplicity free action by a compact Lie group K on a finite dimensional Hermitian vector space V. Letting K act on the Heisenberg group $H_V = V \times \mathbb{R}$ yields a Gelfand pair. The condition that $K : V$ be "well-behaved" establishes a relationship between the associated moment mapping and highest weight vectors occurring in the polynomial ring $\mathbb{C}[V]$. Under this condition an application of the Orbit Method produces a topological embedding of the space of bounded spherical functions for (K, H_V) in the space of K-orbits in the dual of the Lie algebra for H_V. In Part I of this work it was shown that every irreducible multiplicity free action is well-behaved. Here we extend this result to encompass all multiplicity free actions. Our proof uses case-by-case analysis of multiplicity free actions which are indecomposable but not irreducible.

1. Introduction

Let $V \cong \mathbb{C}^n$ be a finite dimensional complex vector space with Hermitian inner product $\langle \cdot, \cdot \rangle$ and K be a compact Lie group acting on $(V, \langle \cdot, \cdot \rangle)$ by some unitary representation. The group K acts by automorphisms on the associated Heisenberg group

$$H_V = V \times \mathbb{R} \quad \text{with product} \quad (z,t)(z',t') = \left(z + z', \ t + t' - \frac{1}{2} Im \langle z, z' \rangle \right)$$

via

$$k \cdot (z,t) = (k \cdot z, t).$$

(K, H_V) is said to be a *Gelfand pair* if the convolution algebra $L^1_K(H_V)$ of integrable K-invariant functions on H_V is commutative. As is well known, this is the case if and only if $K : V$ is a linear *multiplicity free action* [8]. That is, if and only if the representation of K in the space $\mathbb{C}[V]$ of holomorphic polynomial functions on V,

$$(k \cdot p)(z) = p(k^{-1} \cdot z),$$

is multiplicity free.

In this context the spectrum, or Gelfand space, for $L^1_K(H_V)$ can be identified, via integration, with the set $\Delta(K, H_V)$ of bounded K-spherical functions on H_V endowed with the compact-open topology. An application of the *Orbit Method*, given in [5],

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produces an injective mapping $\Psi : \Delta(K, H_V) \to \mathfrak{h}_V^*/K$ from $\Delta(K, H_V)$ to the set of K-orbits in the dual of the Lie algebra for H_V. Giving $\mathfrak{h}_V^*/K$ the quotient topology our main result here is that

Theorem 1.1. *$\Psi : \Delta(K, H_V) \to \mathfrak{h}_V^*/K$ is a homeomorphism onto its image.*

Theorem 1.1 was conjectured, in more general form, in [5] and is the focus of [6] and [2]. This paper is a continuation of these works. We proved in [6] that Ψ is indeed a homeomorphism whenever $K : V$ is a *well-behaved* multiplicity free action. (See Definition 2.3 below.) Thus Theorem 1.1 is a direct consequence of the following.

Theorem 1.2. *Every linear multiplicity free action is well-behaved.*

If $K_1 : V_1$ and $K_2 : V_2$ are multiplicity free actions then so is the product action $(K_1 \times K_2) : (V_1 \oplus V_2)$. Lemma 3.1 below shows, moreover, that if both $K_1 : V_1$ and $K_2 : V_2$ are well-behaved then so is $(K_1 \times K_2) : (V_1 \oplus V_2)$. So to prove Theorem 1.2 it suffices to verify that every *indecomposable* multiplicity free action is well-behaved. These are the multiplicity free actions which do not decompose as product actions.

The papers [13, 3, 15] classify indecomposable multiplicity free actions up to geometric equivalence. Kac's paper [13] gives all multiplicity free actions $K : V$ in which K acts irreducibly on V. In [2] we applied this classification to show that each irreducible multiplicity free action is well-behaved. Here we complete the proof of Theorem 1.2 by analyzing the reducible but indecomposable actions given in [3, 15]. As explained in [2, Section 3.7] a byproduct of our calculations is that the orbital model for $\Delta(K, H_V)$, provided by Theorem 1.1, becomes relatively explicit in each case.

2. Preliminaries and background results

Let $K : V$ denote a fixed multiplicity free action. We summarize below some results from [2], retaining the notational conventions established there. In particular

- $\mathfrak{k}$ is the Lie algebra for K, $T \subset K$ a maximal torus, $\mathfrak{t} \subset \mathfrak{k}$ its Lie algebra and $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}}$. Choosing a system of positive roots we decompose $\mathfrak{k}_{\mathbb{C}} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ and let $B := HN = HN_+$ be the resulting Borel subgroup in $K_{\mathbb{C}}$.
- $\Lambda \subset \mathfrak{h}^*$ is the set of B-highest weights for irreducible representations of $K_{\mathbb{C}}$ occurring in $\mathbb{C}[V]$. For each $\alpha \in \Lambda$ choose a B-highest weight vector $h_\alpha \in \mathbb{C}[V]$ with weight α (unique modulo $\mathbb{C}^\times$) and
- let $\alpha_{\mathfrak{k}} \in \mathfrak{k}^*$ be the (real valued) linear functional on $\mathfrak{k}$ with $\alpha_{\mathfrak{k}}|_{\mathfrak{t}} = -i\alpha$ and $\alpha_{\mathfrak{k}} \equiv 0$ on the orthogonal complement to $\mathfrak{t}$ in $\mathfrak{k}$ with respect to an $Ad(K)$-invariant inner product.
- $\tau : V \to \mathfrak{k}^*$ is the (unnormalized) *moment map* for $K : V$, namely $\tau(v)(A) := i\langle A \cdot v, v\rangle$.

2.1. **Spherical points and well-behaved multiplicity free actions.**

Definition 2.1. [6] A point $v_\alpha \in V$ is said to be a *spherical point* for the highest weight $\alpha \in \Lambda$ if $\tau(v_\alpha) = \alpha_{\mathfrak{k}}$.

Results from [1, 9] ensure that a spherical point v_α exists for each $\alpha \in \Lambda$ and that the K-orbit $\mathcal{K}_\alpha = K \cdot v_\alpha$ is uniquely determined. The following result facilitates the calculation of spherical points in examples.

Lemma 2.2 ([2, Lemma 3.1])**.** *$v_\alpha \in V$ is a spherical point for $\alpha \in \Lambda$ if and only if*

$$\left\{ \begin{array}{ll} \langle X \cdot v_\alpha, v_\alpha \rangle = -\alpha(X) & \text{for all } X \in \mathfrak{h} \text{ and} \\ \langle X \cdot v_\alpha, v_\alpha \rangle = 0 & \text{for all } X \in \mathfrak{n}_+ \oplus \mathfrak{n}_- \end{array} \right\} \tag{2.1}$$

As (2.1) is linear in X this amounts to a system of $\dim(\mathfrak{k})$ quadratic equations whose solutions give all spherical points for α.

Definition 2.3. [6] Given $\alpha \in \Lambda$ we say that a spherical point v_α for α is *well-adapted to h_α* when the following conditions hold.

(i) $h_\alpha(v_\alpha) \neq 0$, and
(ii) $\big(\partial_w h_\alpha\big)(v_\alpha) = \langle w, v_\alpha \rangle\, h_\alpha(v_\alpha)$ for all $w \in V$.

The multiplicity free action $K : V$ is said to be *well-behaved* if for every $\alpha \in \Lambda$ one can choose a spherical point v_α well-adapted to h_α.

Lemma 2.4 ([2, Lemma 3.7])**.** *Let $K_1 : V$ be a multiplicity free action obtained by restricting a multiplicity free action $K_2 : V$ to a closed Lie subgroup $K_1 \subset K_2$. Assume, moreover, that $\mathbb{C}[V]$ shares a common decomposition under the associated representations of K_1 and K_2. Then $K_1 : V$ is well-behaved if and only if $K_2 : V$ is well-behaved.*

2.2. **A limiting procedure.** Recall that $\alpha \in \Lambda$ is a *fundamental* highest weight for $K : V$ when h_α is an irreducible polynomial. The fundamental highest weights are finite in number and freely generate Λ as an additive semigroup [12]. The number of fundamental highest weights for $K : V$ is its *rank*. A technical Lemma from [2] is used to study examples.

Lemma 2.5 ([2, Lemma 3.5])**.** *Let $K : V$ be a rank r multiplicity free action with fundamental highest weights $\{\alpha_1, \dots, \alpha_r\}$ and associated fundamental highest weight vectors $h_j = h_{\alpha_j}$. Suppose that for all positive real numbers $x_1, \dots, x_r > 0$ there is a point $v(\mathbf{x}) = v(x_1, \dots, x_r)$ in V which satisfies the following four conditions:*

(1) $\left\{ \begin{array}{ll} \langle X \cdot v(\mathbf{x}), v(\mathbf{x}) \rangle = -(x_1\alpha_1 + \cdots + x_r\alpha_r)(X) & \text{for all } X \in \mathfrak{h} \text{ and} \\ \langle X \cdot v(\mathbf{x}), v(\mathbf{x}) \rangle = 0 & \text{for all } X \in \mathfrak{n}_+ \oplus \mathfrak{n}_- \end{array} \right\}.$

(2) *$h_i(v(\mathbf{x})) \neq 0$ for $1 \le i \le r$.*

(3) *For each $1 \le k < r$ and indices $1 \le j_1 < j_2 < \cdots < j_k \le r$ the limit*

$$\lim_{x_{j_k} \to 0^+} \cdots \lim_{x_{j_1} \to 0^+} v(x_1, \dots, x_r)$$

exists in V, and

(4) $\lim_{x_{j_k}\to 0^+}\cdots\lim_{x_{j_1}\to 0^+} h_i\big(v(x_1,\ldots,x_r)\big)\neq 0$ *for each* $i\in\{1,\ldots,r\}\backslash\{j_1,\ldots,j_k\}$.

Then $K:V$ *is a well-behaved multiplicity free action.*

Definition 2.6. We call a point $v(\mathbf{x})=v(x_1,\ldots,x_r)\in V$ satisfying condition (1) in Lemma 2.5 a *generalized spherical point*. Moreover $v(\mathbf{x})$ is said to be a *generic generalized spherical point* when each parameter x_j is non-zero.

As $\Lambda=\big\{x_1\alpha_1+\cdots+x_r\alpha_r \ : \ x_j\in\mathbb{Z}, x_j\geq 0\big\}$, Lemma 2.2 shows $v(\mathbf{x})$ to be a spherical point for weight $(\alpha=x_1\alpha_1+\cdots+x_r\alpha_r)\in\Lambda$ whenever each x_j in a non-negative integer. If each x_j is a positive integer then we call the weight α *generic*.

3. Product actions

A routine Lemma enables us to reduce the proof of Theorem 1.2 to the study of indecomposable multiplicity free actions.

Lemma 3.1. *Products of well-behaved multiplicity free actions are well-behaved.*

Proof. Consider a product action $(K_1\times K_2):(V_1\oplus V_2)$ where $K_1:V_1$ and $K_2:V_2$ are well-behaved multiplicity free actions in Hermitian vector spaces $(V_j,\langle\cdot,\cdot\rangle_j)$. Equipping $V_1\oplus V_2$ with the direct sum Hermitian inner product it follows that the moment mapping $\tau:V_1\oplus V_2\to(\mathfrak{k}_1\times\mathfrak{k}_2)^*=\mathfrak{k}_1^*\times\mathfrak{k}_2^*$ is just $\tau(v_1,v_2)=\big(\tau_1(v_1),\tau_2(v_2)\big)$ where $\tau_j:V_j\to\mathfrak{k}_j$ is the moment mapping for $K_j:V_j$.

Let $\Lambda_j\subset\mathfrak{h}_j^*$ be the set of B_j-highest weights for representations of K_j occurring in $\mathbb{C}[V_j]$. The set of $(B_1\times B_2)$-highest weights for representations of $K_1\times K_2$ occurring in $\mathbb{C}[V_1\oplus V_2]\cong\mathbb{C}[V_1]\otimes\mathbb{C}[V_2]$ is $\Lambda=\Lambda_1\times\Lambda_2\subset\big((\mathfrak{h}_1\times\mathfrak{h}_2)^*=\mathfrak{h}_1^*\times\mathfrak{h}_2^*\big)$. If $h_{\alpha_j}\in\mathbb{C}[V_j]$ are highest weight vectors with weights $\alpha_j\in\Lambda_j$ then $h_{\alpha_1}\otimes h_{\alpha_2}$ is a highest weight vector in $\mathbb{C}[V_1\oplus V_2]$ with weight $(\alpha_1,\alpha_2)\in\Lambda$. Let $v_{\alpha_j}\in\mathbb{C}[V_j]$ be a spherical point for α_j well-adapted to h_{α_j}. We claim that the spherical point $(v_{\alpha_1},v_{\alpha_2})\in V_1\oplus V_2$ for $(\alpha_1,\alpha_2)\in\Lambda$ is well-adapted to $h_{\alpha_1}\otimes h_{\alpha_2}$. Indeed $\big(h_{\alpha_1}\otimes h_{\alpha_2}\big)(v_{\alpha_1},v_{\alpha_2})=h_{\alpha_1}(v_{\alpha_1})h_{\alpha_2}(v_{\alpha_2})\neq 0$ and by linearity it suffices to check Definition 2.3 condition (ii) for derivatives in directions lying in $V_1\cup V_2\subset V_1\oplus V_2$. If say $w\in V_1$ then one has

$$\begin{aligned}\big(\partial_{(w,0)}(h_{\alpha_1}\otimes h_{\alpha_2})\big)(v_{\alpha_1},v_{\alpha_2})&=\big(\partial_w h_{\alpha_1}\big)(v_{\alpha_1})h_{\alpha_2}(v_{\alpha_2})=\langle w,v_{\alpha_1}\rangle_1\, h_{\alpha_1}(v_{\alpha_1})h_{\alpha_2}(v_{\alpha_2})\\&=\langle(w,0),(v_{\alpha_1},v_{\alpha_2})\rangle\,\big(h_{\alpha_1}\otimes h_{\alpha_2}\big)(v_{\alpha_1},v_{\alpha_2}).\end{aligned}$$

□

4. Reducible but indecomposable multiplicity free actions

Each multiplicity free action splits as a product of indecomposable multiplicity free actions. The indecomposable but non-irreducible multiplicity free actions are classified in [3] and [15]. See also [14]. Scalar actions somewhat complicate the classification. Lemma 2.4 ensures, however, that a multiplicity free action obtained by adding or removing a copy of the scalars $\mathbb{T}$ from a well-behaved multiplicity free action remains well-behaved. So for our purposes it suffices to describe the multiplicity free actions which are fully *saturated*. That is, actions $K : V$ which include a full copy of the scalars acting on each irreducible subspace of V.

