SPACES OF BOUNDED SPHERICAL FUNCTIONS ON HEISENBERG GROUPS: PART I

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Abstract. Consider a linear multiplicity free action by a compact Lie group K on a finite dimensional hermitian vector space V. Letting K act on the associated Heisenberg group $H_V = V \times \mathbb{R}$ yields a Gelfand pair. In previous work we have applied the Orbit Method to produce an injective mapping Ψ from the space $\Delta(K, H_V)$ of bounded K-spherical functions on H_V to the space $\mathfrak{h}_V^*/K$ of K-orbits in the dual of the Lie algebra for H_V. We have shown that Ψ is a homeomorphism onto its image provided that $K : V$ is a "well-behaved" multiplicity free action. In this paper we prove that $K : V$ is well-behaved whenever K acts irreducibly on V. Thus if $K : V$ is an irreducible multiplicity free action then $\Psi : \Delta(K, H_V) \to \mathfrak{h}_V^*/K$ is a homeomorphism onto its image. Our proof involves case-by-case analysis working from the classification of irreducible multiplicity free actions. A sequel to this paper will extend these results to encompass non-irreducible actions.

1. Introduction

Suppose that K is a compact Lie group acting smoothly on a nilpotent Lie group N via automorphisms. One says that (K, N) is a *Gelfand pair* when the convolution algebra $L^1_K(N)$ of integrable K-invariant functions on N is commutative. In this case the spectrum, or Gelfand space, for $L^1_K(N)$ coincides, via integration, with the set $\Delta(K, N)$ of bounded K-spherical functions on N endowed with the compact-open topology. In [3] we used the *Orbit Method* to produce an injective mapping

$$\Psi : \Delta(K, N) \to \mathfrak{n}^*/K$$

from $\Delta(K, N)$ to the set of K-orbits in the dual of the Lie algebra for N. Giving n^*/K the quotient topology we have conjectured the following.

Conjecture 1.1. *The map Ψ is a homeomorphism onto its image.*

This is established in [3] for pairs with N abelian, for the action of the unitary group on the Heisenberg group, and for the action of the orthogonal group on the free 2-step group.

Our paper [5] concerns Conjecture 1.1 for Gelfand pairs associated with Heisenberg groups. We continue this line of investigation here. Throughout, $V \cong \mathbb{C}^n$ will be a

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finite dimensional complex vector space with Hermitian inner product $\langle\cdot,\cdot\rangle$. The associated Heisenberg group is

$$H_V = V \times \mathbb{R} \quad \text{with product} \quad (z,t)(z',t') = \left(z+z',\ t+t' - \frac{1}{2} Im\,\langle z,z'\rangle\right).$$

The unitary group $U(V)$ acts by automorphisms on H_V via

$$k \cdot (z,t) = (kz,t),$$

as a maximal compact connected subgroup of $Aut(H_V)$. We assume that K is a compact Lie group acting on $(V,\langle\cdot,\cdot\rangle)$ by some unitary representation to yield a Gelfand pair (K,H_V), and denote this action as $K : V$. It is well known that (K,H_V) is a Gelfand pair if and only if $K : V$ is a linear *multiplicity free action*. That is, if and only if the associated representation of K in the space $\mathbb{C}[V]$ of holomorphic polynomial functions on V, namely

$$(k \cdot p)(z) = p(k^{-1} \cdot z),$$

is multiplicity free [6]. The papers [12], [2] and [13] classify all such multiplicity free actions.

Conjecture 1.1 is established in [5] for Gelfand pairs (K,H_V) subject to a technical condition.

Theorem 1.2. [5] *If the multiplicity free action $K : V$ is well-behaved then*

$$\Psi : \Delta(K,H_V) \to \mathfrak{h}_V^*/K$$

is a homeomorphism onto its image.

The requirement that a multiplicity free action $K : V$ be *well-behaved* is made precise in Definition 2.2 below. Here and in Part II of this work we will prove that, in fact:

Theorem 1.3. *Every linear multiplicity free action $K : V$ is well-behaved.*

Together Theorems 1.2 and 1.3 prove Conjecture 1.1 for arbitrary Gelfand pairs associated with Heisenberg groups.

Corollary 1.4. *For all Gelfand pairs (K,H_V) the map $\Psi : \Delta(K,H_V) \to \mathfrak{h}_V^*/K$ is a homeomorphism onto its image.*

In this paper we consider only those multiplicity free actions $K : V$ in which the representation of K on V is irreducible. Our main result is as follows.

Theorem 1.5. *Every* irreducible *multiplicity free action is well-behaved.*

Our proof for Theorem 1.5 involves case-by-case analysis working from Kac's classification of irreducible multiplicity free actions [12]. In Part II of this paper we will complete the proof for Theorem 1.3. This requires the study of *indecomposable* but non-irreducible multiplicity free actions, classified in [2] and [13].

The rest of this paper is organized as follows. In Section 2 we introduce notation and the concept of well-behaved multiplicity free actions. Section 3 concerns background and preliminary results on such actions. The proof of Theorem 1.5 is given in Section 4 via case-by-case analysis working from the classification in [12]. In principle, as explained below in subsection 3.7, this analysis renders the orbital models for $\Delta(K, H_V)$ explicit in each case. We made extensive use of a computer algebra system (Maple) to facilitate the analyses of several cases. Section 5 provides further detail on our computer-aided calculations.

2. WELL-BEHAVED MULTIPLICITY FREE ACTIONS

As in the previous section V will denote a finite dimensional complex vector space with Hermitian inner product $\langle \cdot, \cdot \rangle$ and K a compact Lie group acting on $(V, \langle \cdot, \cdot \rangle)$ by some unitary representation. We assume that $K : V$ is a multiplicity free action and write $k \cdot v$ and $A \cdot v$ for the result of applying elements $k \in K$ and $A \in \mathfrak{k} := \mathrm{Lie}(K)$ to $v \in V$. Fixing notation, let

- $T \subset K$ denote a maximal torus in K with Lie algebra $\mathfrak{t} \subset \mathfrak{k}$,
- $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}}$ the resulting Cartan subalgebra in $\mathfrak{k}_{\mathbb{C}}$,
- H the corresponding subgroup in the complexified group $K_{\mathbb{C}}$,
- $B := HN$ a fixed Borel subgroup in $K_{\mathbb{C}}$ with Lie algebra $\mathfrak{b} \subset \mathfrak{k}_{\mathbb{C}}$ and
- $\Lambda \subset \mathfrak{h}^*$ the set of B-highest weights for irreducible representations of $K_{\mathbb{C}}$ (or equivalently of K) occurring in $\mathbb{C}[V]$.
- Moreover, we write $P_\alpha \subset \mathbb{C}[V]$ for the unique irreducible subspace with highest weight $\alpha \in \Lambda$. So
$$\mathbb{C}[V] = \bigoplus_{\alpha \in \Lambda} P_\alpha$$
is the canonical decomposition of $\mathbb{C}[V]$ into irreducible subspaces for the actions of $K_{\mathbb{C}}$ and K.
- Finally, for each $\alpha \in \Lambda$ choose $h_\alpha \in P_\alpha$, a B-highest weight vector (unique modulo $\mathbb{C}^\times$).

An element $\alpha \in \Lambda$ is said to be a *fundamental* highest weight for $K : V$ when h_α is an irreducible polynomial. The fundamental highest weights form a finite $\mathbb{Q}$-linearly independent set
$$\{\alpha_1, \ldots, \alpha_r\}$$
which freely generates Λ as an additive semigroup [10, page 571]. The value r is called the *rank* of the multiplicity free action $K : V$.

As in [1, 3, 5] we use a version of the *Orbit Method* for compact Lie groups to associate a coadjoint orbit $\mathcal{O}_\alpha$ in $\mathfrak{k}^*$ to each irreducible subspace P_α in the decomposition of $\mathbb{C}[V]$. Note that the weight $\alpha \in \Lambda$ takes pure imaginary values on $\mathfrak{t}$. We extend the real valued functional $(1/i)\alpha$ from $\mathfrak{t}$ to all of $\mathfrak{k}$ as follows: Fix an $Ad(K)$-invariant inner product $(\cdot|\cdot)$ on the Lie algebra $\mathfrak{k}$ and let $\mathfrak{t}^\perp$ denote the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{k}$ with respect to $(\cdot|\cdot)$. We let

- $\alpha_\mathfrak{k} \in \mathfrak{k}^*$ be the (real valued) linear functional on $\mathfrak{k}$ satisfying
$$\alpha_\mathfrak{k}(A) = \begin{cases} -i\alpha(A) & \text{if } A \in \mathfrak{t} \\ 0 & \text{if } A \in \mathfrak{t}^\perp \end{cases},$$
and set
- $\mathcal{O}_\alpha = Ad^*(K)\alpha_\mathfrak{k}$.

The unnormalized *moment map* $\tau : V \to \mathfrak{k}^*$ for the action $K : V$ is given by the formula [15]
$$\tau(v)(A) := i\,\langle A \cdot v, v\rangle\,.$$
Note that $\tau(v)$ takes real values because $\mathfrak{k}$ acts on $(V, \langle\cdot,\cdot\rangle)$ by skew-hermitian operators. The moment map intertwines the action of the group K on V with its coadjoint action on $\mathfrak{k}^*$. Hence τ maps K-orbits in V to $Ad^*(K)$-orbits in $\mathfrak{k}^*$. Moreover as $K : V$ is a multiplicity free action it is known that

- τ is one-to-one on K-orbits ([1, Theorem 1,3], [7]), and
- each coadjoint orbit $\mathcal{O}_\alpha$ ($\alpha \in \Lambda$) lies in the image of τ ([1, Proposition 4.1]).

Definition 2.1. [5] The *spherical orbit* $\mathcal{K}_\alpha \in V/K$ for $\alpha \in \Lambda$ is the unique K-orbit in V satisfying $\tau\big(\mathcal{K}_\alpha\big) = \mathcal{O}_\alpha$. One has
$$\mathcal{K}_\alpha = K \cdot v_\alpha \text{ for some } v_\alpha \in V \text{ with } \tau(v_\alpha) = \alpha_\mathfrak{k}.$$
We call any such point $v_\alpha \in V$ a *spherical point* for α.

For vectors $w \in V$ and polynomials $h \in \mathbb{C}[V]$ we let $\partial_w h$ denote the directional derivative
$$\big(\partial_w h\big)(z) := \lim_{t\to 0} \frac{h(z+tw) - h(z)}{t}$$
and make the following definition.

Definition 2.2. [5] Given $\alpha \in \Lambda$ we say that a spherical point v_α for α is *well-adapted to* h_α when the following conditions hold.

(i) $h_\alpha(v_\alpha) \neq 0$, and
(ii) $\big(\partial_w h_\alpha\big)(v_\alpha) = \langle w, v_\alpha\rangle\, h_\alpha(v_\alpha)$ for all $w \in V$.

We say that the multiplicity free action $K : V$ is *well-behaved* if for every $\alpha \in \Lambda$ one can choose a spherical point v_α well-adapted to h_α.

Note the following points as regards these definitions.

- The highest weight $\alpha = 0$ occurs in $\mathbb{C}[V]$ on the constant polynomials. The zero vector $v_0 = 0$ in V is clearly the unique spherical point for this weight and is well-adapted to the highest weight vector $h_0 = 1$.

- If v_α is a spherical point for $\alpha \in \Lambda$ then so is cv_α for any scalar $c \in \mathbb{C}$ of modulus $|c| = 1$. Likewise for each $k \in T$ the point $k \cdot v_\alpha$ is a spherical point for α. Thus when $\alpha \neq 0$ spherical points are non-unique.
- If v_α is a well-adapted to h_α then so are cv_α ($|c| = 1$) and $k \cdot v_\alpha$ ($k \in T$).
- If v_α is well-adapted to h_α then v_α is also well-adapted to any non-zero scalar multiple ch_α ($c \in \mathbb{C}^\times$).

Example 2.3. To clarify Definitions 2.1 and 2.2 consider the most basic example, namely $K = U(n)$ acting on $V = \mathbb{C}^n$ via its defining representation. The space $\mathbb{C}[V]$ decomposes as

$$\mathbb{C}[V] = \bigoplus_{m \geq 0} \mathcal{P}_m(V)$$

where $\mathcal{P}_m(V)$ denotes the space of polynomials homogeneous of degree m. Let T be the usual maximal torus of diagonal matrices in K. Now $\mathfrak{h} = \mathfrak{t}_\mathbb{C}$ is the algebra of diagonal matrices in $\mathfrak{k}_\mathbb{C} = gl(n, \mathbb{C})$ and we take as $B = HN$ the subgroup of lower triangular matrices in $K_\mathbb{C} = GL(n, \mathbb{C})$. With these conventions the highest weight for the representation of K on $\mathcal{P}_m(V)$ is $\alpha = -m\varepsilon_1$, where $\varepsilon_1 \in \mathfrak{h}^*$ is the linear functional

$$\varepsilon_1\big(diag(a_1, \ldots, a_n)\big) = a_1,$$

and the polynomial

$$h_m(z) = z_1^m$$

is a highest weight vector in $\mathcal{P}_m(V)$. Thus we have $\Lambda = \{-m\varepsilon_1 \;:\; m \geq 0\}$ and $K : V$ is a rank 1 multiplicity free action with fundamental highest weight $\alpha_1 = -\varepsilon_1$.

A spherical point for weight $-m\varepsilon_1 \in \Lambda$ is given by

$$v_m = \sqrt{m}\, e_1 = \big(\sqrt{m}, 0, \ldots, 0\big).$$

Indeed, for $A \in (\mathfrak{k} = \mathfrak{u}(n))$ one has

$$\tau(v_m)(A) = i\,\langle Av_m, v_m \rangle = ima_{11},$$

and, in particular, for $A = diag(i\theta_1, \ldots, i\theta_n) \in \mathfrak{t}$ this gives

$$\tau(v_m)(A) = -m\theta_1 = \big(-m\varepsilon_1\big)_\mathfrak{k}(A).$$

Now we verify that each v_m is well-adapted to h_m, so that $K : V$ is a well-behaved multiplicity free action.

(i) First note that $h_m(v_m) = m^{m/2} \neq 0$.

(ii) We compute

$$\big(\partial_1 h_m\big)(v_m) = m\, m^{(n-1)/2} = m^{1/2}\, m^{m/2} = \langle e_1, v_m \rangle\, h_m(v_m)$$

and $\partial_j h_m = 0 = \langle e_j, v_m \rangle$ for $j \geq 2$.

3. Background and preliminary results

3.1. **Recharacterization of spherical points.** We retain all of the notation from Section 2. Lemma 3.1 below provides a characterization of spherical points involving only the complexified Lie algebra $\mathfrak{k}_{\mathbb{C}}$. A suitable ordering on the roots for $\mathfrak{k}_{\mathbb{C}}$ relative to $\mathfrak{h}$ enables one to decompose the Lie algebra for $B = HN = BN_+$ as $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ where $\mathfrak{n}_+$ is the sum of positive root spaces. Moreover $\mathfrak{k}_{\mathbb{C}} = \mathfrak{b} \oplus \mathfrak{n}_- = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ where $\mathfrak{n}_-$ is the sum of negative root spaces.

Lemma 3.1. *Let $K : V$ be a multiplicity free action and $\alpha \in \Lambda$ a highest weight occurring in $\mathbb{C}[V]$. Then $v_\alpha \in V$ is a spherical point for α if and only if*

$$\left\{ \begin{array}{ll} \langle X \cdot v_\alpha, v_\alpha \rangle = -\alpha(X) & \text{for all } X \in \mathfrak{h} \text{ and} \\ \langle X \cdot v_\alpha, v_\alpha \rangle = 0 & \text{for all } X \in \mathfrak{n}_+ \oplus \mathfrak{n}_- \end{array} \right\} \tag{3.1}$$

Proof. The weight $\alpha \in \mathfrak{h}^*$ extends to a linear functional on all of $\mathfrak{k}_{\mathbb{C}}$ as zero on $\mathfrak{n}_+ \oplus \mathfrak{n}_-$. On the other hand, one can extend the real-valued linear functional $\alpha_{\mathfrak{k}} \in \mathfrak{k}^*$ to a complex-linear functional on $\mathfrak{k}_{\mathbb{C}}$. These extensions are related via $\alpha = i\alpha_{\mathfrak{k}}$ on $\mathfrak{k}_{\mathbb{C}}$. Now v_α is a spherical point for α if and only if

$$\alpha_{\mathfrak{k}}(A) = \tau(v_\alpha)(A) = i \langle A \cdot v_\alpha, v_\alpha \rangle$$

holds for all $A \in \mathfrak{k}$. But the right hand side of this expression has an obvious extension to a complex-linear functional on $\mathfrak{k}_{\mathbb{C}}$. We conclude that v_α is a spherical point for α if and only if

$$\alpha(X) = - \langle X \cdot v_\alpha, v_\alpha \rangle$$

holds for all $X \in \mathfrak{k}_{\mathbb{C}}$. □

Remark 3.2. Note that the equations in (3.1) are linear in X. Thus it suffices that they hold as X ranges over chosen bases for $\mathfrak{h}$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$. That is, for a given $\alpha \in \Lambda$, the conditions in (3.1) amount to a system of $\dim(\mathfrak{k})$ quadratic equations whose solutions give all spherical points for α. We make extensive use of this observation in our subsequent spherical point calculations.

3.2. **Observations on Definition 2.2.** Condition (ii) in the definition of "well-adapted" is easy to establish for directional derivatives in directions $w \in \mathfrak{b} \cdot v_\alpha$:

Lemma 3.3. *For all $\alpha \in \Lambda$ and $w \in \mathfrak{b} \cdot v_\alpha$ one has $\big(\partial_w h_\alpha\big)(v_\alpha) = \langle w, v_\alpha \rangle \, h_\alpha(v_\alpha)$.*

Proof. As h_α is a B-highest weight vector we have $X \cdot h_\alpha = \alpha(X) h_\alpha$ for $X \in \mathfrak{b}$ and hence

$$X \cdot h_\alpha = - \langle X \cdot v_\alpha, v_\alpha \rangle \, h_\alpha \quad \text{for } X \in \mathfrak{b},$$

in view of Lemma 3.1. On the other hand

$$(X \cdot h_\alpha)(z) = \left.\frac{d}{dt}\right|_{t=0} h_\alpha\big(\exp(-tX) \cdot z\big) = \left.\frac{d}{dt}\right|_{t=0} h_\alpha\big(z - tX \cdot z + O(t^2)\big) = \big(\partial_{(-X \cdot z)} h_\alpha\big)(z).$$

So for $X \in \mathfrak{b}$ we obtain

$$\big(\partial_{(-X\cdot v_\alpha)} h_\alpha\big)(v_\alpha) = -\left\langle X\cdot v_\alpha, v_\alpha\right\rangle h_\alpha(v_\alpha).$$

Equivalently $\big(\partial_w h_\alpha\big)(v_\alpha) = \left\langle w, v_\alpha\right\rangle h_\alpha(v_\alpha)$ for all $w \in \mathfrak{b}\cdot v_\alpha$. □

As $K : V$ is multiplicity free it is well known that the Borel subgroup B has a Zariski-open dense orbit in the vector space V [14].

Corollary 3.4 (Proposition 2.5, [5])**.** *If a spherical point v_α lies in the open B-orbit then v_α is well-adapted to h_α.*

Proof. Suppose that v_α lies in the open B-orbit. As h_α is a non-zero B-semi-invariant we must have $h_\alpha(v_\alpha) \neq 0$. Moreover as $\mathfrak{b}\cdot v_\alpha = V$ condition (ii) in Definition 2.2 holds by Lemma 3.3. □

3.3. **A limiting procedure.** Now let $\{\alpha_1, \ldots, \alpha_r\} \subset \mathfrak{h}^*$ be the fundamental B-highest weights for $K : V$ and $h_j = h_{\alpha_j}$ $(1 \le j \le r)$ associated highest weight vectors in $\mathbb{C}[V]$. Thus

$$\Lambda = \big\{a_1\alpha_1 + \cdots + a_r\alpha_r \ : \ a_j \in \mathbb{Z}, a_j \ge 0\big\}$$

is the set of highest weights occurring in the representation of K on $\mathbb{C}[V]$ and

$$h_\alpha = h_1^{a_1}\cdots h_r^{a_r}$$

is a highest weight vector in $\mathbb{C}[V]$ with weight $\alpha = a_1\alpha_1+\cdots+a_r\alpha_r \in \Lambda$. We say that α is *generic* when $a_1, \ldots, a_r$ are all non-zero. In our subsequent treatment of certain examples, spherical points for non-generic weights α arise as limits of (generalized) spherical points for generic weights by taking subsets of the parameters $(a_1, \ldots, a_r)$ to zero. This requires formal replacement of our non-negative integer parameters a_j by non-negative real parameters. The following technical Lemma will be of use.