Up to geometric equivalence there are twelve saturated indecomposable non-irreducible multiplicity free actions $K : V$. In each case $V = V_1 \oplus V_2$ is the direct sum of two K-irreducible subspaces, $K = K' \times \mathbb{T} \times \mathbb{T}$ with K' compact semisimple and $K : V_1$, $K : V_2$ are irreducible multiplicity free actions. The possibilities for $K' : V$ are listed in Table 1, which follows notational conventions from [3].[1] To complete the proof of

§	Action	rank
(5.2)	$\mathbf{SU(n)} \oplus_{\mathbf{SU(n)}} \mathbf{SU(n)}$ $(n \geq 2)$	3
(5.3)	$\mathbf{SU(n)} \oplus_{\mathbf{SU(n)}} \mathbf{SU(n)}^*$ $(n \geq 3)$	3
(5.4)	$(\mathbf{SU(n)} \otimes \mathbf{SU(2)}) \oplus_{\mathbf{SU(2)}} (\mathbf{SU(2)} \otimes \mathbf{SU(m)})$ $(n, m \geq 2)$	5
(5.5)	$(\mathbf{Sp(2n)} \otimes \mathbf{SU(2)}) \oplus_{\mathbf{SU(2)}} (\mathbf{SU(2)} \otimes \mathbf{SU(m)})$ $(n, m \geq 2)$	6
(5.6)	$(\mathbf{Sp(2n)} \otimes \mathbf{SU(2)}) \oplus_{\mathbf{SU(2)}} (\mathbf{SU(2)} \otimes \mathbf{Sp(2m)})$ $(n, m \geq 2)$	7
(5.7)	$\mathbf{SU(2)} \oplus_{\mathbf{SU(2)}} (\mathbf{SU(2)} \otimes \mathbf{Sp(2n)})$ $(n \geq 2)$	5
(5.8)	$\mathbf{Sp(2n)} \oplus_{\mathbf{Sp(2n)}} \mathbf{Sp(2n)}$ $(n \geq 2)$	4
(5.9)	$\mathbf{Spin(8)} \oplus_{\mathbf{Spin(8)}} \mathbf{SO(8)}$	5
(6.1)	$\mathbf{SU(n)} \oplus_{\mathbf{SU(n)}} (\mathbf{SU(n)} \otimes \mathbf{SU(m)})$ $(n, m \geq 2)$	$\min(2n, 2m+1)$
(6.2)	$\mathbf{SU(n)}^* \oplus_{\mathbf{SU(n)}} (\mathbf{SU(n)} \otimes \mathbf{SU(m)})$ $(n \geq 3, m \geq 2)$	$\min(2n, 2m+1)$
(6.3)	$\mathbf{SU(n)} \oplus_{\mathbf{SU(n)}} \mathbf{\Lambda^2(SU(n))}$ $(n \geq 4)$	n
(6.4)	$\mathbf{SU(n)}^* \oplus_{\mathbf{SU(n)}} \mathbf{\Lambda^2(SU(n))}$ $(n \geq 4)$	n

TABLE 1

Theorem 1.2 we will verify that each of these actions is well-behaved. Numbers in the first column of the table refer to subsections treating each example in turn.

5. Case-by-case analysis: Fixed rank examples

In this section we examine the first eight actions in Table 1. Each of these (families of) examples has a fixed rank and one can use brute force calculation. For each action $K : V$ we will apply Lemma 2.5 (the limiting procedure) and proceed as follows.

(a) Give explicit fundamental highest weights α_j and highest weight vectors h_j. This data can be found in [3, 4, 14, 15].
(b) Use Lemma 2.2 to obtain a system of quadratic equations whose solutions are generic generalized spherical points.
(c) Produce one such solution, $v(\mathbf{x})$ say.
(d) Obtain formulas for the $h_j(v(\mathbf{x}))$'s to verify condition (2) in Lemma 2.5.

[1] In fact references [3, 14, 15] concern actions of complex algebraic groups. The table lists compact forms for these.

(e) Take limits as subsets of the parameters $\mathbf{x}$ approach zero in $v(\mathbf{x})$ and in the formulas for the $h_j(v(\mathbf{x}))$'s, verifying conditions (3) and (4) in the lemma.

We made extensive use of a computer algebra system, Maple, to facilitate the calculations. In most cases, however, Maple was unable to solve the equations from step (b) on general symbolic inputs. Considerable experimentation with numerical examples was used to conjecture the expressions given below for a generic generalized spherical point in each case. It is, however, not difficult to check by hand that these points do solve the equations from step (b). Steps (d) and (e) are straightforward in each case. But for a multiplicity free action or rank r there are $2^r - 2$ non-empty proper subsets of the parameters and thus $2^r - 2$ limits to examine in all. We used Maple to perform step (e) and, except for the rank 3 actions, will omit the details. The interested reader can find a Maple worksheet concerning the rank 7 example (5.6) at the first author's web page [7].

5.1. **Notational conventions.** The first seven actions $K : V$ from Table 1 will each be realized in a suitable space of complex matrices, $V = M_{n,m}(\mathbb{C})$ say, and the usual Hermitian inner product on $M_{n,m}(\mathbb{C})$, namely

$$\langle z, w\rangle = tr(zw^*),$$

is K-invariant. For matrices $z \in V$ the notations $z_{i,\bullet}$ and $z_{\bullet,j}$ indicate row and column vectors. The row and column spaces carry their standard inner product and norm.

We fix the following notation concerning highest weight theory for the general linear and symplectic groups.

- B_n will denote the Borel subgroup of lower triangular matrices in $GL(n, \mathbb{C})$, $\mathfrak{h}_n$ the Cartan subalgebra of diagonal matrices in $gl(n, \mathbb{C})$ and $\varepsilon_j \in \mathfrak{h}_n^*$ the functional

$$\varepsilon_j\big(diag(d_1, \ldots, d_n)\big) = d_j, \quad (1 \le j \le n). \tag{5.1}$$

- The compact symplectic group is $Sp(2n) = Sp(2n, \mathbb{C}) \cap U(2n)$ where $Sp(2n, \mathbb{C})$ is the subgroup of $GL(2n, \mathbb{C})$ preserving the symplectic form

$$\omega\big((z_1, \ldots, z_{2n}), (w_1, \ldots, w_{2n})\big) = \sum_{j=1}^{n} (z_j w_{n+j} - z_{n+j} w_j). \tag{5.2}$$

 The group $Sp(2n, \mathbb{C})$ has Lie algebra

$$sp(2n, \mathbb{C}) = \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & -A^t \end{array} \right] \ : \ A, B, C \in gl(n, \mathbb{C}),\ B^t = B,\ C^t = C \right\}.$$

 As Borel subgroup B_{2n}^{Sp} in $Sp(2n, \mathbb{C})$ we choose $B_{2n}^{Sp} = \exp(\mathfrak{b}_{2n}^{Sp})$ where $\mathfrak{b}_{2n}^{Sp}$ is the subalgebra of $sp(2n, \mathbb{C})$ consisting of matrices as above with $B = 0$ and A lower triangular. A Cartan subalgebra in $sp(2n, \mathbb{C})$ is given by

$$\mathfrak{h}_{2n}^{Sp} = \big\{ diag(a_1, \ldots, a_n, -a_1, \ldots, -a_n) \ : \ a_j \in \mathbb{C} \big\}$$

and in this context we let $\varepsilon_j \in (\mathfrak{h}_{2n}^{Sp})^*$ $(1 \le j \le n)$ denote the linear functional

$$\varepsilon_j\big(diag(a_1,\ldots,a_n,-a_1,\ldots,-a_n)\big) = a_j. \tag{5.3}$$

5.2. $\mathbf{SU(n)} \oplus_{\mathbf{SU(n)}} \mathbf{SU(n)}$ $(n \ge 2)$. Here $K = U(n) \times \mathbb{T}$ acts on $V = V_1 \oplus V_2 = \mathbb{C}^n \oplus \mathbb{C}^n \cong M_{n,2}(\mathbb{C})$ via

$$(k,c)\cdot z = kz\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} = \big[ckz_{\bullet,1}\big|kz_{\bullet,2}\big]$$

for $(k,c) \in K$ and $z \in M_{n,2}(\mathbb{C})$.

As Borel subgroup in $K_{\mathbb{C}} = GL(n,\mathbb{C}) \times \mathbb{C}^\times$ we take $B = B_n \times \mathbb{C}^\times$ and as Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_n \times \mathbb{C}$. Let $T = (0,1) \in \mathfrak{h}$ and $\varepsilon_\circ \in \mathfrak{h}^*$ be the functional with $\varepsilon_\circ(T) = 1$, $\varepsilon_\circ|_{\mathfrak{h}_n} = 0$. This is a rank 3 multiplicity free action with fundamental B-highest weights and associated highest weight vectors

$$\left\{\begin{array}{l|l} \alpha_1 = -(\varepsilon_1+\varepsilon_\circ) & h_1(z) = z_{11} \\ \alpha_2 = -\varepsilon_1 & h_2(z) = z_{12} \\ \alpha_3 = -(\varepsilon_1+\varepsilon_2+\varepsilon_\circ) & h_3(z) = \det_2(z) \end{array}\right\}.$$

Here $\det_2(z)$ is the determinant of the 2×2 matrix formed by the first two rows in z. For non-negative integer exponents $h_\alpha = h_1^a h_2^b h_3^c$ is a highest weight vector in $\mathbb{C}[V]$ with weight

$$\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3 = -\big((a+b+c)\varepsilon_1 + c\varepsilon_2 + (a+c)\varepsilon_\circ\big).$$

One has $\langle T\cdot z, z\rangle = \|z_{\bullet,1}\|^2$ and $\langle E_{i,j}\cdot z, z\rangle = \langle z_{j,\bullet}, z_{i,\bullet}\rangle$ for elementary matrices $E_{i,j} \in gl(n,\mathbb{C})$. Thus Lemma 2.2 shows that the matrix entries z_{ij} of a spherical point for α must satisfy

$$\|z_{\bullet,1}\|^2 = a+c, \quad \|z_{1,\bullet}\|^2 = a+b+c, \quad \|z_{2,\bullet}\|^2 = c$$

and $\langle z_{j,\bullet}, z_{i,\bullet}\rangle = 0$ for $i \ne j$.

It is easy to verify that the entries of

$$v(a,b,c) := \begin{bmatrix} \sqrt{\frac{a(a+b+c)}{a+b}} & \sqrt{\frac{b(a+b+c)}{a+b}} \\ -\sqrt{\frac{bc}{a+b}} & \sqrt{\frac{ac}{a+b}} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \tag{5.4}$$

satisfy the above equations for any given real parameters $a,b,c \ge 0$ provided $a+b \ne 0$. In particular (5.4) is a generic generalized spherical point if $a,b,c > 0$. To show that the action $U(n)\times\mathbb{T} : M_{n,2}(\mathbb{C})$ is well-behaved it remains to check conditions (2)-(4) of Lemma 2.5.

Evaluating the fundamental highest weight vectors at $v = v(a,b,c)$ yields

$$h_1(v) = \sqrt{\frac{a(a+b+c)}{a+b}}, \quad h_2(v) = \sqrt{\frac{b(a+b+c)}{a+b}}, \quad h_3(v) = \sqrt{c(a+b+c)}. \tag{5.5}$$

As these values are non-zero for positive parameters (a,b,c) condition (2) in Lemma 2.5 holds here. As regards condition (3) in the lemma we just need to observe that for fixed $c > 0$ the limit

$$\lim_{b\to 0^+}\lim_{a\to 0^+} v(a,b,c) = \lim_{b\to 0^+}\begin{bmatrix} 0 & \sqrt{b+c} \\ -\sqrt{c} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{c} \\ -\sqrt{c} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

does exist. Limiting values for $h_j\big(v(a,b,c)\big)$ $(j = 1,2,3)$ as one or two parameters approach zero are given by

limit	$h_1\big(v(a,b,c)\big)$	$h_2\big(v(a,b,c)\big)$	$h_3\big(v(a,b,c)\big)$
$\lim_{a\to 0^+}$	0	$\sqrt{b+c}$	$\sqrt{c(b+c)}$
$\lim_{b\to 0^+}$	$\sqrt{a+c}$	0	$\sqrt{c(a+c)}$
$\lim_{c\to 0^+}$	$\sqrt{a}$	$\sqrt{b}$	0
$\lim_{b\to 0^+}\lim_{a\to 0^+}$	0	$\sqrt{c}$	c
$\lim_{c\to 0^+}\lim_{a\to 0^+}$	0	$\sqrt{b}$	0
$\lim_{c\to 0^+}\lim_{b\to 0^+}$	$\sqrt{a}$	0	0

These show, in particular, that condition (4) in Lemma 2.5 holds.

5.3. $\mathbf{SU(n)} \oplus_{\mathbf{SU(n)}} \mathbf{SU(n)}^*$ $(n \geq 3)$**.** This is a *twisted* variant of Example 5.2 with $K = U(n) \times \mathbb{T}$ acting on $V = V_1 \oplus V_2 = \mathbb{C}^n \oplus \mathbb{C}^n \cong M_{n,2}(\mathbb{C})$ via

$$(k,c)\cdot z = \big[ckz_{\bullet,1} \mid k^{-t}z_{\bullet,2}\big] \quad \text{where } k^{-t} := (k^{-1})^t.$$

Here the action in the second column is contragredient to the standard action in the first column. One takes $n \geq 3$ here as Examples 5.3 and 5.2 are geometrically equivalent when $n = 2$. This is so because the standard representation for $SU(2)$ is self-contragredient.

Again $K : V$ has rank 3 with fundamental B-highest weights and highest weight vectors

$$\left\{ \begin{array}{l|l} \alpha_1 = -(\varepsilon_1 + \varepsilon_\circ) & h_1(z) = z_{11} \\ \alpha_2 = +\varepsilon_n & h_2(z) = z_{n2} \\ \alpha_3 = -\varepsilon_\circ & h_3(z) = \sum_{i=1}^n z_{i1}z_{i2} = z_{\bullet,1}\cdot z_{\bullet,2} \end{array} \right\}.$$

For non-negative integer exponents $h_\alpha = h_1^a h_2^b h_3^c$ is a highest weight vector in $\mathbb{C}[V]$ with weight

$$\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3 = -\big(a\varepsilon_1 - b\varepsilon_n + (a+c)\varepsilon_\circ\big).$$

Now $\langle T \cdot z, z\rangle = \|z_{\bullet,1}\|^2$ as before but $\langle E_{i,j} \cdot z, z\rangle = z_{j1}\overline{z_{i1}} - z_{i2}\overline{z_{j2}}$ for elementary matrices $E_{i,j} \in gl(n,\mathbb{C})$. So the matrix entries z_{ij} of a spherical point for α must satisfy

$$\|z_{\bullet,1}\|^2 = a + c, \quad |z_{11}|^2 - |z_{12}|^2 = a, \quad |z_{n1}|^2 - |z_{n2}|^2 = -b$$

and $z_{j1}\overline{z_{i1}} - z_{i2}\overline{z_{j2}} = 0$ for $i \neq j$. The matrix entries in

$$v(a,b,c) := \begin{bmatrix} \sqrt{\frac{a(a+b+c)}{a+b}} & \sqrt{\frac{ac}{a+b}} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \sqrt{\frac{bc}{a+b}} & \sqrt{\frac{b(a+b+c)}{a+b}} \end{bmatrix} \tag{5.6}$$

satisfy these equations for any real values $a, b, c \geq 0$ with $a+b \neq 0$. In particular (5.6) is a generic generalized spherical point when $a, b, c > 0$. Evaluating the fundamental highest weight vectors h_j at $v = v(a,b,c)$ again yields Equations 5.5. Thus conditions (2)-(4) from Lemma 2.5 hold as verified in Example 5.2.