Lemma 3.5. *Suppose that for all positive real numbers $x_1, \ldots, x_r > 0$ there is a point*

$$v(\mathbf{x}) = v(x_1, \ldots, x_r)$$

in V which satisfies the following four conditions:

(1) $\left\{\begin{array}{ll} \langle X\cdot v(\mathbf{x}), v(\mathbf{x})\rangle = -(x_1\alpha_1+\cdots+x_r\alpha_r)(X) & \textit{for all } X\in\mathfrak{h} \textit{ and} \\ \langle X\cdot v(\mathbf{x}), v(\mathbf{x})\rangle = 0 & \textit{for all } X \in \mathfrak{n}_+\oplus\mathfrak{n}_- \end{array}\right\}.$

(2) *$h_i(v(\mathbf{x})) \neq 0$ for $1 \le i \le r$.*

(3) *For each $1 \le k < r$ and indices $1 \le j_1 < j_2 < \cdots < j_k \le r$ the limit*

$$\lim_{x_{j_k}\to 0^+} \cdots \lim_{x_{j_1}\to 0^+} v(x_1, \ldots, x_r)$$

exists in V, and

(4) *$\lim_{x_{j_k}\to 0^+}\cdots\lim_{x_{j_1}\to 0^+} h_i\big(v(x_1, \ldots, x_r)\big) \neq 0$ for each $i \in \{1, \ldots, r\}\backslash\{j_1, \ldots, j_k\}$.*

Then $K : V$ is a well-behaved multiplicity free action.

Proof. For fixed real values $x_1, \ldots, x_r > 0$ observe that for each $X \in \mathfrak{b}$ one has

$$X \cdot \big(h_1^{x_1} \cdots h_r^{x_r}\big) = \sum_{i=1}^r x_i \big(h_1^{x_1} \cdots h_i^{x_i - 1} \cdots h_r^{x_r}\big)(X \cdot h_i) = \sum_{i=1}^r x_i \big(h_1^{x_1} \cdots h_i^{x_i-1} \cdots h_r^{x_r}\big)\alpha_i(X) h_i$$
$$= (x_1\alpha_1 + \cdots + x_r\alpha_r)(X) h_1^{x_1} \cdots h_r^{x_r}.$$

In view of hypothesis (1) we may also write

$$X \cdot \big(h_1^{x_1} \cdots h_r^{x_r}\big) = -\langle X \cdot v(\mathbf{x}), v(\mathbf{x})\rangle\, h_1^{x_1} \cdots h_r^{x_r}$$

for $X \in \mathfrak{b}$. As in the proof for Lemma 3.3 we conclude that

$$\partial_w \big(h_1^{x_1} \cdots h_r^{x_r}\big)(v(\mathbf{x})) = \langle w, v(\mathbf{x})\rangle \big(h_1^{x_1} \cdots h_r^{x_r}\big)(v(\mathbf{x})) \tag{3.2}$$

for all $w \in \mathfrak{b} \cdot v(\mathbf{x})$. But hypothesis (2) implies that $v(\mathbf{x})$ lies in the open B-orbit in V. Thus in fact $\mathfrak{b} \cdot v(\mathbf{x}) = V$ and Equation 3.2 holds for all directions $w \in V$ and all real parameters $x_1, \ldots, x_r > 0$.

Now let $a_1, \ldots, a_r \geq 0$ be non-negative integers and consider the weight

$$\big(\alpha = a_1\alpha_1 + \cdots + a_r\alpha_r\big) \in \Lambda.$$

Case 1: Suppose that $a_1, \ldots, a_r$ are all positive and set $v_\alpha := v(a_1, \ldots, a_r)$. Hypothesis (1) and Lemma 3.1 show that v_α is a spherical point for α. Hypothesis (2) gives $h_\alpha(v_\alpha) = \big(h_1(v_\alpha)\big)^{a_1} \cdots \big(h_r(v_\alpha)\big)^{a_r} \neq 0$ and Equation 3.2 shows $\big(\partial_w h_\alpha\big)(v_\alpha) = \langle w, v_\alpha\rangle\, h_\alpha(v_\alpha)$ for all $w \in V$. Thus v_α is well-adapted to h_α.

Case 2: Next suppose that k of the values $a_1, \ldots, a_r$ are zero and the rest positive for some $1 \leq k < r$. To ease notation we may here assume that $a_1 = \cdots = a_k = 0$ and $a_{k+1}, \ldots, a_r > 0$. For positive real parameters $t_1, \ldots, t_k > 0$ let

$$\widetilde{v}(\mathbf{t}) := \widetilde{v}(t_1, \ldots, t_k) = v(t_1, \ldots, t_k, a_{k+1}, \ldots, a_r), \quad v_\alpha := \lim_{t_k \to 0^+} \cdots \lim_{t_1 \to 0^+} \widetilde{v}(t_1, \ldots, t_k).$$

This limit exists in V by hypothesis (3).

Hypothesis (1) shows that for given $X \in \mathfrak{h}$ and all $t_1, \ldots, t_k > 0$ one has

$$\langle X \cdot v(\mathbf{t}), v(\mathbf{t})\rangle = -(t_1\alpha_1 + \cdots + t_k\alpha_k + a_{k+1}\alpha_{k+1} + \cdots + a_r\alpha_r)(X).$$

Taking limits we have, by continuity, that $\langle X \cdot v_\alpha, v_\alpha\rangle = -\alpha(X)$. Likewise we see that $\langle X \cdot v_\alpha, v_\alpha\rangle = 0$ for each $X \in \mathfrak{n}_+ \oplus \mathfrak{n}_-$. Thus v_α is a spherical point for α by Lemma 3.1. Hypothesis (4) shows that $h_i(v_\alpha) \neq 0$ for $i \geq k+1$ and hence $h_\alpha(v_\alpha) = \big(h_{k+1}(v_\alpha)\big)^{a_{k+1}} \cdots \big(h_r(v_\alpha)\big)^{a_r} \neq 0$. Equation 3.2 gives

$$\partial_w \left(h_1^{t_1} \cdots h_k^{t_k}\, h_\alpha\right)(\tilde{v}(\mathbf{t})) = \langle w, \tilde{v}(\mathbf{t})\rangle \left(h_1^{t_1} \cdots h_k^{t_k}\, h_\alpha\right)(\tilde{v}(\mathbf{t}))$$

for all directions $w \in V$ and all $t_1, \ldots, t_k > 0$. Taking limits on both sides as $t_1, \ldots, t_k \to 0^+$ shows $\big(\partial_w h_\alpha\big)(v_\alpha) = \langle w, v_\alpha\rangle\, h_\alpha(v_\alpha)$. Hence v_α is well-adapted to h_α.

Case 3: Finally suppose that $a_1 = \cdots = a_r = 0$. Now $\alpha = 0$ and in this case $v_0 = 0$ is a spherical point for α well-adapted to $h_0 = 1$. □

Remark 3.6. We refer to a vector $v(\mathbf{x}) = v(x_1, \ldots, x_r) \in V$ satisfying condition (1) in Lemma 3.5 as a *generalized* spherical point. In the context of Lemma 3.5 non-generic spherical points are obtained by sending parameters in generalized generic spherical points to zero in succession in a prescribed order. One can consider changing the order in this limit process. In our examples we find that taking limits in different orders always produces (well-adapted) non-generic spherical points but that the resulting limit does depend on the limit order, even though the limit points are always in the same K-orbit.

3.4. **Restriction of multiplicity free actions.** Suppose now that $K_2 : V$ is a multiplicity free action and that $K_1 \subset K_2$ is a closed Lie subgroup for which the restricted action $K_1 : V$ remains multiplicity free. We choose maximal tori $T_j \subset K_j$ and Borel subgroups $B_j \subset (K_j)_{\mathbb{C}}$ to ensure $T_1 \subset T_2$, $B_1 \subset B_2$ and let $\Lambda_j \subset \mathfrak{h}_j^*$ be the sets of B_j-highest weights occurring in the representations of K_1 and K_2 on $\mathbb{C}[V]$. Let $\alpha \in \Lambda_2$ and $h_\alpha \in \mathbb{C}[V]$ be a B_2-highest weight vector with weight α. Clearly h_α is also a B_1-highest weight vector with weight $\alpha|_{\mathfrak{h}_1}$. Thus restriction yields a map

$$\Lambda_2 \to \Lambda_1, \qquad \alpha \mapsto \alpha|_{\mathfrak{h}_1}.$$

It is also transparent that the moment mappings $\tau_1 : V \to \mathfrak{k}_1^*$ and $\tau_2 : V \to \mathfrak{k}_2^*$ for the actions $K_j : V$ are related by restriction. That is, for $v \in V$,

$$\tau_1(v) = \tau_2(v)|_{\mathfrak{k}_1}.$$

Thus if v_α is a spherical point for highest weight $\alpha \in \Lambda_2$ then v_α is also a spherical point for the restricted weight $\alpha|_{\mathfrak{h}_1} \in \Lambda_1$. Moreover, as conditions (i) and (ii) in Definition 2.2 depend only on the point v_α and polynomial $h_\alpha \in \mathbb{C}[V]$ we conclude that v_α is well-adapted to h_α for the action $K_1 : V$ if and only if v_α is well-adapted to h_α for the action $K_2 : V$. This establishes, in particular, the following result.