5.4. $(\mathbf{SU(n)} \otimes \mathbf{SU(2)}) \oplus_{\mathbf{SU(2)}} (\mathbf{SU(2)} \otimes \mathbf{SU(m)})$ $(n, m \geq 2)$**.** Here $K = U(n) \times U(m) \times U(2)$ acts on $V = V_1 \oplus V_2 = (\mathbb{C}^n \otimes \mathbb{C}^2) \oplus (\mathbb{C}^m \otimes \mathbb{C}^2) \cong M_{n+m,2}(\mathbb{C})$ via

$$(k_1, k_2, k_3) \cdot z = \left[\begin{array}{c|c} k_1 & 0 \\ \hline 0 & k_2 \end{array}\right] z k_3^t. \tag{5.7}$$

We take Borel subgroup $B = B_n \times B_m \times B_2$ in $K_{\mathbb{C}} = GL(n,\mathbb{C}) \times GL(m,\mathbb{C}) \times GL(2,\mathbb{C})$ and let ε_j, ε_j', ε_j'' denote functionals on $\mathfrak{h} = \mathfrak{h}_n \times \mathfrak{h}_m \times \mathfrak{h}_2$ as in (5.1) supported on each of the three factors. This is a rank 5 multiplicity free action with fundamental B-highest weights and highest weight vectors

$$\left\{ \begin{array}{l|l} \alpha_1 = -(\varepsilon_1 + \varepsilon_1'') & h_1(z) = z_{11} \\ \alpha_2 = -(\varepsilon_1' + \varepsilon_1'') & h_2(z) = z_{n+1,1} \\ \alpha_3 = -(\varepsilon_1 + \varepsilon_2 + \varepsilon_1'' + \varepsilon_2'') & h_3(z) = \det_2(z) \\ \alpha_4 = -(\varepsilon_1' + \varepsilon_2' + \varepsilon_1'' + \varepsilon_2'') & h_4(z) = \begin{vmatrix} z_{n+1,1} & z_{n+1,2} \\ z_{n+2,1} & z_{n+2,2} \end{vmatrix} \\ \alpha_5 = -(\varepsilon_1 + \varepsilon_1' + \varepsilon_1'' + \varepsilon_2'') & h_5(z) = \begin{vmatrix} z_{11} & z_{12} \\ z_{n+1,1} & z_{n+1,2} \end{vmatrix} \end{array} \right\}. \tag{5.8}$$

For non-negative integer exponents $h_\alpha = h_1^a h_2^b h_3^c h_4^d h_5^e$ is a highest weight vector in $\mathbb{C}[V]$ with weight

$$\begin{aligned} \alpha &= a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + e\alpha_5 \\ &= -\big((a+c+e)\varepsilon_1 + c\varepsilon_2 + (b+d+e)\varepsilon_1' + d\varepsilon_2' + (a+b+c+d+e)\varepsilon_1'' + (c+d+e)\varepsilon_2''\big). \end{aligned}$$

Letting $E_{i,j}$, $E'_{i,j}$ and $E''_{i,j}$ denote elementary matrices in $gl(n,\mathbb{C})$, $gl(m,\mathbb{C})$ and $gl(2,\mathbb{C})$ respectively one has

$$\langle E_{i,j}\cdot z, z\rangle = \langle z_{j,\bullet}, z_{i,\bullet}\rangle\,, \quad \langle E'_{i,j}\cdot z, z\rangle = \langle z_{n+j,\bullet}, z_{n+i,\bullet}\rangle\,, \quad \langle E''_{i,j}\cdot z, z\rangle = \langle z_{\bullet,j}, z_{\bullet,i}\rangle$$

for $z\in M_{n+m,2}(\mathbb{C})$. Thus a spherical point z for weight α must have

- orthogonal columns,
- rows $1\ldots n$ pair-wise orthogonal and rows $n{+}1\ldots n{+}m$ pair-wise orthogonal,
- $\|z_{1\bullet}\|^2 = a+c+e$, $\|z_{2,\bullet}\|^2 = c$, $\|z_{n+1,\bullet}\|^2 = b+d+e$, $\|z_{n+2,\bullet}\|^2 = d$,
- $\|z_{\bullet,1}\|^2 = a+b+c+d+e$, $\|z_{\bullet,2}\|^2 = c+d+e$.

To solve these equations we may set rows other than 1, 2, $n+1$ and $n+2$ to zero and reduce to the case $n=m=2$. So now $K = U(2)\times U(2)\times U(2)$, $V = M_{4,2}(\mathbb{C})$. One can check that the matrix entries of

$$v\big(a,b,c,d,e\big) := \begin{bmatrix} -\sqrt{\frac{a(a+b+e)(a+c+e)}{(a+b)(a+e)}} & \sqrt{\frac{be(a+c+e)}{(a+b)(a+e)}} \\ \sqrt{\frac{bce}{(a+b)(a+e)}} & \sqrt{\frac{ac(a+b+e)}{(a+b)(a+e)}} \\ \sqrt{\frac{b(a+b+e)(b+d+e)}{(a+b)(b+e)}} & \sqrt{\frac{ae(b+d+e)}{(a+b)(b+e)}} \\ -\sqrt{\frac{ade}{(a+b)(b+e)}} & \sqrt{\frac{bd(a+b+e)}{(a+b)(b+e)}} \end{bmatrix}$$

satisfy each of the above conditions for arbitrary positive real parameters. This is a generic generalized spherical point for this example. Evaluating the fundamental highest weight vectors at $v = v(a,b,c,d,e)$ gives

$$\left\{\begin{array}{cc} h_1(v) = -\sqrt{\frac{a(a+b+e)(a+c+e)}{(a+b)(a+e)}} & h_2(v) = \sqrt{\frac{b(a+b+e)(b+d+e)}{(a+b)(b+e)}} \\ h_3(v) = -\sqrt{c(a+c+e)} & h_4(v) = \sqrt{d(b+d+e)} \\ \multicolumn{2}{c}{h_5(v) = -\sqrt{\frac{e(a+b+e)(a+c+e)(b+d+e)}{(a+e)(b+e)}}} \end{array}\right\}.$$

These values are non-zero as required by Lemma 2.5 condition (2). To check conditions (3) and (4) in the Lemma one needs to take limits as one or more parameters approach zero in succession and compute the limiting values of the $h_j(v)$'s. This is routine but there are $2^5-2=30$ limits to examine in all. We used Maple to perform this task. For example one finds

$$\lim_{c\to 0^+}\lim_{a\to 0^+} v(a,b,c,d,e) = \lim_{c\to 0^+}\begin{bmatrix} 0 & \sqrt{c+e} \\ \sqrt{c} & 0 \\ \sqrt{b+d+e} & 0 \\ 0 & \sqrt{d}\end{bmatrix} = \begin{bmatrix} 0 & \sqrt{e} \\ 0 & 0 \\ \sqrt{b+d+e} & 0 \\ 0 & \sqrt{d}\end{bmatrix} = v_\circ$$

say and each of the values

$$h_2(v_\circ) = \sqrt{b+d+e}, \quad h_4(v_\circ) = \sqrt{d(b+d+e)}, \quad h_5(v_\circ) = -\sqrt{e(b+d+e)}$$

are non-zero.

5.5. $(\mathbf{Sp(2n)} \otimes \mathbf{SU(2)}) \oplus_{\mathbf{SU(2)}} (\mathbf{SU(2)} \otimes \mathbf{SU(m)})$ $(n, m \geq 2)$. Now $K = Sp(2n) \times U(m) \times U(2)$ acts on $V = V_1 \oplus V_2 = (\mathbb{C}^{2n} \otimes \mathbb{C}^2) \oplus (\mathbb{C}^m \otimes \mathbb{C}^2) \cong M_{2n+m,2}(\mathbb{C})$ as in Equation 5.7 from the previous example. This is a rank 6 multiplicity free action.

We use Borel subgroup $B = B_{2n}^{Sp} \times B_m \times B_2$ in $K_{\mathbb{C}} = Sp(2n, \mathbb{C}) \times GL(m, \mathbb{C}) \times GL(2, \mathbb{C})$ and Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_{2n}^{Sp} \times \mathfrak{h}_m \times \mathfrak{h}_2$. Let $\varepsilon_j \in \mathfrak{h}^*$ be given by (5.3) on $\mathfrak{h}_{2n}^{Sp}$ and $\varepsilon_j', \varepsilon_j'' \in \mathfrak{h}^*$ be given by (5.1) on the factors $\mathfrak{h}_m$ and $\mathfrak{h}_2$ respectively. With these notational conventions the fundamental highest weights and highest weight vectors are as in Equations 5.8 above (but with "n" replaced by "$2n$" in the formulas for $h_2(z)$, $h_4(z)$, $h_5(z)$) together with

$$\alpha_6 = -(\varepsilon_1'' + \varepsilon_2''), \qquad h_6(z) = \omega(z'_{\bullet,1}, z'_{\bullet,2}),$$

where $z' \in M_{2n,2}(\mathbb{C})$ denotes the first $2n$ rows in $z \in M_{2n+m,2}(\mathbb{C})$ and ω the symplectic inner product (5.2). For non-negative integer exponents $h_\alpha = h_1^a h_2^b h_3^c h_4^d h_5^e h_6^f$ is a highest weight vector in $\mathbb{C}[V]$ with weight

$$\begin{aligned} \alpha &= a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + e\alpha_5 + f\alpha_6 \\ &= -\big((a+c+e)\varepsilon_1 + c\varepsilon_2 + (b+d+e)\varepsilon_1' + d\varepsilon_2' + (a+b+c+d+e+f)\varepsilon_1'' \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (c+d+e+f)\varepsilon_2''\big). \end{aligned}$$

It will suffice, as in the Example 5.4, to consider the case $n = m = 2$. So now $K = Sp(4) \times U(2) \times U(2)$ and $V = M_{6,2}(\mathbb{C})$. From the actions of the two copies of $gl(2, \mathbb{C})$ one obtains $\left\langle E'_{i,j} \cdot z, z \right\rangle = \langle z_{4+j,\bullet}, z_{4+i,\bullet} \rangle$ and $\left\langle E''_{i,j} \cdot z, z \right\rangle = \langle z_{\bullet,j}, z_{\bullet,i} \rangle$. The action of $sp(4, \mathbb{C})$ gives

$$\left\{ \begin{array}{ll} \langle (E_{1,1} - E_{3,3}) \cdot z, z \rangle = \|z_{1,\bullet}\|^2 - \|z_{3,\bullet}\|^2 & \langle E_{1,3} \cdot z, z \rangle = \langle z_{3,\bullet}, z_{1,\bullet} \rangle \\ \langle (E_{2,2} - E_{4,4}) \cdot z, z \rangle = \|z_{2,\bullet}\|^2 - \|z_{4,\bullet}\|^2 & \langle E_{2,4} \cdot z, z \rangle = \langle z_{4,\bullet}, z_{2,\bullet} \rangle \\ \langle (E_{1,2} - E_{4,3}) \cdot z, z \rangle = \langle z_{2,\bullet}, z_{1,\bullet} \rangle - \langle z_{3,\bullet}, z_{4,\bullet} \rangle & \\ \langle (E_{1,4} + E_{2,3}) \cdot z, z \rangle = \langle z_{4,\bullet}, z_{1,\bullet} \rangle + \langle z_{3,\bullet}, z_{2,\bullet} \rangle & \end{array} \right\}.$$

Thus a spherical point $z \in M_{6,2}(\mathbb{C})$ for weight α must have

- $\langle z_{\bullet,1}, z_{\bullet,2} \rangle = 0$,
- $\langle z_{1,\bullet}, z_{3,\bullet} \rangle = 0 = \langle z_{2,\bullet}, z_{4,\bullet} \rangle = \langle z_{5,\bullet}, z_{6,\bullet} \rangle$,
- $\langle z_{1,\bullet}, z_{2,\bullet} \rangle = \langle z_{4,\bullet}, z_{3,\bullet} \rangle$, $\quad \langle z_{1,\bullet}, z_{4,\bullet} \rangle = -\langle z_{2,\bullet}, z_{3,\bullet} \rangle$
- $\|z_{1,\bullet}\|^2 - \|z_{3,\bullet}\|^2 = a + c + e$, $\quad \|z_{2,\bullet}\|^2 - \|z_{4,\bullet}\|^2 = c$,
- $\|z_{5,\bullet}\|^2 = b + d + e$, $\quad \|z_{6,\bullet}\|^2 = d$,
- $\|z_{\bullet,1}\| = a + b + c + d + e + f$, $\quad \|z_{\bullet,2}\|^2 = c + d + e + f$.

A generic generalized spherical point whose matrix entries solve these equations for arbitrary positive values of $a,\dots,f$ is given below.

$$v(a,b,c,d,e,f) := \begin{bmatrix} \sqrt{\frac{a(a+b+e)(a+c+e)(a+2c+e+f)}{(a+b)(a+e)(a+2c+e)}} & -\sqrt{\frac{be(a+c+e)(a+2c+e+f)}{(a+b)(a+e)(a+2c+e)}} \\ \sqrt{\frac{bce(a+2c+e+f)}{(a+b)(a+e)(a+2c+e)}} & \sqrt{\frac{ac(a+b+e)(a+2c+e+f)}{(a+b)(a+e)(a+2c+e)}} \\ \sqrt{\frac{bef(a+c+e)}{(a+b)(a+e)(a+2c+e)}} & \sqrt{\frac{af(a+b+e)(a+c+e)}{(a+b)(a+e)(a+2c+e)}} \\ -\sqrt{\frac{acf(a+b+e)}{(a+b)(a+e)(a+2c+e)}} & \sqrt{\frac{bcef}{(a+b)(a+e)(a+2c+e)}} \\ -\sqrt{\frac{b(a+b+e)(b+d+e)}{(a+b)(b+e)}} & -\sqrt{\frac{ae(b+d+e)}{(a+b)(b+e)}} \\ -\sqrt{\frac{ade}{(a+b)(b+e)}} & \sqrt{\frac{bd(a+b+e)}{(a+b)(b+e)}} \end{bmatrix}$$

Evaluating the $h_j's$ at $v = v(a,b,c,d,e,f)$ yields

$$\left\{ \begin{array}{ll} h_1(v) = \sqrt{\frac{a(a+b+e)(a+c+e)(a+2c+e+f)}{(a+b)(a+e)(a+2c+e)}} & h_2(v) = -\sqrt{\frac{b(a+b+e)(b+d+e)}{(a+b)(b+e)}} \\ h_3(v) = \frac{(a+2c+e+f)\sqrt{c(a+c+e)}}{a+2c+e} & h_4(v) = -\sqrt{d(b+d+e)} \\ h_5(v) = -\sqrt{\frac{e(a+b+e)(a+c+e)(b+d+e)(a+2c+e+f)}{(a+e)(b+e)(a+2c+e)}} & h_6(v) = \sqrt{f(a+2c+e+f)} \end{array} \right\}.$$

upon simplification using a computer algebra system. These show, in particular, that condition (2) in Lemma 2.5 holds. We also used Maple to check conditions (3) and (4) from the lemma, completing the verification for this example. This entails routine examination of $2^6 - 2 = 62$ limits.

5.6. $(\mathbf{Sp(2n)} \otimes \mathbf{SU(2)}) \oplus_{\mathbf{SU(2)}} (\mathbf{SU(2)} \otimes \mathbf{Sp(2m)})$ $(n, m \geq 2)$**.** Next $K = Sp(2n) \times Sp(2m) \times U(2) \times \mathbb{T}$ acts on $V = V_1 \oplus V_2 = (\mathbb{C}^{2n} \otimes \mathbb{C}^2) \oplus (\mathbb{C}^{2m} \otimes \mathbb{C}^2) \cong M_{2(n+m),2}(\mathbb{C})$. Letting z' and z'' denote the first $2n$ and last $2m$ rows of $z \in M_{2(n+m),2}(\mathbb{C})$ we have

$$(k_1, k_2, k_3, c) \cdot z = \left[\begin{array}{c|c} cI_{2n} & 0 \\ \hline 0 & I_{2m} \end{array}\right] \left[\begin{array}{c|c} k_1 & 0 \\ \hline 0 & k_2 \end{array}\right] z k_3^t = \left[\begin{array}{c|c} ck_1 z'_{\bullet,1} & ck_1 z'_{\bullet,2} \\ \hline k_2 z''_{\bullet,1} & k_2 z''_{\bullet,2} \end{array}\right] k_3^t.$$

The factor $\mathbb{T}$ is required to fully saturate this example. In fact this action fails to be multiplicity free if the circle is removed [3, Theorem 6].

We use Borel subgroup $B = B_{2n}^{Sp} \times B_{2m}^{Sp} \times B_2 \times \mathbb{C}^\times$ in $K_{\mathbb{C}}$ and Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_{2n}^{Sp} \times \mathfrak{h}_{2m}^{Sp} \times \mathfrak{h}_2 \times \mathbb{C}$. Let

- $\varepsilon_j, \varepsilon'_j \in \mathfrak{h}^*$ be as in (5.3) on the symplectic factors $\mathfrak{h}_{2n}^{Sp}$ and $\mathfrak{h}_{2m}^{Sp}$,
- $\varepsilon''_j \in \mathfrak{h}^*$ be as in (5.1) on the $\mathfrak{h}_2$ factor and
- $\varepsilon_\circ \in \mathfrak{h}^*$ be dual to $T = (0,0,0,1) \in \mathfrak{h}$.

The action $K : V$ has rank 7 with fundamental highest weights

$$\left\{ \begin{array}{ll} \alpha_1 = -(\varepsilon_1 + \varepsilon_1'' + \varepsilon_\circ) & \alpha_4 = -(\varepsilon_1' + \varepsilon_2' + \varepsilon_1'' + \varepsilon_2'') \\ \alpha_2 = -(\varepsilon_1' + \varepsilon_1'') & \alpha_5 = -(\varepsilon_1 + \varepsilon_1' + \varepsilon_1'' + \varepsilon_2'' + \varepsilon_\circ) \\ \alpha_3 = -(\varepsilon_1 + \varepsilon_2 + \varepsilon_1'' + \varepsilon_2'' + 2\varepsilon_\circ) & \alpha_6 = -(\varepsilon_1'' + \varepsilon_2'' + 2\varepsilon_\circ) \\ \multicolumn{2}{c}{\alpha_7 = -(\varepsilon_1'' + \varepsilon_2'')} \end{array} \right\}.$$

Fundamental highest weight vectors $h_1(z), \ldots, h_6(z)$ with weights $\alpha_1, \ldots, \alpha_6$ are as in Example 5.5. A highest weight vector for α_7 is

$$h_7(z) = \omega(z''_{\bullet,1}, z''_{\bullet,2}).$$

For non-negative integer exponents $h_\alpha = h_1^a h_2^b h_3^c h_4^d h_5^e h_6^f h_7^g$ is a highest weight vector in $\mathbb{C}[V]$ with weight

$$\begin{aligned} \alpha &= a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + e\alpha_5 + f\alpha_6 + g\alpha_7 \\ &= -\big((a+c+e)\varepsilon_1 + c\varepsilon_2 + (b+d+e)\varepsilon_1' + d\varepsilon_2' + (a+b+c+d+e+f+g)\varepsilon_1'' \\ &\qquad + (c+d+e+f+g)\varepsilon_2'' + (a+2c+e+2f)\varepsilon_\circ\big). \end{aligned}$$

As in Examples 5.4, 5.5 we need only consider the case $n = m = 2$. So now $K = Sp(4) \times Sp(4) \times U(2) \times \mathbb{T}$ and $V = M_{8,2}(\mathbb{C})$. Lemma 2.2 yields a system of equations for the matrix entries of a spherical point $z \in M_{8,2}(\mathbb{C})$ for weight α, just as in Example 5.5, namely

- $\langle z_{\bullet,1}, z_{\bullet,2} \rangle = 0$,
- $\langle z_{1,\bullet}, z_{3,\bullet} \rangle = 0 = \langle z_{2,\bullet}, z_{4,\bullet} \rangle = \langle z_{5,\bullet}, z_{7,\bullet} \rangle = \langle z_{6,\bullet}, z_{8,\bullet} \rangle$,
- $\langle z_{1,\bullet}, z_{2,\bullet} \rangle = \langle z_{4,\bullet}, z_{3,\bullet} \rangle, \quad \langle z_{1,\bullet}, z_{4,\bullet} \rangle = -\langle z_{2,\bullet}, z_{3,\bullet} \rangle$
- $\langle z_{5,\bullet}, z_{6,\bullet} \rangle = \langle z_{8,\bullet}, z_{7,\bullet} \rangle, \quad \langle z_{5,\bullet}, z_{8,\bullet} \rangle = -\langle z_{6,\bullet}, z_{7,\bullet} \rangle$
- $\|z_{1,\bullet}\|^2 - \|z_{3,\bullet}\|^2 = a + c + e, \quad \|z_{2,\bullet}\|^2 - \|z_{4,\bullet}\|^2 = c$,
- $\|z_{5,\bullet}\|^2 - \|z_{7,\bullet}\| = b + d + e, \quad \|z_{6,\bullet}\|^2 - \|z_{8,\bullet}\| = d$,
- $\|z_{\bullet,1}\| = a + b + c + d + e + f + g, \quad \|z_{\bullet,2}\|^2 = c + d + e + f + g$,
- $\langle z', z' \rangle = \|z'_{\bullet,1}\|^2 + \|z'_{\bullet,2}\|^2 = a + 2c + e + 2f$.

A generic generalized spherical point whose matrix entries solve these equations for arbitrary positive values of $a,\ldots,g$ is given below.

$$v(a,b,c,d,e,f,g) := \begin{bmatrix} -\sqrt{\frac{a(a+b+e)(a+c+e)(a+2c+e+f)}{(a+b)(a+e)(a+2c+e)}} & \sqrt{\frac{be(a+c+e)(a+2c+e+f)}{(a+b)(a+e)(a+2c+e)}} \\ -\sqrt{\frac{bce(a+2c+e+f)}{(a+b)(a+e)(a+2c+e)}} & -\sqrt{\frac{ac(a+b+e)(a+2c+e+f)}{(a+b)(a+e)(a+2c+e)}} \\ \sqrt{\frac{bef(a+c+e)}{(a+b)(a+e)(a+2c+e)}} & \sqrt{\frac{af(a+b+e)(a+c+e)}{(a+b)(a+e)(a+2c+e)}} \\ -\sqrt{\frac{acf(a+b+e)}{(a+b)(a+e)(a+2c+e)}} & \sqrt{\frac{bcef}{(a+b)(a+e)(a+2c+e)}} \\ -\sqrt{\frac{b(a+b+e)(b+d+e)(b+2d+e+g)}{(a+b)(b+e)(b+2d+e)}} & -\sqrt{\frac{ae(b+d+e)(b+2d+e+g)}{(a+b)(b+e)(b+2d+e)}} \\ -\sqrt{\frac{ade(b+2d+e+g)}{(a+b)(b+e)(b+2d+e)}} & \sqrt{\frac{bd(a+b+e)(b+2d+e+g)}{(a+b)(b+e)(b+2d+e)}} \\ -\sqrt{\frac{aeg(b+d+e)}{(a+b)(b+e)(b+2d+e)}} & \sqrt{\frac{bg(a+b+e)(b+d+e)}{(a+b)(b+e)(b+2d+e)}} \\ \sqrt{\frac{bdg(a+b+e)}{(a+b)(b+e)(b+2d+e)}} & \sqrt{\frac{adeg}{(a+b)(b+e)(b+2d+e)}} \end{bmatrix}.$$

The fundamental highest weight vectors take the following non-zero values at $v = v(a,b,c,d,e,f,g)$.

$$\left\{ \begin{array}{ll} h_1(v) = -\sqrt{\frac{a(a+b+e)(a+c+e)(a+2c+e+f)}{(a+b)(a+e)(a+2c+e)}} & h_2(v) = -\sqrt{\frac{b(a+b+e)(b+d+e)(b+2d+e+g)}{(a+b)(b+e)(b+2d+e)}} \\ h_3(v) = \frac{(a+2c+e+f)\sqrt{c(a+c+e)}}{a+2c+e} & h_4(v) = -\frac{(b+2d+e+g)\sqrt{d(b+d+e)}}{b+2d+e} \\ \multicolumn{2}{c}{h_5(v) = \sqrt{\frac{e(a+b+e)(a+c+e)(b+d+e)(a+2c+e+f)(b+2d+e+g)}{(a+e)(b+e)(a+2c+e)(b+2d+e)}}} \\ h_6(v) = -\sqrt{f(a+2c+e+f)} & h_7(v) = -\sqrt{g(b+2d+e+g)} \end{array} \right\}.$$

To complete the verification that $K : V$ is well-behaved via Lemma 2.5 we used computer calculations to check each of $2^7 - 2 = 126$ relevant limits. A Maple worksheet giving full details can be found at [7].

5.7. $\mathbf{SU(2)} \oplus_{\mathbf{SU(2)}} (\mathbf{SU(2)} \otimes \mathbf{Sp(2n)})$ $(n \geq 2)$**.** Next consider $K = Sp(2n) \times U(2) \times \mathbb{T}$ acting on $V = V_1 \oplus V_2 = (\mathbb{C}^{2n} \otimes \mathbb{C}^2) \oplus \mathbb{C}^2 \cong M_{2n+1,2}(\mathbb{C})$ via

$$(k_1, k_2, c) \cdot z = \left[\begin{array}{c|c} c & 0 \\ \hline 0 & k_1 \end{array}\right] z k_2^t.$$

As Borel subgroup in $K_{\mathbb{C}} = Sp(2n,\mathbb{C}) \times GL(2,\mathbb{C}) \times \mathbb{C}^\times$ we take $B = B_{2n}^{Sp} \times B_2 \times \mathbb{C}^\times$ and as Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_{2n}^{Sp} \times \mathfrak{h}_2 \times \mathbb{C}$. Let $\varepsilon_j \in \mathfrak{h}^*$ be as in (5.3) on the symplectic factor $\mathfrak{h}_{2n}^{Sp}$, ε_j' be as in (5.1) on the $\mathfrak{h}_2$ factor and $\varepsilon_\circ$ dual to $T = (0,0,1)$.

This is a rank 5 multiplicity free action with the following fundamental B-highest weights and associated highest weight vectors. Here we let $z' \in M_{2n,2}(\mathbb{C})$ be the matrix obtained by removing the first row from $z \in M_{2n+1,2}(\mathbb{C})$.

$$\left\{ \begin{array}{l|l} \alpha_1 = -(\varepsilon_1' + \varepsilon_\circ) & h_1(z) = z_{11} \\ \alpha_2 = -(\varepsilon_1 + \varepsilon_1') & h_2(z) = z_{21} \\ \alpha_3 = -(\varepsilon_1 + \varepsilon_1' + \varepsilon_2' + \varepsilon_\circ) & h_3(z) = \det_2(z) \\ \alpha_4 = -(\varepsilon_1 + \varepsilon_2 + \varepsilon_1' + \varepsilon_2') & h_4(z) = \det_2(z') \\ \alpha_5 = -(\varepsilon_1' + \varepsilon_2') & h_5(z) = \omega(z'_{\bullet,1}, z'_{\bullet,2}) \end{array} \right\}.$$

For non-negative integer exponents $h_\alpha = h_1^a h_2^b h_3^c h_4^d h_5^e$ is a highest weight vector in $\mathbb{C}[V]$ with weight

$$\begin{aligned} \alpha &= a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + e\alpha_5 \\ &= -\big((b+c+d)\varepsilon_1 + d\varepsilon_2 + (a+b+c+d+e)\varepsilon_1' + (c+d+e)\varepsilon_2' + (a+c)\varepsilon_\circ\big). \end{aligned}$$

As in prior examples it suffices to examine the case $n = 2$. So henceforth $K = Sp(4) \times U(2) \times \mathbb{T}$ and $V = M_{5,2}(\mathbb{C})$. Applying Lemma 2.2 one obtains the following equations for the matrix entries of a spherical point $z \in M_{5,2}(\mathbb{C})$ for weight α.

$$\left\{ \begin{array}{ll} \|z_{\bullet,1}\|^2 = a+b+c+d+e & \langle z_{\bullet,1}, z_{\bullet,2}\rangle = 0 \\ \|z_{\bullet,2}\|^2 = c+d+e & \langle z_{2,\bullet}, z_{4,\bullet}\rangle = 0 \\ \|z_{1,\bullet}\|^2 = a+c & \langle z_{3,\bullet}, z_{5,\bullet}\rangle = 0 \\ \|z_{2,\bullet}\|^2 - \|z_{4,\bullet}\|^2 = b+c+d & \langle z_{2,\bullet}, z_{3,\bullet}\rangle = \langle z_{5,\bullet}, z_{4,\bullet}\rangle \\ \|z_{3,\bullet}\|^2 - \|z_{5,\bullet}\|^2 = d & \langle z_{2,\bullet}, z_{5,\bullet}\rangle = -\langle z_{3,\bullet}, z_{4,\bullet}\rangle \end{array} \right\}.$$

One can check that the entries of

$$v(a,b,c,d,e) := \begin{bmatrix} \sqrt{\frac{a(a+b+c)}{a+b}} & -\sqrt{\frac{bc}{a+b}} \\ -\sqrt{\frac{b(a+b+c)(b+c+d)(b+c+2\,d+e)}{(b+c+2\,d)(b+c)(a+b)}} & -\sqrt{\frac{ac(b+c+d)(b+c+2\,d+e)}{(a+b)(b+c)(b+c+2\,d)}} \\ \sqrt{\frac{acd(b+c+2\,d+e)}{(a+b)(b+c)(b+c+2\,d)}} & -\sqrt{\frac{bd(a+b+c)(b+c+2\,d+e)}{(a+b)(b+c)(b+c+2\,d)}} \\ \sqrt{\frac{ace(b+c+d)}{(a+b)(b+c)(b+c+2\,d)}} & -\sqrt{\frac{be(a+b+c)(b+c+d)}{(a+b)(b+c)(b+c+2\,d)}} \\ \sqrt{\frac{bde(a+b+c)}{(a+b)(b+c)(b+c+2\,d)}} & \sqrt{\frac{acde}{(a+b)(b+c)(b+c+2\,d)}} \end{bmatrix}$$

solve these equations for arbitrary positive parameters. This is our generic generalized spherical point. Evaluating the fundamental highest weight vectors at $v = v(a,b,c,d,e)$ gives the values

$$\left\{ \begin{array}{ll} h_1(v) = \sqrt{\frac{a(a+b+c)}{a+b}} & h_2(v) = -\sqrt{\frac{b(a+b+c)(b+c+d)(b+c+2\,d+e)}{(b+c+2\,d)(b+c)(a+b)}} \\ h_3(v) = -\sqrt{\frac{c(a+b+c)(b+c+d)(b+c+2d+e)}{(b+c)(b+c+2d)}} & h_4(v) = \frac{(b+c+2d+e)\sqrt{d(b+c+d)}}{b+c+2d} \\ \multicolumn{2}{c}{h_5(v) = \sqrt{e(b+c+2d+e)}} \end{array} \right\}.$$

which are, in particular, non-zero. A computer algebra system was used to check conditions (3) and (4) from Lemma 2.5. The action $K : V$ is indeed well-behaved.

5.8. $\mathbf{Sp(2n)} \oplus_{\mathbf{Sp(2n)}} \mathbf{Sp(2n)}$ ($n \geq 2$)**.** Next $K = Sp(2n)\times\mathbb{T}\times\mathbb{T}$ acts on $V = V_1 \oplus V_2 = \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \cong M_{2n,2}(\mathbb{C})$ via

$$(k, c_1, c_2)\cdot z = kz \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} = \big[c_1 k z_{\bullet,1} \big| c_2 k z_{\bullet,2}\big].$$

We take $B = B_{2n}^{Sp} \times \mathbb{C}^\times \times \mathbb{C}^\times$, $\mathfrak{h} = \mathfrak{h}_{2n}^{Sp} \times \mathbb{C} \times \mathbb{C}$, let $\varepsilon_j \in \mathfrak{h}^*$ be as in (5.3) supported on the $\mathfrak{h}_{2n}^{Sp}$ factor, $\varepsilon_\circ \in \mathfrak{h}^*$ dual to $T_1 = (0,1,0) \in \mathfrak{h}$ and $\varepsilon_{\circ\circ} \in \mathfrak{h}^*$ dual to $T_2 = (0,0,1) \in \mathfrak{h}$. This is a rank 4 multiplicity free action with fundamental highest weights and highest weight vectors

$$\left\{\begin{array}{l|l} \alpha_1 = -(\varepsilon_1 + \varepsilon_\circ) & h_1(z) = z_{11} \\ \alpha_2 = -(\varepsilon_1 + \varepsilon_{\circ\circ}) & h_2(z) = z_{12} \\ \alpha_3 = -(\varepsilon_1 + \varepsilon_2 + \varepsilon_\circ + \varepsilon_{\circ\circ}) & h_3(z) = \det_2(z) \\ \alpha_4 = -(\varepsilon_\circ + \varepsilon_{\circ\circ}) & h_4(z) = \omega(z_{\bullet,1}, z_{\bullet,2}) \end{array}\right\}.$$

For non-negative integer exponents $h_\alpha = h_1^a h_2^b h_3^c h_4^d$ has weight

$$\begin{aligned}\alpha &= a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 \\ &= -\big((a+b+c)\varepsilon_1 + c\varepsilon_2 + (a+c+d)\varepsilon_\circ + (b+c+d)\varepsilon_{\circ\circ}\big).\end{aligned}$$