Lemma 3.7. *Let $K_1 : V$ be a multiplicity free action obtained by restricting a multiplicity free action $K_2 : V$ to a closed Lie subgroup $K_1 \subset K_2$. Assume, moreover, that $\mathbb{C}[V]$ shares a common decomposition under the associated representations of K_1 and K_2. Then $K_1 : V$ is well-behaved if and only if $K_2 : V$ is well-behaved.*

Example 3.8. Let $K : V$ be a multiplicity free action. Replacing K by its image in $U(V)$ and using an orthonormal basis to identify V with $\mathbb{C}^n$ we may regard K as a subgroup of $U(n)$. In view of Example 2.3, for each $m \geq 0$,

- $\alpha = -m\varepsilon_1|_{\mathfrak{h}}$ belongs to Λ,
- $h_m(z) = z_1^m$ is a highest weight vector with weight α, and
- $v_m = \sqrt{m}\, e_1$ is a spherical point for α well-adapted to h_m.

3.5. **Scalar actions.** Let $K : V$ be a multiplicity free action. Letting the circle $\mathbb{T}$ act on V by scalar multiplication produces a multiplicity free action $K \times \mathbb{T} : V$. As the decompositions of $\mathbb{C}[V]$ under K and $K \times \mathbb{T}$ coincide Lemma 3.7 yields the following corollary.

Corollary 3.9. *$K : V$ is well-behaved if and only if $K \times \mathbb{T} : V$ is well-behaved.*

3.6. **Hermitian symmetric spaces.** Some interesting multiplicity free actions arise in connection with Hermitian symmetric spaces. Let G/K be a Hermitian symmetric space of non-compact type and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denote the Cartan decomposition for the Lie algebra of G. The real vector space $\mathfrak{p}$ inherits a complex Hermitian structure and K acts unitarily on $\mathfrak{p}$ via Ad. The action $K : \mathfrak{p}$ is, in fact, multiplicity free and the rank of $K : \mathfrak{p}$ coincides with the rank of G/K as a symmetric space. (See [11].)

Proposition 3.10 (Theorem 1.3, [5])**.** *$K : \mathfrak{p}$ is well-behaved.*

In broad outline the situation is a follows. For the proof one can assume that G/K is irreducible and identify the action $K : \mathfrak{p}$ with the adjoint action of K on $\mathfrak{p}_+$, the $(+i)$-eigenspace in $\mathfrak{p}_{\mathbb{C}}$ for the complexified complex structure on $\mathfrak{p}$. A Cartan subalgebra $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{k}_C$ is simultaneously a Cartan subalgebra for $\mathfrak{g}_{\mathbb{C}}$.

Theorem 3.11. [11] *The fundamental highest weights occurring in $\mathbb{C}[\mathfrak{p}_+]$, relative to a suitable Borel subgroup, can be written as*

$$\big\{\alpha_j := -(\delta_1 + \cdots + \delta_j) \;:\; 1 \leq j \leq r\big\},$$

where $\{\delta_1, \ldots, \delta_r\}$ is a certain maximal ordered set of strongly orthogonal non-compact positive roots for $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{h}$. Moreover, the representation in $\mathbb{C}[\mathfrak{p}_+]$ with highest weight α_j occurs in degree j.

Given a highest weight $\big(\alpha = a_1\alpha_1 + \cdots + a_r\alpha_r\big) \in \Lambda$ take

$$v_\alpha := \sqrt{a_1 + \cdots + a_r}\, X_1 + \sqrt{a_2 + \cdots + a_r}\, X_2 + \cdots + \sqrt{a_r} X_r$$

where X_j is a unit vector in the root space for δ_j. Now v_α is a spherical point for α well-adapted to a highest weight vector $h_\alpha \in \mathbb{C}[\mathfrak{p}_+]$ with weight α. See [5] for details.

3.7. **The mapping $\Psi : \Delta(K, H_V) \to \mathfrak{h}_V^*/K$.** Recall that a multiplicity free action $K : V$ yields a Gelfand pair (K, H_V) with H_V the Heisenberg group $H_V = V \times \mathbb{R}$. The bounded spherical functions $\varphi \in \Delta(K, H_V)$ are of two distinct *types*. The type 1 spherical functions $\phi_{\lambda,\alpha}$ are indexed by pairs $(\lambda, \alpha) \in \Lambda \times \mathbb{R}^\times$ whereas the type 2 spherical functions $\eta_{K\cdot w}$ are indexed by K-orbits $K \cdot w \in V/K$. The mapping $\Psi : \Delta(K, H_V) \to \mathfrak{h}_V^*/K$, discussed in Section 1, is written concretely in [5] as

$$\begin{aligned} \Psi\big(\phi_{\lambda,\alpha}\big) &= \mathcal{K}_{\lambda,\alpha} := \sqrt{2|\lambda|}\, \mathcal{K}_\alpha \times \{\lambda\} \qquad (\lambda \in \mathbb{R}^\times,\ \alpha \in \Lambda), \\ \Psi\big(\eta_{K\cdot w}\big) &= (K \cdot w) \times \{0\}, \end{aligned}$$

where $\mathcal{K}_\alpha = K \cdot v_\alpha$ is the spherical orbit as in Definition 2.1. Here we have used the $U(V)$-equivariant isomorphism $\mathfrak{h}_V \to \mathfrak{h}_V^*$

$$(z, t) \mapsto \ell_{(z,t)} \quad \text{where} \quad \ell_{(z,t)}(z', t') := \operatorname{Im} \langle z, z' \rangle + tt'$$

to identify $\mathfrak{h}_V^*$ with $\mathfrak{h}_V$ and $\mathfrak{h}_V^*/K$ with $\mathfrak{h}_V/K$.

Case	Group K	Vector space V	Degrees of fundamental highest weight vectors (rank)
(a)	$U(n)$	$\mathbb{C}^n$ $(n \geq 1)$	1 (1)
(b)	$SO(n) \times \mathbb{T}$	$\mathbb{C}^n$ $(n \geq 3)$	1,2 (2)
(c)	$Sp(2n) \times \mathbb{T}$	$\mathbb{C}^{2n}$ $(n \geq 2)$	1 (1)
(d)	$U(n)$	$S^2(\mathbb{C}^n)$ $(n \geq 2)$	$1, 2, \ldots, n$ (n)
(e)	$U(n)$	$\Lambda^2(\mathbb{C}^n)$ $(n \geq 4)$	$1, 2, \ldots \lfloor n/2 \rfloor$ $(\lfloor n/2 \rfloor)$
(f)	$U(n) \times U(m)$	$\mathbb{C}^n \otimes \mathbb{C}^m$ $(n, m \geq 2)$	$1, 2 \ldots, \min(n, m)$ $(\min(n, m))$
(g)	$Sp(2n) \times U(2)$	$\mathbb{C}^{2n} \otimes \mathbb{C}^2$ $(n \geq 2)$	1,2,2 (3)
(h-1) (h-2)	$Sp(2n) \times U(3)$ $Sp(4) \times U(3)$	$\mathbb{C}^{2n} \otimes \mathbb{C}^3$ $(n \geq 3)$ $\mathbb{C}^4 \otimes \mathbb{C}^3$	1,2,2,3,3,4 (6) 1,2,2,3,4 (5)
(i)	$Sp(4) \times U(m)$	$\mathbb{C}^{2n} \otimes \mathbb{C}^2$ $(m \geq 4)$	1,2,2,3,4,4 (6)
(j)	$Spin(7) \times \mathbb{T}$	$\Lambda(\mathbb{C}^3) \cong \mathbb{C}^8$	1,2 (2)
(k)	$Spin(9) \times \mathbb{T}$	$\Lambda(\mathbb{C}^4) \cong \mathbb{C}^{16}$	1,2,2 (3)
(l)	$Spin(10) \times \mathbb{T}$	$\Lambda^{\text{even}}(\mathbb{C}^5) \cong \mathbb{C}^{16}$	1,2 (2)
(m)	$G_2 \times \mathbb{T}$	$\mathbb{C}^7$	1,2 (2)
(n)	$E_6 \times \mathbb{T}$	$\mathbb{C}^{27}$	1,2,3 (3)

TABLE 1. Irreducible multiplicity free actions $K : V$

To render the orbital model for $\Delta(K, H_V)$ explicit in a given example requires finding a spherical point $v_\alpha \in V$ for each weight $\alpha \in \Lambda$. The case-by-case analysis given below accomplishes this task for the irreducible multiplicity free actions.

4. Case-by-case analysis

In this section we prove Theorem 1.5 working case-by-case from the known classification of irreducible (linear) multiplicity free actions. Victor Kac classified all irreducible multiplicity free actions of connected complex algebraic groups up to geometric equivalence in [12]. Corollary 3.9 shows that it suffices to examine actions $K : V$ whose image in $U(V)$ include a copy of the scalars $\mathbb{T}$. Table 1 lists compact forms for the actions in [12] which include a copy of the scalars.[1] The conditions on n and m in the table are imposed to eliminate redundancies in low dimensions. Data in the final column are drawn from [10, page 612]. To prove Theorem 1.5 it suffices to verify that each action in Table 1 is well-behaved.

4.1. **Case (a).** $U(n) : \mathbb{C}^n$ is well-behaved, as shown in Example 2.3.

[1] Of these actions the following remain multiplicity free upon removal of the scalars: Case (a) with $n \geq 2$, case (c), case (e) with n odd, case (f) with $n \neq m$, case (i) with $m \geq 5$ and case (l).

4.2. **Cases (b, d, e, f, l, n).** The actions

$$\left\{ \begin{array}{c|c|c} \big(SO(n)\times\mathbb{T}\big) : \mathbb{C}^n & U(n) : S^2(\mathbb{C}^n) & U(n) : \Lambda^2(\mathbb{C}^n) \\ \hline \big(U(n)\times U(m)\big) : \big(\mathbb{C}^n\otimes\mathbb{C}^m\big) & \big(Spin(10)\times\mathbb{T}\big) : \mathbb{C}^{16} & \big(E_6\times\mathbb{T}\big) : \mathbb{C}^{27} \end{array} \right\}$$

arise, up to geometric equivalence, in connection with irreducible Hermitian symmetric spaces. (See [11] and [9].) These are well-behaved by Proposition 3.10.

4.3. **Case (c).** The space $\mathcal{P}_m(V)$ of homogeneous polynomials of degree m on $V = \mathbb{C}^{2n}$ is irreducible under $K = Sp(2n)\times\mathbb{T}$. Thus the decompositions for $\mathbb{C}[V]$ under the actions of K and $U(2n)$ coincide and Lemma 3.7 shows that $K : V$ is well-behaved.