We may take $n = 2$, $K = Sp(4)\times\mathbb{T}\times\mathbb{T}$, $V = M_{4,2}(\mathbb{C})$ here. The Lemma 2.2 equations on the entries in a spherical point $z \in M_{4,2}(\mathbb{C})$ for weight α read

$$\left\{\begin{array}{l|l|l} \langle z_{1,\bullet}, z_{3,\bullet}\rangle = 0 = \langle z_{2,\bullet}, z_{4,\bullet}\rangle & \langle z_{1,\bullet}, z_{2,\bullet}\rangle = \langle z_{4,\bullet}, z_{3,\bullet}\rangle & \langle z_{1,\bullet}, z_{4,\bullet}\rangle = -\langle z_{2,\bullet}, z_{3,\bullet}\rangle \\ \|z_{1,\bullet}\|^2 - \|z_{3,\bullet}\|^2 = a+b+c & \|z_{2,\bullet}\|^2 - \|z_{4,\bullet}\|^2 = c & \\ \|z_{\bullet,1}\| = a+c+d & \|z_{\bullet,2}\|^2 = b+c+d & \end{array}\right\}.$$

One generic generalized spherical point whose matrix entries solve these equations for arbitrary positive values of the parameters is

$$v(a,b,c,d) := \begin{bmatrix} -\sqrt{\frac{a(a+b+c)(a+b+2c+d)}{(a+b)(a+b+2c)}} & -\sqrt{\frac{b(a+b+c)(a+b+2c+d)}{(a+b)(a+b+2c)}} \\ \sqrt{\frac{bc(a+b+2c+d)}{(a+b)(a+b+2c)}} & -\sqrt{\frac{ac(a+b+2c+d)}{(a+b)(a+b+2c)}} \\ -\sqrt{\frac{bd(a+b+c)}{(a+b)(a+b+2c)}} & \sqrt{\frac{ad(a+b+c)}{(a+b)(a+b+2c)}} \\ -\sqrt{\frac{acd}{(a+b)(a+b+2c)}} & -\sqrt{\frac{bcd}{(a+b)(a+b+2c)}} \end{bmatrix}$$

and the fundamental highest weight vectors take values

$$\left\{\begin{array}{ll} h_1(v) = -\sqrt{\frac{a(a+b+c)(a+b+2c+d)}{(a+b)(a+b+2c)}} & h_2(v) = -\sqrt{\frac{b(a+b+c)(a+b+2c+d)}{(a+b)(a+b+2c)}} \\ h_3(v) = \frac{(a+b+2c+d)\sqrt{c(a+b+c)}}{a+b+2c} & h_4(v) = -\sqrt{d(a+b+2c+d)} \end{array}\right\}$$

at $v = v(a,b,c,d)$. Thus Lemma 2.5 condition (2) holds and it is not difficult to check conditions (3) and (4) for each of $2^4 - 2 = 14$ relevant limits.

5.9. **Spin(8) $\oplus_{\mathbf{Spin(8)}}$ SO(8).** The compact group $Spin(8)$ has three inequivalent irreducible representations of dimension eight, the natural representation via $SO(8)$ and two half spin representations. Consider $Spin(8)$ acting on a sixteen dimensional space via the direct sum of two of these representations. As the triality automorphism permutes the eight dimensional irreducibles (see [10, §20.3]) it makes no difference, up to geometric equivalence, which pair are used. It is convenient to choose the two half spin representations.

The positive and negative half spin representations, $\sigma_\pm$ say, can be realized in

$$\Lambda^{even}(\mathbb{C}^4) = \Lambda^0(\mathbb{C}^4) \oplus \Lambda^2(\mathbb{C}^4) \oplus \Lambda^4(\mathbb{C}^4) \quad \text{and} \quad \Lambda^{odd}(\mathbb{C}^4) = \Lambda^1(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4)$$

respectively. We take $V = V_1 \oplus V_2 = \Lambda^{even}(\mathbb{C}^4) \oplus \Lambda^{odd}(\mathbb{C}^4) = \Lambda(\mathbb{C}^4)$ and let $K = Spin(8) \times \mathbb{T} \times \mathbb{T}$ act via

$$(k, c_1, c_2) \cdot (v_1, v_2) = \big(c_1\sigma_+(k)(v_1), c_2\sigma_-(k)(v_2)\big) \qquad \big((v_1, v_2) \in V_1 \oplus V_2\big).$$

We adopt some notation from [2, Section 4.7]. $V = \Lambda(\mathbb{C}^4) = \sum_{j=0}^4 \Lambda^j(\mathbb{C}^4)$ carries its usual Hermitian inner product and letting $e_{j_1\cdots j_k} = e_{j_1} \wedge \cdots \wedge e_{j_k}$ denotes a wedge product of standard basis vectors in $\mathbb{C}^4$,

$$\mathcal{B} = \{1,\, e_1,\, e_2,\, e_3,\, e_4,\, e_{12},\, e_{13},\, e_{14},\, e_{23},\, e_{24},\, e_{34},\, e_{234},\, e_{134},\, e_{124},\, e_{123},\, e_{1234}\}$$

is an orthonormal basis. We write

$$\big(z_\emptyset,\, z_1,\, z_2,\, z_3,\, z_4,\, z_{12},\, z_{13},\, z_{14},\, z_{23},\, z_{24},\, z_{34},\, z_{234},\, z_{134},\, z_{124},\, z_{123},\, z_{1234}\big)$$

for coordinates with respect to $\mathcal{B}$. The compact group $Spin(8)$ acts unitarily on V via $\sigma_+ \oplus \sigma_-$. The image of the derived representation of $so(8)$ in $u(V)$ is given explicitly in [11, Chapter 3] and elsewhere. Complexifying yields a copy of $so(8, \mathbb{C})$ inside $gl(V)$. This is the $\mathbb{C}$-span of the 28 operators

$$\left\{ \begin{array}{c} H_k = \frac{1}{2}(D_kW_k - W_kD_k)\ (1 \le k \le 4), \\ W_kD_\ell\ (1 \le k \ne \ell \le 4), \quad W_kW_\ell\ (1 \le k < \ell \le 4), \quad D_kD_\ell\ (1 \le k < \ell \le 4) \end{array} \right\}.$$

Here W_k is the operator $W_k(v) = e_k \wedge v$ and D_k its adjoint, contraction by e_k. As Cartan subalgebra $\mathfrak{h}_8$ and Borel subalgebra $\mathfrak{b}_8 = \mathfrak{h}_8 \oplus \mathfrak{n}_8$ in this copy of $so(8, \mathbb{C})$ we take $\mathfrak{h}_8 = \mathbb{C}\text{-Span}\{H_1, \ldots, H_4\}$,

$$\mathfrak{n}_8 = \mathbb{C}\text{-Span}\Big(\{W_kD_\ell \ :\ 1 \le k < \ell \le 4\} \cup \{W_kW_\ell \ :\ 1 \le k < \ell \le 4\}\Big).$$

K has complexified Lie algebra $\mathfrak{k}_\mathbb{C} = so(8, \mathbb{C}) \times \mathbb{C} \times \mathbb{C}$ with Cartan and Borel subalgebras $\mathfrak{h} = \mathfrak{h}_8 \times \mathbb{C} \times \mathbb{C}$, $\mathfrak{b} = \mathfrak{b}_8 \times \mathbb{C} \times \mathbb{C}$. Let $T_1 = (0, 1, 0) \in \mathfrak{h}$, $T_2 = (0, 0, 1) \in \mathfrak{h}$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_\circ, \varepsilon_{\circ\circ} \in \mathfrak{h}^*$ denote the functionals

$$\left\{ \begin{array}{c} \varepsilon_j\big(a_1H_1 + \cdots + a_4H_4 + b_1T_1 + b_2T_2\big) = a_j \\ \varepsilon_\circ\big(a_1H_1 + \cdots + a_4H_4 + b_1T_1 + b_2T_2\big) = b_1 \\ \varepsilon_{\circ\circ}\big(a_1H_1 + \cdots + a_4H_4 + b_1T_1 + b_2T_2\big) = b_2 \end{array} \right\}.$$

$K : V$ is a rank 5 multiplicity free action with fundamental highest weights and highest weight vectors

$$\left\{ \begin{array}{l|l} \alpha_1 = -\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)-\varepsilon_\circ & h_1(z) = z_\emptyset \\ \alpha_2 = -\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4)-\varepsilon_{\circ\circ} & h_1(z) = z_4 \\ \alpha_3 = -2\varepsilon_\circ & h_3(z) = z_\emptyset z_{234} - z_{12}z_{34} + z_{13}z_{24} - z_{14}z_{23} \\ \alpha_4 = -2\varepsilon_{\circ\circ} & h_4(z) = z_1 z_{234} - z_2 z_{134} + z_3 z_{124} - z_4 z_{123} \\ \alpha_5 = -(\varepsilon_1+\varepsilon_\circ+\varepsilon_{\circ\circ}) & h_5(z) = z_2 z_{34} - z_3 z_{24} + z_4 z_{23} - z_\emptyset z_{234} \end{array} \right\}.$$

For non-negative integer exponents $h_\alpha = h_1^a h_2^b h_3^c h_4^d h_5^e$ is a highest weight vector in $\mathbb{C}[V]$ with weight

$$\begin{aligned} \alpha &= a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + e\alpha_5 \\ &= -\Bigg[\left(\frac{1}{2}a+\frac{1}{2}b+e\right)\varepsilon_1 + \left(\frac{1}{2}a+\frac{1}{2}b\right)\varepsilon_2 + \left(\frac{1}{2}a+\frac{1}{2}b\right)\varepsilon_3 + \left(\frac{1}{2}a-\frac{1}{2}b\right)\varepsilon_4 \\ &\qquad + (a+2c+e)\varepsilon_\circ + (b+2d+e)\varepsilon_{\circ\circ}\Bigg]. \end{aligned}$$

Lemma 2.2 gives a system of 18 equations for the coordinates $(z_\emptyset, \ldots, z_{1234})$ of a spherical point for weight α. These are obtained by letting X in (2.1) range over the basis given above for $\mathfrak{b}_8$ together with T_1 and T_2. Numerical experimentation with a computer algebra system reveals that this system has generic solutions in which eight of the coordinates vanish, namely $z_2, z_3, z_{12}, z_{13}, z_{24}, z_{34}, z_{134}, z_{124}$. Setting these coordinate variables to zero reduces the system to the following eight equations in the remaining eight variables.

$$\left\{ \begin{array}{c} z_4\overline{z_1} + z_{234}\overline{z_{123}} = 0 \\ z_\emptyset\overline{z_{14}} + z_{23}\overline{z_{1234}} = 0 \\ z_\emptyset\overline{z_{23}} + z_4\overline{z_{234}} + z_1\overline{z_{123}} + z_{14}\overline{z_{1234}} = 0 \\ \|z_\emptyset\|^2 - \|z_1\|^2 + \|z_4\|^2 - \|z_{14}\|^2 + \|z_{23}\|^2 + \|z_{234}\|^2 - \|z_{123}\|^2 - \|z_{1234}\|^2 = a+b+2\,e \\ \|z_\emptyset\|^2 + \|z_1\|^2 + \|z_4\|^2 + \|z_{14}\|^2 - \|z_{23}\|^2 - \|z_{234}\|^2 - \|z_{123}\|^2 - \|z_{1234}\|^2 = a+b \\ \|z_\emptyset\|^2 + \|z_1\|^2 - \|z_4\|^2 - \|z_{14}\|^2 + \|z_{23}\|^2 - \|z_{234}\|^2 + \|z_{123}\|^2 - \|z_{1234}\|^2 = a-b \\ \|z_\emptyset\|^2 + \|z_{14}\|^2 + \|z_{23}\|^2 + \|z_{1234}\|^2 = a+2\,c+e \\ \|z_1\|^2 + \|z_4\|^2 + \|z_{234}\|^2 + \|z_{123}\|^2 = b+2\,d+e \end{array} \right\}.$$

These arise by taking $X = W_1D_4$, W_1W_4, W_2W_3, H_1, H_2, H_4, T_1, T_2 in (2.1). One can check that

$$\begin{aligned} v(a,b,c,d,e) := &\sqrt{\frac{a(a+b+e)(a+c+e)}{(a+b)(a+e)}}\;1 + \sqrt{\frac{ade}{(a+b)(b+e)}}\;e_1 \\ &+ \sqrt{\frac{b(a+b+e)(b+d+e)}{(a+b)(b+e)}}\;e_4 - \sqrt{\frac{bce}{(a+b)(a+e)}}\;e_{14} + \sqrt{\frac{be\,(a+c+e)}{(a+b)(a+e)}}\;e_{23} \end{aligned}$$

$$- \sqrt{\frac{ae\,(b+d+e)}{(a+b)(b+e)}}\; e_{234} + \sqrt{\frac{bd\,(a+b+e)}{(a+b)(b+e)}}\; e_{123} + \sqrt{\frac{ac\,(a+b+e)}{(a+b)(a+e)}}\; e_{1234}$$

is a generic generalized spherical point solving the above system for arbitrary positive parameter values. Evaluating the fundamental highest weight vectors at $v = v(a,b,c,d,e)$ gives

$$\left\{ \begin{array}{cc} h_1(v) = \sqrt{\frac{a(a+b+e)(a+c+e)}{(a+b)(a+e)}} & h_2(v) = \sqrt{\frac{b(a+b+e)(b+d+e)}{(a+b)(b+e)}} \\ h_3(v) = \sqrt{c(a+c+e)} & h_4(v) = -\sqrt{d(b+d+e)} \\ \multicolumn{2}{c}{h_5(v) = \sqrt{\frac{e(a+b+e)(a+c+e)(b+d+e)}{(a+e)(b+e)}}} \end{array} \right\},$$

which verifies condition (2) in Lemma 2.5. Maple was used to carry out the routine calculations required to verify the two remaining conditions.

6. Case-by-case analysis: Variable rank examples

The last four entries in Table 1 are infinite families of multiplicity free actions with increasing ranks. Lemma 2.5 will be used to show that these are well-behaved. We will discuss these actions together as their spherical points are closely related. Proofs will be provided for the first of these examples. Proofs for the remaining examples are similar, and are omitted for brevity. We begin by tabulating basic data concerning these examples.

6.1. $\mathbf{SU(n)} \oplus_{\mathbf{SU(n)}} (\mathbf{SU(n)} \otimes \mathbf{SU(m)})$ (Table 2). The group $K = U(n) \times U(m)$ acts on $V = V_1 \oplus V_2 = \mathbb{C}^n \oplus M_{n,m}(\mathbb{C})$ as $(k_1, k_2) \cdot (\xi, z') = \big(k_1\xi, k_1 z' k_2^t\big)$. We identify V with $M_{n,m+1}(\mathbb{C})$ by adjoining the column vector ξ to z',

$$(\xi, z') \leftrightarrow z = \big[\xi | z'\big],$$

and will number the columns of $z \in V$ by 0 through m. By embedding $U(m)$ in $U(m+1)$ as $\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & U(m) \end{array}\right]$ the action of K is realized by restriction of the usual action of $U(n) \times U(m+1)$ on $M_{n,m+1}(\mathbb{C})$. The standard Hermitian inner product on $V = M_{n,m+1}(\mathbb{C})$ is K-invariant. Fundamental highest weights, associated highest weight vectors and the set Λ of all highest weights that occur in $\mathbb{C}[V]$ are listed in Table 2. Here we use Borel subgroup $B = B_n \times B_m$ and let ε_j, ε'_j denote functionals on $\mathfrak{h} = \mathfrak{h}_n \times \mathfrak{h}_m$ as in (5.1) supported on the two factors. We write $\det_j$ for the determinant of the first j rows and columns of a matrix. Action $K : V$ has rank $2n$ when $m \geq n$ and rank $2m+1$ when $m < n$. For purposes of verifying that these actions are well-behaved we assume henceforth that either $m = n$ or $m = n-1$. The final entry in Table 2 gives criteria, derived from Lemma 2.2, for a matrix of size $n \times (m+1)$ to be a spherical point for weight $-\Big(\sum_j \lambda_j \varepsilon_j + \sum_j \mu_j \varepsilon'_j\Big) \in \Lambda$.