4.4. **Case (j).** Here $Spin(7)$ acts on $V = \Lambda(\mathbb{C}^3) \cong \mathbb{C}^8$ via its half-spin representation. The space V admits a $Spin(7)$-invariant inner product which embeds $Spin(7)$ in $SO(8)$. As the multiplicity free actions of $Spin(7)\times\mathbb{T}$ and $SO(8)\times\mathbb{T}$ on V both have rank 2, there will be no splitting of irreducibles when restricted to the smaller group. It follows that the decompositions for $\mathbb{C}[V]$ under these actions coincide. Hence $Spin(7)\times\mathbb{T} : V$ is well-behaved by Lemma 3.7.

4.5. **Case (m).** Another application of Lemma 3.7 shows that this action is well-behaved. The compact exceptional group G_2 acts on $V = \mathbb{C}^7$ as a subgroup of $SO(7)$ and $G_2\times\mathbb{T} : \mathbb{C}^7$ is a rank 2 multiplicity free action. Thus the decompositions for $\mathbb{C}[V]$ under $G_2\times\mathbb{T}$ and $SO(7)\times\mathbb{T}$ coincide.

4.6. **Actions $Sp(2n)\times U(m) : \mathbb{C}^{2n}\otimes\mathbb{C}^m$.** These actions are multiplicity free if either $m \le 3$ or $n \le 2$. We may also require that both $m \ge 2$ and $n \ge 2$ as otherwise we are in Case (a), (c) or (f). So the actions at issue are Cases (g), (h) and (i) in Table 1. To discuss these cases in detail we must establish some notational conventions.

Identifying $\mathbb{C}^{2n}\otimes\mathbb{C}^m$ with the space $V = M_{2n,m}(\mathbb{C})$ of $2n\times m$ complex matrices the compact group $K = Sp(2n)\times U(m)$ acts via

$$(k_1, k_2)\cdot z = k_1 z k_2^t,$$

and the usual Hermitian inner product on V, namely

$$\langle z, w\rangle = tr(zw^*),$$

is K-invariant. The moment map is given by

$$\tau(z)(X, Y) = i\left\langle Xz + zY^t, z\right\rangle$$

for $X \in sp(2n)$, $Y \in u(m)$, $z \in V$.

Here $Sp(2n) = Sp(2n,\mathbb{C}) \cap U(2n)$ where $Sp(2n,\mathbb{C})$ is the subgroup of $GL(2n,\mathbb{C})$ preserving the symplectic form

$$\omega\big((z_1,\ldots,z_{2n}),(w_1,\ldots,w_{2n})\big) = \sum_{j=1}^{n}(z_j w_{n+j} - z_{n+j} w_j).$$

The group $Sp(2n,\mathbb{C})$ has Lie algebra

$$sp(2n,\mathbb{C}) = \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & -A^t \end{array} \right] \ : \ A,B,C \in gl(n,\mathbb{C}),\ B^t = B,\ C^t = C \right\}.$$

As Borel subgroup B_{2n} in $Sp(2n,\mathbb{C})$ we choose $B_{2n} = \exp(\mathfrak{b}_{2n})$ where $\mathfrak{b}_{2n}$ is the subalgebra of $sp(2n,\mathbb{C})$ consisting of matrices as above with $B = 0$ and A lower triangular. A Cartan subalgebra in $sp(2n,\mathbb{C})$ is given by the diagonal matrices,

$$\mathfrak{h}_{2n} = \big\{ diag(a_1,\ldots,a_n,-a_1,\ldots,-a_n) \ : \ a_j \in \mathbb{C} \big\}.$$

We let $\varepsilon_j \in \mathfrak{h}_{2n}^*$ $(1 \le j \le n)$ denote the linear functional

$$\varepsilon_j\big(diag(a_1,\ldots,a_n,-a_1,\ldots,-a_n)\big) = a_j.$$

Moreover we let B'_m denote the Borel subgroup of lower triangular matrices in $GL(m,\mathbb{C}) = U(m)_{\mathbb{C}}$, write $\mathfrak{h}'_m$ for the Cartan subalgebra of diagonal matrices in $gl(m,\mathbb{C})$ and let $\varepsilon'_j \in (\mathfrak{h}'_m)^*$ be the functional

$$\varepsilon'_j\big(diag(b_1,\ldots,b_m)\big) = b_j, \quad (1 \le j \le m).$$

Now $B = B_{2n} \times B'_m$ is our chosen Borel subgroup in $K_{\mathbb{C}} = Sp(2n,\mathbb{C}) \times GL(m,\mathbb{C})$. We use these preliminaries to study cases (g), (h), and (i).

4.6.1. **Case (g)**. Here $K = Sp(2n) \times U(2)$ acts on the space $V = M_{2n,2}(\mathbb{C})$ as above. This is a rank 3 multiplicity free action with fundamental highest weights, given in [10, Section 11.6],

$$\left\{ \begin{array}{l} \alpha_1 = -(\varepsilon_1 + \varepsilon'_1) \\ \alpha_2 = -(\varepsilon_1 + \varepsilon_2 + \varepsilon'_1 + \varepsilon'_2) \\ \alpha_3 = -(\varepsilon'_1 + \varepsilon'_2) \end{array} \right\}.$$

We remark that in reference [10] the authors twist the actions of both $Sp(2n)$ and $U(2)$ by $k \mapsto (k^t)^{-1}$. The resulting action is geometrically equivalent to the more standard action here but the effect is to remove the minus signs in weights α_j. Also in [10] upper triangular matrices are used in place of lower in the choice of Borel subgroup. This observation applies also to our subsequent discussion of Cases (h, i).

We will first show the result for $n = 2$ and then reduce the general case to that particular one. So now $K = Sp(4) \times U(2)$ and $V = M_{4,2}(\mathbb{C})$. Highest weight vectors h_j for the fundamental weights α_j are, from [10],

$$\left\{ \begin{array}{c} h_1(z) = z_{11}, \qquad h_2(z) = \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \\ h_3(z) = \omega\big(z_{\bullet,1}, z_{\bullet,2}\big) = z_{11}z_{32} + z_{21}z_{42} - z_{31}z_{12} - z_{41}z_{22} \end{array} \right\}.$$

We use the notation $z_{i,\bullet}$ for the i'th row of z, and $z_{\bullet,j}$ for the j'th column. The set Λ of highest weights occurring in $\mathbb{C}[V]$ is thus

$$\Lambda = \big\{ a\alpha_1 + b\alpha_2 + c\alpha_3 = -(a+b)\varepsilon_1 - b\varepsilon_2 - (a+b+c)\varepsilon'_1 - (b+c)\varepsilon'_2 \ : \ a,b,c \in \mathbb{Z}_{\ge 0} \big\}$$

and $h_\alpha = h_1^a h_2^b h_3^c$ is a highest weight vector with weight $(\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3) \in \Lambda$.

$$\left\{\begin{array}{c|c|c}
E_{11}-E_{33} & |z_{11}|^2+|z_{12}|^2-|z_{31}|^2-|z_{32}|^2=a+b & \|z_{1,\bullet}\|^2-\|z_{3,\bullet}\|^2=a+b \\
E_{22}-E_{44} & |z_{21}|^2+|z_{22}|^2-|z_{41}|^2-|z_{42}|^2=b & \|z_{2,\bullet}\|^2-\|z_{4,\bullet}\|^2=b \\
E'_{11} & |z_{11}|^2+|z_{21}|^2+|z_{31}|^2+|z_{41}|^2=a+b+c & \|z_{\bullet,1}\|^2=a+b+c \\
E'_{22} & |z_{12}|^2+|z_{22}|^2+|z_{32}|^2+|z_{42}|^2=b+c & \|z_{\bullet,2}\|^2=b+c \\
\hline
E_{31} & z_{11}\overline{z_{31}}+z_{12}\overline{z_{32}}=0 & \langle z_{1,\bullet},z_{3,\bullet}\rangle=0 \\
E_{42} & z_{21}\overline{z_{41}}+z_{22}\overline{z_{42}}=0 & \langle z_{2,\bullet},z_{4,\bullet}\rangle=0 \\
E_{41}+E_{32} & z_{11}\overline{z_{41}}+z_{12}\overline{z_{42}}+z_{21}\overline{z_{31}}+z_{22}\overline{z_{32}}=0 & \langle z_{1,\bullet},z_{4,\bullet}\rangle=-\langle z_{2,\bullet},z_{3,\bullet}\rangle \\
E_{21}-E_{34} & z_{11}\overline{z_{21}}+z_{12}\overline{z_{22}}-z_{41}\overline{z_{31}}-z_{42}\overline{z_{32}}=0 & \langle z_{1,\bullet},z_{2,\bullet}\rangle=\langle z_{4,\bullet},z_{3,\bullet}\rangle \\
E'_{21} & z_{11}\overline{z_{12}}+z_{21}\overline{z_{22}}+z_{31}\overline{z_{32}}+z_{41}\overline{z_{42}}=0 & \langle z_{\bullet,1},z_{\bullet,2}\rangle=0
\end{array}\right\}$$

TABLE 2. Spherical point equations for Case (g)

The matrix entries z_{ij} $(1 \le i \le 4,\ 1 \le j \le 2)$ of a spherical point for weight $\big(\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3\big) \in \Lambda$ satisfy a system of equations obtained from Lemma 3.1 by letting X range over the usual bases for $\mathfrak{h}$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$. This produces 14 equations of which only 9 are distinct. These are listed in Table 2. The first column in the table specifies a Lie algebra element $X \in sp(4,\mathbb{C}) \times gl(2,\mathbb{C})$ yielding each equation. Here E_{ij} and E'_{ij} denote elementary matrices of size 4×4 and 2×2 respectively. The first four equations arise from the action of $\mathfrak{h}$ and the last five from $\mathfrak{n}_+ \oplus \mathfrak{n}_-$. Each equation amounts to a constraint on the rows $z_{i,\bullet}$ or columns $z_{\bullet,j}$ of $z \in V$, listed in the third column. Note that the minus sign in (3.1) cancels the minus signs in our highest weights.