Group	$U(n) \times U(m) \subset U(n) \times U(m+1),\ n, m \geq 2$
Vector Space	$\mathbb{C}^n \oplus M_{n,m}(\mathbb{C}) \cong M_{n,m+1}(\mathbb{C}),$ rows $1 \ldots n,$ columns $0 \ldots m$
Action	$(k_1, k_2) \cdot z = k_1 z k_2^t$
Rank	$2n$ when $m \geq n$; $2m+1$ when $m < n$
Cases	$m = n,\ m = n-1$
Fundamental Highest Weights	$-\big(\varepsilon_1 + \cdots + \varepsilon_j + \varepsilon_1' + \cdots + \varepsilon_j'\big) \quad (1 \leq j \leq \min(n, m))$ $-\big(\varepsilon_1 + \cdots + \varepsilon_j + \varepsilon_1' + \cdots + \varepsilon_{j-1}'\big) \quad (1 \leq j \leq \min(n, m+1))$
Fundamental H. W. Vectors	$h_j'(z) = \det_j(z') \quad (1 \leq j \leq \min(n, m))$ $h_j(z) = \det_j(z) \quad (1 \leq j \leq \min(n, m+1))$
Spectrum Λ	$-\Big(\sum_j \lambda_j \varepsilon_j + \sum_j \mu_j \varepsilon_j'\Big);\ \lambda_j, \mu_j \in \mathbb{Z},$ $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n \geq \mu_n \geq 0,\ m = n$ $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n \geq 0,\ m = n-1$
Spherical Points	rows are pairwise orthogonal with norms $\lambda_1^{1/2}, \ldots, \lambda_n^{1/2}$ columns 1 through m are pairwise orthogonal with norms $\mu_1^{1/2}, \ldots, \mu_m^{1/2}$

TABLE 2. Data for $\mathbf{SU(n)} \oplus_{\mathbf{SU(n)}} (\mathbf{SU(n)} \otimes \mathbf{SU(m)})$

6.2. $\mathbf{SU(n)^*} \oplus_{\mathbf{SU(n)}} (\mathbf{SU(n)} \otimes \mathbf{SU(m)})$ **(Table 3).** This is a twisted variant of Example 6.1 with $K = U(n) \times U(m)$ acting on $V = V_1 \oplus V_2 = \mathbb{C}^n \oplus M_{n,m}(\mathbb{C})$ via $(k_1, k_2) \cdot (\xi, z') = \big(k_1^{-t}\xi, k_1 z' k_2^t\big)$. We identify V with $M_{n,m+1}(\mathbb{C})$ as in the previous example and number the columns of matrices $z \in V$ by 0 through m. Table 3 lists relevant data for this action. In the formula for highest weight vector $\widetilde{h}_j(z) = \det_j(\widetilde{z})$ the matrix $\widetilde{z} \in M_{n+1,m}(\mathbb{C})$ is defined as

$$\widetilde{z} := \left[\begin{array}{ccc} \xi^t z'_{\bullet,1} & \cdots & \xi^t z'_{\bullet,m} \\ \hline & z' & \end{array} \right] \quad \text{for} \quad z = \left[\ \xi \mid z'\ \right].$$

Entries in the first row of $\widetilde{z}$ are dot products of ξ with the columns of z'. Note that $\widetilde{h}_j$ is a polynomial of degree $j+1$. Thus the number and degrees of the fundamental highest weight vectors agree with those in the untwisted example. For purposes of verifying that these actions are well-behaved it suffices to assume that either $m = n$ or $m = n-1$.

6.3. $\mathbf{SU(n)} \oplus_{\mathbf{SU(n)}} \mathbf{\Lambda^2}(\mathbf{SU(n)})$ **(Table 4).** Identifying $\Lambda^2(\mathbb{C}^n)$ with the space $Skew(n, \mathbb{C})$ of $n \times n$ skew symmetric matrices the group $K = U(n)$ acts on $V = V_1 \oplus V_2 = \mathbb{C}^n \oplus Skew(n, \mathbb{C})$ via $k \cdot (\xi, z') = \big(k\xi, kz'k^t\big)$. We further identify V with $Skew(n+1, \mathbb{C})$ via the isomorphism $(\xi, z') \leftrightarrow \left[\begin{array}{c|c} 0 & \xi^t \\ \hline -\xi & z' \end{array} \right]$ and number rows and columns as 0 through n. In this model the action of $U(n)$ on $V_1 \oplus V_2$ is realized as a restriction of the usual action of $U(n+1)$ on $Skew(n+1, \mathbb{C})$. The space $V = Skew(n+1, \mathbb{C})$ carries the

Group	$U(n)\times U(m),\ n\geq 3, m\geq 2$
Vector Space	$(\mathbb{C}^n)^*\oplus M_{n,m}(\mathbb{C})\cong\mathbb{C}^n\times M_{n,m}(\mathbb{C})\cong M_{n,m+1}(\mathbb{C})$ rows $1\ldots n$, columns $0\ldots m$
Action	$(k_1,k_2)\cdot(\xi,z')=(k_1^{-t}\xi,k_1z'k_2^t)\quad(k_1^{-t}:=(k_1^t)^{-1})$
Rank	$2n$ when $m\geq n$; $2m+1$ when $m<n$
Cases	$m=n,\ m=n-1$
Fundamental Highest Weights	$+\varepsilon_n$ $-\big(\varepsilon_1+\cdots+\varepsilon_j+\varepsilon'_1+\cdots+\varepsilon'_j\big)\quad(1\leq j\leq\min(n,m))$ $-\big(\varepsilon_1+\cdots+\varepsilon_j+\varepsilon'_1+\cdots+\varepsilon'_{j-1}\big)\quad(1\leq j\leq\min(n-1,m))$
Fundamental H. W. Vectors	$h_\circ(z)=\xi_n$ $h'_j(z)=\det_j(z')\quad(1\leq j\leq\min(n,m))$ $\widetilde{h}_j(z)=\det_j(\widetilde{z})\quad(1\leq j\leq\min(n-1,m))$
Spectrum Λ	$-\left(\sum_j\lambda_j\varepsilon_j+\sum_j\mu_j\varepsilon'_j\right);\ \lambda_j,\mu_j\in\mathbb{Z}$ $\mu_1\geq\lambda_1\geq\mu_2\geq\lambda_2\geq\cdots\geq\mu_n\geq\lambda_n,\ \mu_n\geq 0,\ m=n$ $\mu_1\geq\lambda_1\geq\mu_2\geq\lambda_2\geq\cdots\geq\mu_{n-1}\geq\lambda_{n-1}\geq 0\geq\lambda_n,\ m=n-1$
Spherical Points	$-z_{i,0}\overline{z_{j,0}}+\sum_{k=1}^m z_{j,k}\overline{z_{i,k}}=0,\ 1\leq i\neq j\leq n$ $-\lvert z_{i,0}\rvert^2+\sum_{k=1}^m\lvert z_{i,k}\rvert^2=\lambda_i^2,\ 1\leq i\leq n$ columns 1 through m are pairwise orthogonal with norms $\mu_1^{1/2},\ldots,\mu_m^{1/2}$

TABLE 3. Data for $\mathbf{SU(n)}^*\oplus_{\mathbf{SU(n)}}(\mathbf{SU(n)}\otimes\mathbf{SU(m)})$

K-invariant Hermitian inner product

$$\langle z,w\rangle=\frac{1}{2}tr(zw^*)=\sum_{i<j}z_{i,j}\overline{w_{i,j}}.$$

Table 4 summarizes data for this action. Here Pf_j denotes the Pfaffian of the first $2j$ rows and columns of a skew symmetric matrix and $z'\in Skew(n,\mathbb{C})$ denotes the last n rows and columns of z. In our subsequent analysis for this example we will distinguish the cases n even and n odd.

Remark 6.1. In contrast to all previous examples this action fails to be fully saturated since the scalars in $U(n)$ act diagonally. This is none-the-less a multiplicity free action [3, Theorem 6]. In view of Lemma 2.4 we choose to work with this non-saturated action. This remark applies equally to Example 6.4 which follows.

6.4. $\mathbf{SU(n)}^*\oplus_{\mathbf{SU(n)}}\mathbf{\Lambda^2(SU(n))}$ **(Table 5).** This is the twisted variant of Example 6.3 with $K=U(n)$ acting on $V=V_1\oplus V_2=\mathbb{C}^n\oplus Skew(n,\mathbb{C})$ via $k\cdot(\xi,z')=\big(k^{-t}\xi,kz'k^t\big)$. As in the previous example we identify V with $Skew(n+1,\mathbb{C})$ and number rows and columns as 0 through n. Explicitly we have

$$k\cdot z=\left[\begin{array}{c|c}0&\xi^tk^{-1}\\\hline -k^{-t}\xi&kz'k^t\end{array}\right]\quad\text{for}\quad k\in U(n),\ z=\left[\begin{array}{c|c}0&\xi^t\\\hline -\xi&z'\end{array}\right]\in Skew(n+1,\mathbb{C}).$$

Group	$U(n),\ n \geq 4$
Vector Space	$\mathbb{C}^n \oplus \Lambda^2(\mathbb{C}^n) \cong \Lambda^2(\mathbb{C}^{n+1}) \cong Skew(n+1,\mathbb{C})$ rows and columns $0\ldots n$
Action	$k\cdot(\xi,z') = \big(k\xi, kz'k^t\big)$
Rank	n
Cases	$n=2m,\ n=2m-1$
Fundamental Highest Weights	$-\big(\varepsilon_1+\cdots+\varepsilon_{2j}\big) \quad (1\leq j\leq \lfloor n/2\rfloor)$ $-\big(\varepsilon_1+\cdots+\varepsilon_{2j-1}\big) \quad (1\leq j\leq \lfloor (n+1)/2\rfloor)$
Fundamental H. W. Vectors	$h'_j(z) = Pf_j(z') \quad (1\leq j\leq \lfloor n/2\rfloor)$ $h_j(z) = Pf_j(z) \quad (1\leq j\leq \lfloor (n+1)/2\rfloor)$
Spectrum Λ	$-\sum_{i=1}^n c_i\varepsilon_i\ :\ c_i\in\mathbb{Z},\ c_1\geq c_2\geq\cdots\geq c_n\geq 0$
Spherical Points	rows 1 through n are pairwise orthogonal with norms $c_1^{1/2},\ldots,c_n^{1/2}$

TABLE 4. Data for $\mathbf{SU(n)} \oplus_{\mathbf{SU(n)}} \mathbf{\Lambda^2(SU(n))}$

In Table 5 for this example matrix $\widetilde{z} \in Skew(n+1,\mathbb{C})$ is defined as

$$\widetilde{z} := \left[\begin{array}{c|ccc} 0 & \xi^t z'_{\bullet,1} & \cdots & \xi^t z'_{\bullet,n} \\ \hline z'_{1,\bullet}\xi & & & \\ \vdots & & z' & \\ z'_{n,\bullet}\xi & & & \end{array}\right] \quad \text{for} \quad z = \left[\begin{array}{c|c} 0 & \xi^t \\ \hline -\xi & z' \end{array}\right].$$

Entries in the first row of $\widetilde{z}$ are dot products of ξ with the columns $z'_{\bullet,j}$ of z'.

6.5. **Spherical points.** Generalized generic spherical points are given below for each of Examples 6.1-6.4. Full justification for the spherical point formulas will be provided for Example 6.1. Given sequences of distinct real parameters $\lambda_1,\ldots,\lambda_N$ and $\mu_1,\ldots\mu_N$ we set, for $1\leq i,j\leq N$,

$$\text{(6.1)}\quad \mathbf{z}_{i,0} = \left|\frac{\prod_{k=1}^n(\lambda_i-\mu_k)}{\prod_{k\neq i}(\lambda_i-\lambda_k)}\right|^{1/2} \quad \text{and} \quad \mathbf{z}_{i,j} = \left|\frac{\mu_j\prod_{k\neq j}(\lambda_i-\mu_k)\prod_{k\neq i}(\mu_j-\lambda_k)}{\prod_{k\neq i}(\lambda_i-\lambda_k)\prod_{k\neq j}(\mu_j-\mu_k)}\right|^{1/2}.$$

6.5.1. $\mathbf{SU(n)} \oplus_{\mathbf{SU(n)}} (\mathbf{SU(n)}\otimes\mathbf{SU(m)})$ (**Table 2**). We must consider the cases $m=n$ and $m=n-1$.

For $m=n$, generic weights in Λ are indexed by (integral) parameters $\boldsymbol{\lambda},\boldsymbol{\mu}$ with $\lambda_1>\mu_1>\lambda_2>\mu_2>\cdots>\lambda_{n-1}>\mu_{n-1}>\lambda_n>\mu_n>0$. Let $v=v(\boldsymbol{\lambda},\boldsymbol{\mu})\in V$ be the matrix with entries indexed by $(1\leq i\leq n,\ 0\leq j\leq n)$ given by $\mathbf{z}_{i,0}$ and $sign(i,j)\mathbf{z}_{i,j}$ for $j\geq 1$, where

$$\text{(6.2)}\qquad sgn(i,j) = \begin{cases} -1 & \text{if } i>j \\ +1 & \text{if } i\leq j \end{cases},$$

Group	$U(n),\ n \geq 4$
Vector Space	$(\mathbb{C}^n)^* \oplus \Lambda^2(\mathbb{C}^n) \cong \Lambda^2(\mathbb{C}^{n+1}) \cong Skew(n+1,\mathbb{C})$ rows and columns $0 \ldots n$
Action	$k \cdot (\xi, z') = \left(k^{-t}\xi, kz'k^t\right)$
Rank	n
Cases	$n = 2m,\ n = 2m-1$
Fundamental Highest Weights	$+\varepsilon_n$ $-\left(\varepsilon_1 + \cdots + \varepsilon_{2j}\right) \quad (1 \leq j \leq \lfloor n/2 \rfloor)$ $-\left(\varepsilon_1 + \cdots + \varepsilon_{2j-1}\right) \quad (1 \leq j \leq \lfloor (n-1)/2 \rfloor)$
Fundamental H. W. Vectors	$h_\circ(z) = \xi_n, \quad h'_j(z) = Pf_j(z'), \quad \widetilde{h}_j(z) = Pf_j(\widetilde{z})$
Spectrum Λ	$-\sum_{i=1}^n c_i\varepsilon_i,\ c_i \in \mathbb{Z}$ $c_1 \geq c_2 \geq \cdots \geq c_n$ $c_{n-1} \geq 0$ and $0 \geq c_n$ for n odd
Spherical Points	$-\lvert\xi_i\rvert^2 + \sum_{k=1}^n \lvert z'_{i,k}\rvert^2 = c_i^2,\ (1 \leq i \leq n)$ $-\xi_i\overline{\xi_j} + \sum_{k=1}^n z'_{j,k}\overline{z'_{i,k}} = 0,\ (1 \leq i \neq j \leq n)$

TABLE 5. Data for $\mathbf{SU(n)}^* \oplus_{\mathbf{SU(n)}} \mathbf{\Lambda^2(SU(n))}$

and $\mathbf{z}_{i,0}$ and $\mathbf{z}_{i,j}$ are as in Equations 6.1. It is show below that this is a generic generalized spherical point for this example. That is, the rows of $v(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are pairwise orthogonal with norms $\lambda_1^{1/2}, \ldots, \lambda_n^{1/2}$ and columns 1 through n are pairwise orthogonal with norms $\mu_1^{1/2}, \ldots, \mu_n^{1/2}$.

For $m = n-1$, generic weights in Λ are indexed by parameters $\boldsymbol{\lambda}, \boldsymbol{\mu}$ with $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n > 0$. Let $v(\boldsymbol{\lambda}, \boldsymbol{\mu})$ be obtained by setting $\mu_n = 0$ in the formulas for the case $m = n$ discussed above and deleting the last column. This gives a generic generalized spherical point for the case $m = n-1$. All of the identities needed to confirm this may also be derived by setting $\mu_n = 0$ in the arguments for the case $m = n$. (See Section 6.7.) One obtains all lower-dimensional examples by successively setting $\mu_n, \lambda_{n-1}, \mu_{n-1}, \ldots$ equal to zero and deleting a row or column.