One checks easily that given any real parameters $a, b, c \ge 0$ the matrix entries of

$$v(a,b,c) := \begin{bmatrix} \sqrt{\frac{(a+b)(a+2b+c)}{a+2b}} & 0 \\ 0 & \sqrt{\frac{b(a+2b+c)}{a+2b}} \\ 0 & \sqrt{\frac{c(a+b)}{a+2b}} \\ -\sqrt{\frac{bc}{a+2b}} & 0 \end{bmatrix} \tag{4.1}$$

satisfy the nine equations in Table 2 provided $a+2b \neq 0$. In particular, $v_\alpha = v(a,b,c)$ is a spherical point for weight $\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3$ with a, b, c non-negative integers satisfying $a \neq 0$ or $b \neq 0$. To show that $K : V$ is well-behaved we will apply Lemma 3.5. Evaluating the fundamental highest weight vectors h_j at $v(a,b,c)$ yields

$$\left\{ \begin{array}{c} h_1\big(v(a,b,c)\big) = \sqrt{\frac{(a+b)(a+2b+c)}{a+2b}}, \quad h_2\big(v(a,b,c)\big) = \frac{a+2b+c}{a+2b}\sqrt{b(a+b)} \\ h_3\big(v(a,b,c)\big) = \sqrt{c(a+2b+c)} \end{array} \right\}.$$

Note that these values are each non-zero when a, b, c are all positive. Thus condition (2) in Lemma 3.5 holds here. In particular for a generic weight $\alpha \in \Lambda$ the spherical point v_α belongs to the open B-orbit in V and is well-adapted to h_α by Corollary 3.4.

As regards condition (3) in the lemma we just need to observe that for fixed $c > 0$ the limit

$$\lim_{b\to 0^+}\lim_{a\to 0^+} v(a,b,c) = \lim_{b\to 0^+}\begin{bmatrix} \sqrt{(2b+c)/2} & 0 \\ 0 & \sqrt{(2b+c)/2} \\ 0 & \sqrt{c/2} \\ -\sqrt{c/2} & \end{bmatrix} = \begin{bmatrix} \sqrt{c/2} & 0 \\ 0 & \sqrt{c/2} \\ 0 & \sqrt{c/2} \\ -\sqrt{c/2} & \end{bmatrix}$$

does exist. For non-negative integers c this limit is a spherical point for weight $c\alpha_3 \in \Lambda$. This fact follows from Lemma 3.5 but is also easy to see directly using the equations in Table 2. We remark that interchanging the limit order here results in

$$\lim_{a\to 0^+}\lim_{b\to 0^+} v(a,b,c) = \begin{bmatrix} \sqrt{c} & 0 \\ 0 & 0 \\ 0 & \sqrt{c} \\ 0 & 0 \end{bmatrix},$$

an alternate spherical point for $c\alpha_3$. This illustrates Remark 3.6. Limiting values for $h_j\big(v(a,b,c)\big)$ $(j = 1,2,3)$ as one or two parameters approach zero are given by

limit	$h_1\big(v(a,b,c)\big)$	$h_2\big(v(a,b,c)\big)$	$h_3\big(v(a,b,c)\big)$
$lim_{a\to 0^+}$	$\sqrt{(2b+c)/2}$	$(2b+c)/2$	$\sqrt{c(2b+c)/2}$
$lim_{b\to 0^+}$	$\sqrt{a+c}$	0	$\sqrt{c(a+c)}$
$lim_{c\to 0^+}$	$\sqrt{a+b}$	$\sqrt{b(a+b)}$	0
$lim_{b\to 0^+} lim_{a\to 0^+}$	$\sqrt{c/2}$	$c/2$	c
$lim_{c\to 0^+} lim_{a\to 0^+}$	$\sqrt{b}$	b	0
$lim_{c\to 0^+} lim_{b\to 0^+}$	$\sqrt{a}$	0	0

These show, in particular, that condition (4) in Lemma 3.5 holds, completing the verification that $K : V$ is a well-behaved multiplicity free action.

Finally we consider the action $Sp(2n) \times U(2) : M_{2n,2}(\mathbb{C})$ with $n > 2$. If $[z_{ij}] \in M_{4,2}(\mathbb{C})$ is a spherical point for $Sp(4) \times U(2) : M_{4,2}(\mathbb{C})$ well-adapted to $\alpha \in \Lambda$, as above say, then the same is true for

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ z_{31} & z_{32} \\ z_{41} & z_{42} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \in M_{2n,2}(\mathbb{C})$$

with respect to the action $Sp(2n) \times U(2) : M_{2n,2}(\mathbb{C})$. This just results from the way in which $Sp(4)$ embeds in $Sp(2n)$, namely

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \mapsto \left[\begin{array}{c|c|c|c} A & O & B & O \\ \hline O & I_{n-2} & O & O \\ \hline C & O & D & O \\ \hline O & O & O & I_{n-2} \end{array}\right].$$

4.6.2. **Case (h-1)**. Here $K = Sp(2n) \times U(3)$ acts on $V = M_{2n,3}(\mathbb{C})$. For $n \geq 3$ this is a rank 6 multiplicity free action with fundamental highest weights [10, Section 11.7]

$$\left\{ \begin{array}{lll} \alpha_1 = -(\varepsilon_1 + \varepsilon_1'), & \alpha_2 = -(\varepsilon_1 + \varepsilon_2 + \varepsilon_1' + \varepsilon_2'), & \alpha_3 = -(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_1' + \varepsilon_2' + \varepsilon_3'), \\ \alpha_4 = -(\varepsilon_1' + \varepsilon_2'), & \alpha_5 = -(\varepsilon_1 + \varepsilon_1' + \varepsilon_2' + \varepsilon_3'), & \alpha_6 = -(\varepsilon_1 + \varepsilon_2 + 2\varepsilon_1' + \varepsilon_2' + \varepsilon_3') \end{array} \right\}.$$

As in the previous case the story for $n \geq 3$ reduces to that for $n = 3$. So we henceforth take $K = Sp(6) \times U(3)$ and $V = M_{6,3}(\mathbb{C})$. Fundamental highest weight vectors h_j in $\mathbb{C}[V]$ for weights α_j are [10, Section 11.7]

$$\left\{ \begin{array}{l} h_1(z) = z_{11}, \quad h_2(z) = \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \quad h_3(z) = \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix}, \\ h_4(z) = \omega(z_{\bullet,1}, z_{\bullet,2}), \\ h_5(z) = z_{11}\omega(z_{\bullet,2}, z_{\bullet,3}) - z_{12}\omega(z_{\bullet,1}, z_{\bullet,3}) + z_{13}\omega(z_{\bullet,1}, z_{\bullet,2}), \\ h_6(z) = \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix} \omega(z_{\bullet,1}, z_{\bullet,2}) - \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} \omega(z_{\bullet,1}, z_{\bullet,3}) \end{array} \right\},$$

with degrees 1,2,3,2,3,4 respectively. The set Λ of highest weights occurring in $\mathbb{C}[V]$ is $\Lambda = \{\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + e\alpha_5 + f\alpha_6 \ : \ a,b,c,d,e,f \in \mathbb{Z}_{\geq 0}\}$ and $h_\alpha = h_1^a h_2^b h_3^c h_4^d h_5^e h_6^f$ is a highest weight vector with weight $\alpha \in \Lambda$.

Applying Lemma 3.1, as in the preceding case, one obtains a system of 18 quadratic equations for the entries z_{ij} of a spherical point for $\alpha \in \Lambda$. These are given in Table 3, as conditions on the rows and columns of $[z_{ij}]$. Using a computer algebra system

$$\left\{ \begin{array}{c} \|z_{1,\bullet}\|^2 - \|z_{4,\bullet}\|^2 = a + b + c + e + f \\ \|z_{2,\bullet}\|^2 - \|z_{5,\bullet}\|^2 = b + c + f \\ \|z_{3,\bullet}\|^2 - \|z_{6,\bullet}\|^2 = c \\ \|z_{\bullet,1}\|^2 = a + b + c + d + e + 2f \\ \|z_{\bullet,2}\|^2 = b + c + d + e + f \\ \|z_{\bullet,3}\|^2 = c + e + f \\ \hline \langle z_{1,\bullet}, z_{4,\bullet}\rangle = 0, \quad \langle z_{2,\bullet}, z_{5,\bullet}\rangle = 0, \quad \langle z_{3,\bullet}, z_{6,\bullet}\rangle = 0 \\ \langle z_{1,\bullet}, z_{5,\bullet}\rangle = -\langle z_{2,\bullet}, z_{4,\bullet}\rangle, \quad \langle z_{1,\bullet}, z_{6,\bullet}\rangle = -\langle z_{3,\bullet}, z_{4,\bullet}\rangle, \quad \langle z_{2,\bullet}, z_{6,\bullet}\rangle = -\langle z_{3,\bullet}, z_{5,\bullet}\rangle \\ \langle z_{1,\bullet}, z_{2,\bullet}\rangle = \langle z_{5,\bullet}, z_{4,\bullet}\rangle, \quad \langle z_{1,\bullet}, z_{3,\bullet}\rangle = \langle z_{6,\bullet}, z_{4,\bullet}\rangle, \quad \langle z_{2,\bullet}, z_{3,\bullet}\rangle = \langle z_{6,\bullet}, z_{5,\bullet}\rangle \\ \langle z_{\bullet,1}, z_{\bullet,2}\rangle = 0, \quad \langle z_{\bullet,1}, z_{\bullet,3}\rangle = 0, \quad \langle z_{\bullet,2}, z_{\bullet,3}\rangle = 0 \end{array} \right\}.$$