6.5.2. $\mathbf{SU(n)}^* \oplus_{\mathbf{SU(n)}} (\mathbf{SU(n)} \otimes \mathbf{SU(m)})$ (**Table 3**). For $m = n$, generic weights in Λ are indexed by parameters $\boldsymbol{\lambda}, \boldsymbol{\mu}$ with $\mu_1 > \lambda_1 > \mu_2 > \lambda_2 > \cdots > \mu_n > \lambda_n$ and $\mu_n \geq 0$. Let $v(\boldsymbol{\lambda}, \boldsymbol{\mu})$ be the matrix with entries given by $v_{i,0} = \mathbf{z}_{i,0}$ and $v_{i,j} = sign(i,j)\mathbf{z}_{i,j}$ for $j \geq 1$ as in Equations 6.1 but where now

$$sgn(i,j) = \begin{cases} -1 & \text{if } i \geq j \\ +1 & \text{if } i < j \end{cases} . \tag{6.3}$$

This gives a generic generalized spherical point.

For $m = n-1$, generic weights in Λ are indexed by parameters $\boldsymbol{\lambda}, \boldsymbol{\mu}$ with $\mu_1 > \lambda_1 > \mu_2 > \lambda_2 > \cdots > \mu_{n-1} > \lambda_{n-1} > \lambda_n$ and $\lambda_{n-1} > 0 > \lambda_n$. One can obtain a generic generalized spherical point for this data by setting $\mu_n = 0$ in the formulas for the case $m = n$ discussed above and deleting the last column. One obtains all lower-dimensional examples by successively setting $\mu_n, \lambda_{n-1}, \mu_{n-1}, \ldots$ equal to zero and deleting a row or column.

6.5.3. $\mathbf{SU(n)} \oplus_{\mathbf{SU(n)}} \mathbf{\Lambda^2(SU(n))}$ (**Table 4**). Generic weights in Λ are indexed by strictly decreasing (integral) sequences $c_1 > c_2 > \cdots > c_n > 0$. First suppose that n is even, $n = 2m$ say, and let $\lambda_j = c_{2j-1}$, $\mu_j = c_{2j}$, so that $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 \cdots > \lambda_m > \mu_m > 0$. A generic generalized spherical point $v(\boldsymbol{\lambda}, \boldsymbol{\mu})$ for such data is the skew symmetric matrix with entries indexed by $0 \le i \le n,\ 0 \le j \le n$, defined as follows for $i < j$:

- $v_{i,j} = 0$ if i and j have equal parity.
- Non-zero entries on row 0 are

$$v_{0,2j-1} = \mathbf{z}_{j,0} \quad \text{for } 1 \le j \le m. \tag{6.4}$$

- Below row 0 one has

$$v_{2i,2j-1} = \mathbf{z}_{j,i} \tag{6.5}$$

  for $1 \le i < j \le m$
- and

$$v_{2i-1,2j} = \mathbf{z}_{i,j} \tag{6.6}$$

  for $1 \le i \le j \le m$.

Setting $\mu_m = 0$ in formulas (6.4-6.6) and deleting the last row and column produces the spherical point for the case $n = 2m-1$ with data $(\lambda_1, \ldots, \lambda_m; \mu_1, \ldots, \mu_{m-1})$.

6.5.4. $\mathbf{SU(n)^*} \oplus_{\mathbf{SU(n)}} \mathbf{\Lambda^2(SU(n))}$ (**Table 5**). For $n = 2m$, we have parameters $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ with $\mu_1 > \lambda_1 > \cdots > \mu_m > \lambda_m$ and $\mu_m \ge 0$. Let $v = v(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in Skew(n+1, \mathbb{C})$ have entries $z_{i,j}$ defined as follows for $i < j$:

- In row 0 we have $v_{0,2j-1} = 0$ and

$$v_{0,2j} = \mathbf{z}_{j,0} \quad \text{for } 1 \le j \le m. \tag{6.7}$$

- $z_{i,j} = 0$ for $i \ge 2$ if i and j have equal parity and

$$v_{2i,2j-1} = \mathbf{z}_{i,j} \tag{6.8}$$

  for $1 \le i < j \le m$
- and

$$v_{2i-1,2j} = \mathbf{z}_{j,i} \tag{6.9}$$

  for $1 \le i \le j \le m$.

This is a generic generalized spherical point for the data $(\boldsymbol{\lambda}, \boldsymbol{\mu})$.

For $n = 2m-1$ generic weights are indexed by data $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ with $\mu_1 > \lambda_1 > \cdots > \mu_{m-1} > \lambda_{m-1} > \lambda_m$ and $\lambda_{m-1} > 0 > \lambda_m$. Form the generic generalized spherical point for the case $n = 2m$ as above with data $(\lambda_1, \ldots, \lambda_m;\ \mu_1, \ldots, \mu_{m-1}, 0)$ and delete the *second* last row and column. This gives a generic generalized spherical point for the case $n = 2m-1$.

6.6. **A combinatorial lemma.** To justify the formulas given above for generic generalized spherical points in Examples 6.1-6.4 we make extensive use of the following lemma.

Lemma 6.2. *Let $p(x) = \sum_{j=0}^{n-1} p_j x^j$ be a polynomial of degree at most $n-1$ and $a_1, \ldots a_n$ be distinct real numbers. Then*

$$\sum_{i=1}^{n} \frac{p(a_i)}{\prod_{k \neq i}(a_i - a_k)} = p_{n-1}.$$

Proof. For non-negative integers j let

$$Q_j(\mathbf{a}) := \begin{vmatrix} a_1^j & a_1^{n-2} & \ldots & a_1 & 1 \\ a_2^j & a_2^{n-2} & \ldots & a_2 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n^j & a_n^{n-2} & \ldots & a_n & 1 \end{vmatrix}, \tag{6.10}$$

so that $Q_j(\mathbf{a}) = 0$ for $j < n-1$ and $Q_{n-1}(\mathbf{a})$ is the Vandermonde determinant,

$$Q_{n-1}(\mathbf{a}) = V_n(\mathbf{a}) = \prod_{k<\ell}(a_k - a_\ell).$$

Expanding along the first column, we obtain

$$Q_j(\mathbf{a}) = \sum_{i=1}^{n}(-1)^{i-1} a_i^j V_{n-1}(\widehat{\mathbf{a}}_i) = \sum_{i=1}^{n}(-1)^{i-1} a_i^j \prod_{\substack{k<\ell \\ k,\ell \neq i}} (a_k - a_\ell)$$

where $V_{n-1}(\widehat{\mathbf{a}}_i)$ is the $(n-1) \times (n-1)$ Vandermonde determinant obtained by eliminating a_i. Hence also

$$\frac{Q_j(\mathbf{a})}{V_n(\mathbf{a})} = \sum_{i=1}^{n}(-1)^{i-1} \frac{a_i^j}{\prod_{k<i}(a_k - a_i) \prod_{k>i}(a_i - a_k)} = \sum_{i=1}^{n} \frac{a_i^j}{\prod_{k \neq i}(a_i - a_k)}.$$

So

$$\sum_{i=1}^{n} \frac{p(a_i)}{\prod_{k \neq i}(a_i - a_k)} = \sum_{i=1}^{n} \sum_{j=0}^{n-1} \frac{p_j a_i^j}{\prod_{k \neq i}(a_i - a_k)} = \sum_{j=0}^{n-1} p_j \frac{Q_j(\mathbf{a})}{V_n(\mathbf{a})} = \sum_{j=0}^{n-1} p_j \delta_{j,n-1} = p_{n-1}$$

as claimed. □

Remark 6.3. Taking $j > n-1$ in (6.10) the quotient $Q_j(\mathbf{a})/V_n(\mathbf{a})$ becomes the Schur function $s_{(j-n+1,0,\dots,0)}(\mathbf{a})$. This coincides with a complete symmetric function, explicitly

$$\frac{Q_j(\mathbf{a})}{V_n(\mathbf{a})} = \sum_{|\boldsymbol{k}|=j-n+1} a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}$$

for $j > n-1$. (See, for example, [16, §1.15].)

6.7. **Justification of the spherical point formulas.** We concentrate on Example 6.1 (see Table 2) with $m = n$, as proofs for all other examples are similar. For the matrix $v(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in M_{n,n+1}(\mathbb{C})$ given in (6.5.1) we must verify that

- the rows are pairwise orthogonal with norms $\lambda_1^{1/2}, \dots, \lambda_n^{1/2}$ and
- columns 1 through n are pairwise orthogonal with norms $\mu_1^{1/2}, \dots, \mu_n^{1/2}$.

6.7.1. *Row norms.* First consider the polynomial

$$E(x) := \prod_k (x - \mu_k) + \sum_j \frac{\mu_j \prod_{k\neq j}(x-\mu_k) \prod_{k\geq 2}(\mu_j - \lambda_k)}{\prod_{k\neq j}(\mu_j - \mu_k)}$$

Setting $x = \lambda_2$, we get

$$\begin{aligned}
E(\lambda_2) &= \prod_k (\lambda_2 - \mu_k) + \sum_j \frac{\mu_j \prod_{k\neq j}(\lambda_2 - \mu_k)(\mu_j - \lambda_2) \prod_{k\geq 3}(\mu_j - \lambda_k)}{\prod_{k\neq j}(\mu_j - \mu_k)} \\
&= \prod_k (\lambda_2 - \mu_k) - \sum_j \frac{\mu_j \prod_k (\lambda_2 - \mu_k) \prod_{k\geq 3}(\mu_j - \mu_k)}{\prod_{k\neq j}(\mu_j - \mu_k)} \\
&= \prod_k (\lambda_2 - \mu_k) \left[1 - \sum_j \frac{\mu_j \prod_{k\geq 3}(\mu_j - \lambda_k)}{\prod_{k\neq j}(\mu_j - \mu_k)} \right] \\
&= 0
\end{aligned}$$

by Lemma 6.2. Likewise, by symmetry, $E(\lambda_k) = 0$ for all $k \geq 2$. Also,

$$\begin{aligned}
E(0) &= \prod_k (-\mu_k) + \sum_j \frac{\mu_j \prod_{k\neq j}(-\mu_k) \prod_{k\geq 2}(\mu_j - \lambda_l)}{\prod_{k\neq j}(\mu_j - \mu_k)} \\
&= \prod_k (-\mu_k) \left[1 - \sum_j \frac{\prod_{k\geq 2}(\mu_j - \lambda_k)}{\prod_{k\neq j}(\mu_j - \mu_k)} \right] \\
&= 0.
\end{aligned}$$

Thus $E(x)$ has zeros at $0, \lambda_2, \dots \lambda_n$. Since the highest order term is x^n, we conclude that in fact

$$E(x) = x \prod_{k\geq 2} (x - \lambda_k).$$

One can check that, for Example 6.1, the expressions inside the absolute values in Equations 6.1 are positive. Thus the first row of $v = v(\boldsymbol{\lambda}, \boldsymbol{\mu})$ satisfies

$$\begin{aligned}\|v_{1,\bullet}\|^2 = v_{1,0}^2 + \sum_j v_{1,j}^2 &= \frac{\prod_k(\lambda_1 - \mu_k)}{\prod_{k\geq 2}(\lambda_1 - \lambda_k)} + \sum_j \frac{\mu_j \prod_{k\neq j}(\lambda_1 - \mu_k)\prod_{k\geq 2}(\mu_j - \lambda_k)}{\prod_{k\geq 2}(\lambda_1 - \lambda_k)\prod_{k\neq j}(\mu_j - \mu_k)} \\ &= \frac{1}{\prod_{k\geq 2}(\lambda_1 - \lambda_k)} E(\lambda_1) \\ &= \lambda_1.\end{aligned}$$

A similar argument shows that $\|v_{i,\bullet}\|^2 = \lambda_i$ for $2 \leq i \leq n$. □

6.7.2. *Column norms.* An argument similar to that used above for the row norms shows that

$$E(x) := \sum_i \frac{\prod_{k\neq j}(\lambda_i - \mu_k)\prod_{k\neq i}(x - \lambda_k)}{\prod_{k\neq i}(\lambda_i - \lambda_k)} = \prod_{k\neq j}(x - \mu_k).$$

So now

$$\begin{aligned}\|v_{\bullet,j}\|^2 = \sum_i v_{i,j}^2 &= \sum_i \frac{\mu_j \prod_{k\neq j}(\lambda_i - \mu_k)\prod_{k\neq i}(\mu_j - \lambda_k)}{\prod_{k\neq i}(\lambda_i - \lambda_k)\prod_{k\neq j}(\mu_j - \mu_k)} \\ &= \frac{\mu_j}{\prod_{k\neq j}(\mu_j - \mu_l)} \sum_i \frac{\prod_{k\neq j}(\lambda_i - \mu_k)\prod_{k\neq i}(\mu_j - \lambda_k)}{\prod_{k\neq i}(\lambda_i - \lambda_k)} \\ &= \frac{\mu_j}{\prod_{k\neq j}(\mu_j - \mu_l)} E(\mu_j) \\ &= \mu_j\end{aligned}$$

as required. □

6.7.3. *Row and column inner products.* Next we will show that the rows and columns of $v = v(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are pair-wise orthogonal. For $1 \leq a < b \leq n$ one has

$$\begin{aligned}\langle v_{a,\bullet}, v_{b,\bullet}\rangle &= v_{a,0}v_{b,0} + \sum_j v_{a,j}v_{b,j} \\ &= \sqrt{\left|\frac{\prod_k(\lambda_a - \mu_k)}{\prod_{k\neq a}(\lambda_a - \lambda_k)}\frac{\prod_k(\lambda_b - \mu_k)}{\prod_{k\neq b}(\lambda_b - \lambda_k)}\right|} \\ &\quad + \sum_j sgn(a,j)sgn(b,j)\sqrt{\left|\frac{\prod_k(\lambda_a - \mu_k)}{\prod_{k\neq a}(\lambda_a - \lambda_k)}\frac{\prod_k(\lambda_b - \mu_k)}{\prod_{k\neq b}(\lambda_b - \lambda_k)}\right|} \\ &\qquad \times \left|\frac{\mu_j \prod_k(\mu_j - \lambda_k)}{(\lambda_a - \mu_j)(\lambda_b - \mu_j)\prod_{k\neq j}(\mu_j - \mu_l)}\right|\end{aligned}$$

$$= \sqrt{\left| \frac{\prod_k(\lambda_a - \mu_k)}{\prod_{k\neq a}(\lambda_a - \lambda_k)} \frac{\prod_k(\lambda_b - \mu_k)}{\prod_{k\neq b}(\lambda_b - \lambda_k)} \right|}$$
$$\times \left(1 + \sum_j sgn(a,j)sgn(b,j) \left| \frac{\mu_j \prod_{k\neq a,b}(\mu_j - \lambda_k)}{\prod_{k\neq j}(\mu_j - \mu_l)} \right| \right).$$

For $a < b$, the signs have been chosen so that

$$sgn(a,j)sgn(b,j) \frac{\mu_j \prod_{k\neq a,b}(\mu_j - \lambda_k)}{\prod_{k\neq j}(\mu_j - \mu_k)} < 0.$$

Hence

$$1 + \sum_j sgn(a,j)sgn(b,j) \left| \frac{\mu_j \prod_{k\neq a,b}(\mu_j - \lambda_k)}{\prod_{k\neq j}(\mu_j - \mu_l)} \right| = 1 - \sum_j \frac{\mu_j \prod_{k\neq a,b}(\mu_j - \lambda_l)}{\prod_{k\neq j}(\mu_j - \mu_l)} = 0,$$

by another application of Lemma 6.2, and thus $\langle v_{a,\bullet}, v_{b,\bullet}\rangle = 0$ as required. The proof that $\langle v_{\bullet,a}, v_{\bullet,b}\rangle = 0$ for $1 \leq a < b \leq n$ is similar. □

6.8. **Evaluation of the highest weight vectors.** Lemma 2.5 will be used to show that Examples 6.1-6.4 are well-behaved. We continue to focus on Example 6.1 as the calculations for the remaining examples are similar. First suppose that $m = n$. The fundamental highest weight vectors are $h'_r(z) = \det_r(z')$ and $h_r(z) = \det_r(z)$ $(1 \leq r \leq n)$. Below we evaluate these, up to sign, at the generic generalized spherical point $z = v(\boldsymbol{\lambda}, \boldsymbol{\mu})$ to show that $h'_r\big(v(\boldsymbol{\lambda}, \boldsymbol{\mu})\big) \neq 0 \neq h_r\big(v(\boldsymbol{\lambda}, \boldsymbol{\mu})\big)$, as required by Lemma 2.5 condition (2).

6.8.1. *Calculation of* $\det_r(z')$ *for* $z = v(\boldsymbol{\lambda}, \boldsymbol{\mu})$ *in case* $m = n$. Pulling common factors from the first r rows and columns of z' gives

$$\det_r(z') = \det_r \left[sgn(i,j) \sqrt{\left| \frac{\mu_j \prod_{k\neq j}(\lambda_i - \mu_k) \prod_{k\neq i}(\mu_j - \lambda_k)}{\prod_{k\neq i}(\lambda_i - \lambda_k) \prod_{k\neq j}(\mu_j - \mu_k)} \right|} \right]$$
$$= \sqrt{\left| \frac{\prod_{j\leq r} \mu_j}{\prod_{1\leq i\leq r,\ 1\leq k\leq n,\ k\neq i}(\lambda_i - \lambda_k) \prod_{1\leq j\leq r,\ 1\leq k\leq n,\ k\neq j}(\mu_j - \mu_k)} \right|}$$
$$\times \det_r \left[sgn(i,j) \sqrt{\left| \frac{\prod_k(\lambda_i - \mu_k) \prod_k(\mu_j - \lambda_k)}{(\lambda_i - \mu_j)(\mu_j - \lambda_i)} \right|} \right]$$
$$= \frac{\prod_{j\leq r} \sqrt{\mu_j} \prod_{i,k\leq r} |\lambda_i - \mu_k|}{\prod_{i<k\leq r} |\lambda_i - \lambda_k| \prod_{j<k\leq r} |\mu_j - \mu_k|}$$

$$\times \sqrt{\left|\frac{\prod_{i\le r<k}(\lambda_i-\mu_k)\prod_{j\le r<k}(\mu_j-\lambda_k)}{\prod_{i\le r<k}(\lambda_i-\lambda_k)\prod_{j\le r<k}(\mu_j-\mu_k)}\right|}\det_r\left[sgn(i,j)\left|\frac{1}{\lambda_i-\mu_j}\right|\right]$$

Since $sgn(i,j)$ and $(\lambda_i-\mu_j)$ have opposite signs one has

$$\det_r\left[sgn(i,j)\left|\frac{1}{\lambda_i-\mu_j}\right|\right]=(-1)^r\det_r\left[\frac{1}{\lambda_i-\mu_j}\right]$$

where $\det_r\big[1/(\lambda_i-\mu_j)\big]$ is a *Cauchy determinant*. As is well-kown [16, page 397],

$$\det_r\left[\frac{1}{\lambda_i-\mu_j}\right]=\frac{\prod_{a<b\le r}(\lambda_b-\lambda_a)(\mu_a-\mu_b)}{\prod_{a,b\le r}(\lambda_a-\mu_b)}.$$

So now

$$\big|h_r'\big(v(\boldsymbol{\lambda},\boldsymbol{\mu})\big)\big|=\prod_{j\le r}\sqrt{\mu_j}\sqrt{\left|\frac{\prod_{i\le r<k}(\lambda_i-\mu_k)\prod_{j\le r<k}(\mu_j-\lambda_k)}{\prod_{i\le r<k}(\lambda_i-\lambda_k)\prod_{j\le r<k}(\mu_j-\mu_k)}\right|}. \tag{6.11}$$

Remark 6.4. Thus, in particular, $\big|h_n'\big(v(\boldsymbol{\lambda},\boldsymbol{\mu})\big)\big|=\sqrt{\mu_1\cdots\mu_n}$. In fact this is clear since z' is an orthogonal matrix whose jth column has norm $\sqrt{\mu_j}$.

6.8.2. *Calculation of* $\det_r(z)$ *for* $z=v(\boldsymbol{\lambda},\boldsymbol{\mu})$ *in case* $m=n$. Pulling common factors from the first r rows and columns of z gives

$$\begin{aligned}\det_r(z)&=\det_r\left[\;\sqrt{\left|\tfrac{\prod_{k=1}^n(\lambda_i-\mu_k)}{\prod_{k\ne i}(\lambda_i-\lambda_k)}\right|}\;\middle|\;\left[sgn(i,j)\sqrt{\left|\tfrac{\mu_j\prod_{k\ne j}(\lambda_i-\mu_k)\prod_{k\ne i}(\mu_j-\lambda_k)}{\prod_{k\ne i}(\lambda_i-\lambda_k)\prod_{k\ne j}(\mu_j-\mu_k)}\right|}\right]_{\substack{1\le i\le r\\1\le j\le r-1}}\;\right]\\&=\frac{\prod_{j\le r-1}\sqrt{\mu_j}\prod_{1\le i\le r,\,1\le k\le r-1}|\lambda_i-\mu_k|}{\prod_{i<k\le r}|\lambda_i-\lambda_k|\prod_{j<k\le r-1}|\mu_j-\mu_k|}\\&\qquad\times\sqrt{\left|\frac{\prod_{i\le r\le k}(\lambda_i-\mu_k)\prod_{j<r<k}(\mu_j-\lambda_k)}{\prod_{i\le r<k}(\lambda_i-\lambda_k)\prod_{j<r\le k}(\mu_j-\mu_k)}\right|}\det(w)\end{aligned}$$

where w is the $r\times r$ matrix

$$w=\left[\begin{array}{c|c}1&\\\vdots&\left[\frac{sgn(i,j)}{|\lambda_i-\mu_j|}\right]_{\substack{1\le i\le r\\1\le j\le r-1}}\\1&\end{array}\right]=\left[\begin{array}{c|c}1&\\\vdots&\left[\frac{-1}{\lambda_i-\mu_j}\right]_{\substack{1\le i\le r\\1\le j\le r-1}}\\1&\end{array}\right].$$

Subtracting the r'th row of w from the first $r-1$ rows reduces the calculation of $\det(w)$ to that for a Cauchy determinant. Explicitly

$$\det(w) = (-1)^{r-1}\begin{vmatrix} \begin{matrix}1\\ \vdots\\ 1\end{matrix} & \left[\frac{1}{\lambda_i-\mu_j}\right]_{\substack{1\le i\le r\\ 1\le j\le r-1}}\end{vmatrix} = (-1)^{r-1}\left|\begin{array}{c|c} \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \left[\frac{\lambda_r-\lambda_i}{(\lambda_i-\mu_j)(\lambda_r-\mu_j)}\right]_{1\le i,j\le r-1} \\ \hline 1 & \frac{1}{\lambda_r-\mu_1} \quad \cdots \quad \frac{1}{\lambda_r-\mu_{r-1}}\end{array}\right|$$

$$= \det\left(\left[\frac{\lambda_r-\lambda_i}{(\lambda_i-\mu_j)(\lambda_r-\mu_j)}\right]_{1\le i,j\le r-1}\right)$$

$$= \frac{\prod_{1\le i\le r-1}(\lambda_r-\lambda_i)}{\prod_{1\le j\le r-1}(\lambda_r-\mu_j)}\det_{r-1}\left[\frac{1}{\lambda_i-\mu_j}\right]$$

$$= \frac{\prod_{1\le i\le r-1}(\lambda_r-\lambda_i)}{\prod_{1\le j\le r-1}(\lambda_r-\mu_j)}\times\frac{\prod_{a<b\le r-1}(\lambda_b-\lambda_a)(\mu_a-\mu_b)}{\prod_{a,b\le r-1}(\lambda_a-\mu_b)}$$

$$= \frac{\prod_{a<b\le r}(\lambda_b-\lambda_a)\prod_{a<b\le r-1}(\mu_a-\mu_b)}{\prod_{1\le a\le r,\,1\le b\le r-1}(\lambda_a-\mu_b)}$$

Thus we obtain

$$\left|h_r\big(v(\boldsymbol{\lambda},\boldsymbol{\mu})\big)\right| = \prod_{j<r}\sqrt{\mu_j}\sqrt{\left|\frac{\prod_{i\le r\le k}(\lambda_i-\mu_k)\prod_{j<r<k}(\mu_j-\lambda_k)}{\prod_{i\le r<k}(\lambda_i-\lambda_k)\prod_{j<r\le k}(\mu_j-\mu_k)}\right|}. \tag{6.12}$$

6.8.3. *Calculation of* $\det_r(z)$ *and* $\det_r(z')$ *for* $z = v(\boldsymbol{\lambda},\boldsymbol{\mu})$ *in case* $m = n-1$. Next consider Example 6.1 with $m = n-1$. We have fundamental highest weight vectors $h_r(z) = \det_r(z)$ for $1 \le r \le n$ and $h_r(z') = \det_r(z')$ for $1 \le r \le n-1$. Values for these at the generic generalized spherical point $z = v(\boldsymbol{\lambda},\boldsymbol{\mu})$ may be obtained by setting $\mu_n = 0$ in Equations 6.12 and 6.11 respectively. This gives

$$\left|h_r\big(v(\boldsymbol{\lambda},\boldsymbol{\mu})\big)\right| = \prod_{i\le r}\sqrt{\lambda_i}\sqrt{\left|\frac{\prod_{i\le r\le k}(\lambda_i-\mu_k)\prod_{j<r<k}(\mu_j-\lambda_k)}{\prod_{i\le r<k}(\lambda_i-\lambda_k)\prod_{j<r\le k}(\mu_j-\mu_k)}\right|}, \tag{6.13}$$

$$\left|h'_r\big(v(\boldsymbol{\lambda},\boldsymbol{\mu})\big)\right| = \prod_{i\le r}\sqrt{\lambda_i}\sqrt{\left|\frac{\prod_{i\le r<k}(\lambda_i-\mu_k)\prod_{j\le r<k}(\mu_j-\lambda_k)}{\prod_{i\le r<k}(\lambda_i-\lambda_k)\prod_{j\le r<k}(\mu_j-\mu_k)}\right|}. \tag{6.14}$$

6.9. **Limit conditions.** Finally we verify conditions (3) and (4) from Lemma 2.5 for Example 6.1 with $m = n$. (See Table 2.) This works by induction on n and m. For non-negative integer exponents a_j, b_j the polynomial $h_\alpha = h_1^{a_1}\cdots h_n^{a_n}(h'_1)^{b_1}\cdots(h'_n)^{b_n}$ is a highest weight vector in $\mathbb{C}[V]$ with weight

$$\alpha = -\left(\sum_{i=1}^n \lambda_i\varepsilon_i + \sum_{i=1}^n \mu_i\varepsilon'_i\right)$$

where $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \cdots \geq \lambda_n \geq \mu_n \geq 0$ are given by

$$\lambda_i = \sum_{j \geq i} a_j + \sum_{j \geq i} b_j, \qquad \mu_i = \sum_{j \geq i+1} a_j + \sum_{j \geq i} b_j. \tag{6.15}$$

The highest weight vector h_α has all exponents positive if and only if the weight coefficients satisfy $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 \cdots > \lambda_n > \mu_n > 0$. Let $v(\boldsymbol{\lambda}, \boldsymbol{\mu})$ be the generic generalized spherical point from (6.5.1) with $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ determined by Equations 6.15 and positive real parameters $(\mathbf{a}, \mathbf{b})$.

Since $\mu_n = b_n$ the limit as $b_n \to 0$ is just the limit as $\mu_n \to 0$. As we have already discussed, when the last parameter $\mu_n \to 0$, the last column of $v(\boldsymbol{\lambda}, \boldsymbol{\mu})$ becomes zero and the remaining matrix is a spherical point for the case $m = n - 1$. Thus this limit exists. The limiting values for $h_r\big(v(\boldsymbol{\lambda}, \boldsymbol{\mu})\big)$ $(1 \leq r \leq n)$ and $h'_r\big(v(\boldsymbol{\lambda}, \boldsymbol{\mu})\big)$ $(1 \leq r \leq n-1)$ are given, up to sign, by Equations 6.13 and 6.14 respectively. In particular, these limiting values are non-zero as required by Lemma 2.5 condition (4).

Other limits as parameters a_i, b_i approach zero are equivalent to two adjacent parameters merging in $(\boldsymbol{\lambda}, \boldsymbol{\mu})$. Indeed $a_i \to 0$ $(1 \leq i \leq n)$ corresponds to $\lambda_i \to \mu_i$ and $b_i \to 0$ $(1 \leq i \leq n-1)$ to $\mu_i \to \lambda_{i+1}$. Taking, for example, the limit as $a_1 \to 0$ one obtains

$$\lim_{a_1 \to 0} v(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \lim_{\lambda_1 \to \mu_1} v(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \begin{bmatrix} 0 & \sqrt{\mu_1} & 0 \\ \zeta_0 & 0 & \zeta \end{bmatrix},$$

where $[\zeta_0|\zeta] = v\big(\lambda_2, \ldots, \lambda_n;\, \mu_2, \ldots, \mu_n\big) \in M_{n-1,n}(\mathbb{C})$ is a generic generalized spherical point for Example 6.1 with n and m reduced by one and data $\lambda_2 > \mu_2 > \ldots > \mu_n > 0$. In particular, this limit exists. Moreover $\lim_{a_1 \to 0} h'_1\big(v(\boldsymbol{\lambda}, \boldsymbol{\mu})\big) = \sqrt{\mu_1} \neq 0$,

$$\lim_{a_1 \to 0} h'_r\big(v(\boldsymbol{\lambda}, \boldsymbol{\mu})\big) = \sqrt{\mu_1}\, \det_{r-1}(\zeta) = \sqrt{\mu_1}\, h'_{r-1}\big([\zeta_0|\zeta]\big) \neq 0$$

for $2 \leq r \leq n$ and

$$\lim_{a_1 \to 0} h_r\big(v(\boldsymbol{\lambda}, \boldsymbol{\mu})\big) = -\sqrt{\mu_1}\, h_{r-1}\big([\zeta_0|\zeta]\big) \neq 0$$

for $2 \leq r \leq n$, as required. Similarly one has

$$\lim_{b_1 \to 0} v(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \lim_{\mu_1 \to \lambda_2} v(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \begin{bmatrix} z^\circ_{1,0} & 0 & \zeta_1^t \\ 0 & -\sqrt{\lambda_2} & 0 \\ \zeta_0 & 0 & \zeta \end{bmatrix}$$

where $\left[\begin{array}{c|c} z^\circ_{1,0} & \zeta_1^t \\ \hline \zeta_0 & \zeta \end{array}\right] = v\big(\lambda_1, \lambda_3, \ldots, \lambda_n;\, \mu_2, \ldots, \mu_n\big) \in M_{n-1,n}(\mathbb{C})$ is a generic generalized spherical point for Example 6.1 with n and m reduced by one. Here

$$\lim_{b_1 \to 0} h'_r\big(v(\boldsymbol{\lambda}, \boldsymbol{\mu})\big) = \sqrt{\lambda_2}\, h'_{r-1}\left(\left[\begin{array}{c|c} z^\circ_{1,0} & \zeta_1^t \\ \hline \zeta_0 & \zeta \end{array}\right]\right) \neq 0$$

for $2 \le r \le n$, $\lim_{b_1\to 0} h_1\big(v(\boldsymbol{\lambda},\boldsymbol{\mu})\big) = z^\circ_{1,0} \neq 0$ and

$$\lim_{b_1\to 0} h_r\big(v(\boldsymbol{\lambda},\boldsymbol{\mu})\big) = -\sqrt{\lambda_2}\, h'_{r-1}\left(\left[\begin{array}{c|c} z^\circ_{1,0} & \zeta_1^t \\ \hline \zeta_0 & \zeta \end{array}\right]\right) \neq 0$$

for $2 \le r \le n$. Limits as $a_i \to 0$ for $2 \le i \le n$ and $b_i \to 0$ for $2 \le i \le n-1$ behave in a similar manner.

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Dept of Mathematics, East Carolina University, Greenville, NC 27858

E-mail address: bensonf@ecu.edu, ratcliffg@ecu.edu