TABLE 3. Spherical point equations for Case (h)

one can verify that the entries of the following matrix $v = v(a, b, c, d, e, f)$ solves the equations in Table 3 for all positive real values $a, b, c, d, e, f > 0$. Namely

$$v := \begin{bmatrix} \sqrt{\frac{a s_1 s_2 s_3 s_4 s_5 s_6}{s_7 s_8 s_9 s_{10} s_{11} s_{12}}} & -\sqrt{\frac{bef s_{13} s_3 s_5 s_6}{s_8 s_9 s_{10} s_{14} s_{11} s_{12}}} & \sqrt{\frac{ed s_{13} s_1 s_{15} s_4 s_6}{s_7 s_8 s_9 s_{14} s_{11} s_{12}}} \\ \sqrt{\frac{ef s_2 s_{15} s_{16} s_4 s_5}{s_7 s_9 s_{10} s_{17} s_{18} s_{12}}} & \sqrt{\frac{ab s_{13} s_1 s_{15} s_{16} s_5}{s_9 s_{10} s_{14} s_{17} s_{18} s_{12}}} & -\sqrt{\frac{adf s_{13} s_{16} s_3 s_4}{s_7 s_9 s_{14} s_{17} s_{18} s_{12}}} \\ -\sqrt{\frac{abced s_4 s_5}{s_7 s_8 s_{10} s_{17} s_{18} s_{11}}} & \sqrt{\frac{cdf s_{13} s_1 s_2 s_5}{s_8 s_{10} s_{14} s_{17} s_{18} s_{11}}} & \sqrt{\frac{bc s_{13} s_2 s_{15} s_3 s_4}{s_7 s_8 s_{14} s_{17} s_{18} s_{11}}} \\ \sqrt{\frac{bedf s_{13} s_{15} s_6}{s_7 s_8 s_9 s_{10} s_{11} s_{12}}} & \sqrt{\frac{ad s_1 s_2 s_{15} s_4 s_6}{s_8 s_9 s_{10} s_{14} s_{11} s_{12}}} & \sqrt{\frac{abf s_2 s_3 s_5 s_6}{s_7 s_8 s_9 s_{14} s_{11} s_{12}}} \\ -\sqrt{\frac{abd s_{13} s_1 s_{16} s_3}{s_7 s_9 s_{10} s_{17} s_{18} s_{12}}} & \sqrt{\frac{edf s_2 s_{16} s_3 s_4}{s_9 s_{10} s_{14} s_{17} s_{18} s_{12}}} & \sqrt{\frac{be s_1 s_2 s_{15} s_{16} s_5}{s_7 s_9 s_{14} s_{17} s_{18} s_{12}}} \\ -\sqrt{\frac{cf s_{13} s_1 s_2 s_{15} s_3}{s_7 s_8 s_{10} s_{17} s_{18} s_{11}}} & -\sqrt{\frac{abce s_{15} s_3 s_4}{s_8 s_{10} s_{14} s_{17} s_{18} s_{11}}} & \sqrt{\frac{acedf s_1 s_5}{s_7 s_8 s_{14} s_{17} s_{18} s_{11}}} \end{bmatrix}, \tag{4.2}$$

where $s_1, \ldots, s_{18}$ are certain linear combinations of $a, \ldots, f$ with coefficients 1 and 2:

$$\left\{ \begin{array}{lll} s_1 = a + e + f, & s_2 = a + b + f, & s_3 = a + b + d + e + f, \\ s_4 = a + b + 2\,c + e + 2\,f, & s_5 = a + 2\,b + 2\,c + d + e + 2\,f, & s_6 = a + b + c + e + f, \\ s_7 = a + b + d + f, & s_8 = a + b + e + f, & s_9 = a + e, \\ s_{10} = a + f, & s_{11} = a + b + 2\,c + e + f, & s_{12} = a + 2\,b + 2\,c + e + 2\,f, \\ s_{13} = b + 2\,c + e + f, & s_{14} = b + d, & s_{15} = b + d + f, \\ s_{16} = b + c + f, & s_{17} = b + f, & s_{18} = b + 2\,c + f \end{array} \right\}.$$

Taking positive integer values for $a \ldots, f$, the point $v_\alpha = v(a, \ldots, f)$ is a spherical point for generic weight $(\alpha = a\alpha_1 + \cdots + f\alpha_6) \in \Lambda$. Each entry in $v(a, \ldots, f)$ is a non-zero real number; the signed square root of a quotient of 7 factors from $\{a, \ldots, f, s_1, \ldots, s_{18}\}$ by 6 factors from $\{s_1, \ldots, s_{18}\}$. The interested reader can download a Maple worksheet fully justifying the details for this example [4].

Evaluating the fundamental highest weight vectors $h_1, \ldots, h_6$ at the point $v = v(a, \ldots, f)$ yields the following expressions upon computer-aided simplification. Note that the values $h_j\big(v(a, \ldots, f)\big)$ are non-zero for all positive real values of the parameters $a, \ldots, f$. This verifies condition (2) in Lemma 3.5. Thus for generic weights $\alpha \in \Lambda$ our spherical point v_α lies in the open B-orbit and is, by Corollary 3.4, well-adapted to h_α.

$$\left\{\begin{array}{l} h_1(v) = \sqrt{\frac{a(a+e+f)(a+b+f)(a+b+d+e+f)(a+b+2\,c+e+2\,f)(a+2\,b+2\,c+d+e+2\,f)(a+b+c+e+f)}{(a+e)(a+f)(a+b+d+f)(a+b+e+f)(a+b+2\,c+e+f)(a+2\,b+2\,c+e+2\,f)}} \\ h_2(v) = \frac{a+2\,b+2\,c+d+e+2\,f}{a+2\,b+2\,c+e+2\,f}\sqrt{\frac{b(a+b+c+e+f)(a+b+2\,c+e+2\,f)(a+b+d+e+f)(a+b+f)}{(b+d)(b+f)(a+b+2\,c+e+f)(a+b+d+f)}} \\ \qquad\qquad \times\sqrt{\frac{(b+c+f)(b+2\,c+e+f)(b+d+f)}{(a+b+e+f)(b+2\,c+f)}} \\ h_3(v) = \frac{(a+b+2\,c+e+2\,f)(a+2\,b+2\,c+d+e+2\,f)(b+2\,c+e+f)}{(a+b+2\,c+e+f)(a+2\,b+2\,c+e+2\,f)(b+2\,c+f)}\sqrt{c(a+b+c+e+f)(b+c+f)} \\ h_4(v) = \sqrt{\frac{d(a+b+e+d+f)(a+2\,b+2\,c+d+e+2\,f)(b+d+f)}{(b+d)(a+b+d+f)}} \\ h_5(v) = \sqrt{\frac{e(a+b+c+e+f)(a+b+d+e+f)(a+b+2\,c+e+2\,f)(a+2\,b+2\,c+d+e+2\,f)(a+e+f)(b+2\,c+e+f)}{(a+e)(a+b+2\,c+e+f)(a+2\,b+2\,c+e+2\,f)(a+b+e+f)}} \\ h_6(v) = -\frac{(a+b+2\,c+e+2\,f)(a+2\,b+2\,c+d+e+2\,f)}{(a+2\,b+2\,c+e+2\,f)} \\ \qquad\qquad \times\sqrt{\frac{f(a+b+c+e+f)(a+b+d+e+f)(a+b+f)(a+e+f)(b+c+f)(b+2\,c+e+f)(b+d+f)}{(a+f)(b+f)(a+b+d+f)(a+b+e+f)(a+b+2\,c+e+f)(b+2\,c+f)}} \end{array}\right\}$$

To complete the proof that $Sp(6) \times U(3) : M_{6,3}(\mathbb{C})$ is a well-behaved multiplicity free action it remains to check limit conditions (3) and (4) from Lemma 3.5. Using a computer algebra system one verifies these conditions for the successive limits associated with each of the 62 non-empty proper subsets of the parameter set $\{a, \ldots, f\}$. Full details are given in [4]. One obtains, for example,

$$\lim_{b\to 0^+}\lim_{a\to 0^+} v\big(a,b,c,d,e,f\big) = \begin{bmatrix} 0 & 0 & \sqrt{c+e+f} \\ \sqrt{\frac{(c+f)(2\,c+d+e+2\,f)}{2\,c+f}} & 0 & 0 \\ 0 & \sqrt{\frac{c(2\,c+d+e+2\,f)}{2\,c+f}} & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{\frac{(c+f)(e+d+f)}{2\,c+f}} & 0 \\ -\sqrt{\frac{c(e+d+f)}{2\,c+f}} & 0 & 0 \end{bmatrix}$$

for fixed $c, d, e, f > 0$. The right hand side is a spherical point for weight $c\alpha_3 + d\alpha_4 + e\alpha_5 + f\alpha_6$ when $c, \ldots, f$ are positive integers. Evaluating $h_3(z), \ldots, h_6(z)$ at this limit point gives

$$\left\{\begin{array}{c} h_3 = \frac{(2\,c+d+e+2\,f)}{2\,c+f}\sqrt{c(c+f)(c+e+f)}, \qquad h_4 = \sqrt{(2\,c+d+e+2\,f)(e+d+f)}, \\ h_5 = \sqrt{(c+e+f)(2\,c+d+e+2\,f)(e+d+f)}, \\ h_6 = -\,(2\,c+d+e+2\,f)\sqrt{\frac{(c+f)(c+e+f)(e+d+f)}{2\,c+f}} \end{array}\right\},$$

each of which is non-zero, as required in condition (4) of Lemma 3.5.

4.6.3. **Case (h-2)**. The action $Sp(4)\times U(3) : M_{4,3}(\mathbb{C})$ has rank 5. The fundamental highest weights α_j and highest weight vectors $h_j(z)$ are as in Case (h-1) except that one must drop α_3 and $h_3(z)$ from the list [10]. In view of the embedding $Sp(4)\hookrightarrow Sp(6)$ we see that a (generalized) generic spherical point $v(a,b,d,e,f)$ for this example can be obtained from (4.2) by setting parameter c to zero and deleting the third and sixth row. Our treatment of Case (h-1) thus encompasses this action as well, showing it to be well-behaved.

4.6.4. **Case (i)**. Next consider the action of $Sp(4)\times U(m)$ on $V=M_{4,m}(\mathbb{C})$ with $m\geq 4$. This is a rank 6 multiplicity free action with fundamental highest weights [10, Section 11.8]

$$\left\{\begin{array}{lll} \alpha_1=-(\varepsilon_1+\varepsilon_1'), & \alpha_2=-(\varepsilon_1+\varepsilon_2+\varepsilon_1'+\varepsilon_2'), & \alpha_3=-(\varepsilon_1+\varepsilon_1'+\varepsilon_2'+\varepsilon_3'), \\ \alpha_4=-(\varepsilon_1'+\varepsilon_2'), & \alpha_5=-(\varepsilon_1+\varepsilon_2+2\varepsilon_1'+\varepsilon_2'+\varepsilon_3'), & \alpha_6=-(\varepsilon_1'+\varepsilon_2'+\varepsilon_3'+\varepsilon_4') \end{array}\right\}.$$

With the standard embeddings $M_{4,4}(\mathbb{C})\hookrightarrow M_{4,m}(\mathbb{C})$, $U(4)\hookrightarrow U(m)$ in mind, it suffices to take $m=4$, $K=Sp(4)\times U(4)$, $V=M_{4,4}(\mathbb{C})$ here. Fundamental highest weight vectors $h_j\in\mathbb{C}[V]$ for weights α_j are [10, Section 11.8]

$$\left\{\begin{array}{l} h_1(z)=z_{11}, \quad h_2(z)=\begin{vmatrix} z_{11} & z_{12}\\ z_{21} & z_{22}\end{vmatrix}, \quad h_3(z)=\begin{vmatrix} z_{11} & z_{12} & z_{13}\\ z_{21} & z_{22} & z_{23}\\ z_{41} & z_{42} & z_{43}\end{vmatrix}, \\ h_4(z)=\omega(z_{\bullet,1},z_{\bullet,2}), \\ h_5(z)=\begin{vmatrix} z_{11} & z_{13}\\ z_{21} & z_{23}\end{vmatrix}\omega(z_{\bullet,1},z_{\bullet,2})-\begin{vmatrix} z_{11} & z_{12}\\ z_{21} & z_{22}\end{vmatrix}\omega(z_{\bullet,1},z_{\bullet,3}), \\ h_6(z)=\det(z) \end{array}\right\},$$

in degrees 1,2,3,2,4,4 respectively. So $\Lambda=\{\alpha=a\alpha_1+b\alpha_2+c\alpha_3+d\alpha_4+e\alpha_5+f\alpha_6 \;:\; a,b,c,d,e,f\in\mathbb{Z}_{\geq 0}\}$ is the set of highest weights occurring in $\mathbb{C}[V]$ and $h_\alpha=h_1^a h_2^b h_3^c h_4^d h_5^e h_6^f$ is a highest weight vector with weight $\alpha\in\Lambda$. Lemma 3.1 produces a system of 16 equations for the matrix entries z_{ij} in a spherical point for weight α, namely

$$\left\{\begin{array}{c} \|z_{1,\bullet}\|^2-\|z_{3,\bullet}\|^2=a+b+c+e, \quad \|z_{2,\bullet}\|^2-\|z_{4,\bullet}\|^2=b+e \\ \|z_{\bullet,1}\|^2=a+b+c+d+2e+f, \quad \|z_{\bullet,2}\|^2=b+c+d+e+f \\ \|z_{\bullet,3}\|^2=c+e+f, \quad \|z_{\bullet,4}\|^2=f \\ \hline \langle z_{1,\bullet},z_{3,\bullet}\rangle=0, \quad \langle z_{2,\bullet},z_{4,\bullet}\rangle=0 \\ \langle z_{1,\bullet},z_{4,\bullet}\rangle=-\langle z_{2,\bullet},z_{3,\bullet}\rangle, \quad \langle z_{1,\bullet},z_{2,\bullet}\rangle=\langle z_{4,\bullet},z_{3,\bullet}\rangle \\ \langle z_{\bullet,1},z_{\bullet,2}\rangle=0, \quad \langle z_{\bullet,1},z_{\bullet,3}\rangle=0, \quad \langle z_{\bullet,1},z_{\bullet,4}\rangle=0 \\ \langle z_{\bullet,2},z_{\bullet,3}\rangle=0, \quad \langle z_{\bullet,2},z_{\bullet,4}\rangle=0, \quad \langle z_{\bullet,3},z_{\bullet,4}\rangle=0 \end{array}\right\}.$$

The results here are similar in flavor to those for Case (h). Using a computer algebra system one can show that the matrix $v = v(a,b,c,d,e,f)$ given by

$$\begin{bmatrix} \sqrt{\frac{as_1s_2s_3s_4s_5s_6}{s_7s_8s_9s_{10}s_{11}s_{12}}} & -\sqrt{\frac{bces_5s_4s_{13}s_{15}}{s_7s_8s_{16}s_9s_{11}s_{18}}} & -\sqrt{\frac{cds_3s_2s_{13}s_{14}s_{19}}{s_7s_{16}s_{20}s_9s_{11}s_{10}}} & -\sqrt{\frac{abdefs_1s_{14}}{s_7s_{20}s_9s_{11}s_{12}s_{18}}} \\ -\sqrt{\frac{ces_1s_3s_4s_{14}s_6}{s_7s_8s_{17}s_{11}s_{12}s_{10}}} & -\sqrt{\frac{abs_4s_2s_{13}s_{14}s_{15}}{s_7s_8s_{16}s_{17}s_{11}s_{18}}} & -\sqrt{\frac{ades_3s_5s_{13}s_{19}}{s_7s_{16}s_{17}s_{20}s_{11}s_{10}}} & \sqrt{\frac{bcdfs_1s_5s_2}{s_7s_{17}s_{20}s_{11}s_{12}s_{18}}} \\ \sqrt{\frac{bcdes_{13}s_{14}s_6}{s_7s_8s_9s_{10}s_{11}s_{12}}} & \sqrt{\frac{ads_1s_3s_2s_{14}s_{15}}{s_7s_8s_{16}s_9s_{11}s_{18}}} & -\sqrt{\frac{abes_5s_4s_1s_{19}}{s_7s_{16}s_{20}s_9s_{11}s_{10}}} & \sqrt{\frac{cfs_3s_5s_4s_2s_{13}}{s_7s_{20}s_9s_{11}s_{12}s_{18}}} \\ \sqrt{\frac{abds_5s_2s_{13}s_6}{s_7s_8s_{17}s_{11}s_{12}s_{10}}} & -\sqrt{\frac{cdes_3s_5s_1s_{15}}{s_7s_8s_{16}s_{17}s_{11}s_{18}}} & \sqrt{\frac{bcs_4s_1s_2s_{14}s_{19}}{s_7s_{16}s_{17}s_{20}s_{11}s_{10}}} & \sqrt{\frac{aefs_3s_4s_{14}s_{13}}{s_7s_{17}s_{20}s_{11}s_{12}s_{18}}} \end{bmatrix}$$

solves the preceding equations for all positive real parameters $a,\ldots,f$. Here $s_1,\ldots,s_{20}$ denote

$$\left\{ \begin{array}{lll} s_1 = a+b+e, & s_2 = a+c+e, & s_3 = a+b+c+2\,e, \\ s_4 = a+2\,b+c+d+2\,e, & s_5 = a+b+c+d+e, & s_6 = a+b+c+d+2\,e+f, \\ s_7 = a+c, & s_8 = a+e, & s_9 = a+b+c+e, \\ s_{10} = a+b+d+e, & s_{11} = a+2\,b+c+2\,e, & s_{12} = a+b+c+d+2\,e, \\ s_{13} = b+c+e, & s_{14} = b+d+e, & s_{15} = b+c+d+e+f, \\ s_{16} = b+d, & s_{17} = b+e, & s_{18} = b+c+d+e, \\ s_{19} = c+e+f, & s_{20} = c+e & \end{array} \right\}.$$

Plugging $v(a,\ldots,f)$ into the highest weight vectors $h_j(z)$ and using a computer to simplify yields

$$\left\{ \begin{array}{l} h_1(v) = \sqrt{\frac{a(a+b+e)(a+c+e)(a+b+c+2\,e)(a+b+c+d+e)(a+2\,b+c+d+2\,e)(a+b+c+d+2\,e+f)}{(a+c)(a+e)(a+b+c+e)(a+b+d+e)(a+2\,b+c+2\,e)(a+b+c+d+2\,e)}} \\ h_2(v) = -\frac{a+2\,b+c+d+2\,e}{a+2\,b+c+2\,e}\sqrt{\frac{b(a+b+e)(a+b+c+2\,e)(a+b+c+d+e)}{(b+d)(b+e)(a+b+c+e)}} \\ \qquad\qquad \times\sqrt{\frac{(a+b+c+d+2\,e+f)(b+c+e)(b+d+e)(b+c+d+e+f)}{(a+b+c+d+2\,e)(a+b+d+e)(b+c+d+e)}} \\ h_3(v) = -\sqrt{\frac{c(a+c+e)(a+b+c+2\,e)(a+b+c+d+e)(a+2\,b+c+d+2\,e)}{(a+c)(c+e)(a+b+c+e)(a+2\,b+c+2\,e)}} \\ \qquad\qquad \times\sqrt{\frac{(a+b+c+d+2\,e+f)(b+c+e)(b+c+d+e+f)(c+e+f)}{(a+b+c+d+2\,e)(b+c+d+e)}} \\ h_4(v) = \sqrt{\frac{d(a+b+c+d+e)(a+2\,b+c+d+2\,e)(a+b+c+d+2\,e+f)(b+d+e)(b+c+d+e+f)}{(b+d)(a+b+d+e)(a+b+c+d+2\,e)(b+c+d+e)}} \\ h_5(v) = -\frac{(a+b+c+2\,e)(a+2\,b+c+d+2\,e)(a+b+c+d+2\,e+f)}{(a+2\,b+c+2\,e)(a+b+c+d+2\,e)} \\ \qquad\qquad \times\sqrt{\frac{e(a+b+e)(a+c+e)(a+b+c+d+e)(b+c+e)(b+d+e)(b+c+d+e+f)(c+e+f)}{(a+e)(b+e)(c+e)(a+b+c+e)(a+b+d+e)(b+c+d+e)}} \\ h_6(v) = \sqrt{f(a+b+c+d+2\,e+f)(b+c+d+e+f)(c+e+f)} \end{array} \right\}$$

Note that these values are all non-zero as required in condition (2) of Lemma 3.5. One can also use a computer to verify conditions (3) and (4) in the lemma, completing the proof that this action is well-behaved. All of these calculations are presented in [4]. For example we compute that for given $a, c, f > 0$

$$\lim_{e\to 0^+}\lim_{d\to 0^+}\lim_{b\to 0^+} v(a,b,c,d,e,f) = \begin{bmatrix} \sqrt{a+c+f} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{c+f} & 0 \\ 0 & 0 & 0 & \sqrt{f} \\ 0 & -\sqrt{c+f} & 0 & 0 \end{bmatrix}$$

and the fundamental highest weight vectors h_1, h_3, h_6 take non-zero values at this limit point, namely

$$h_1 = \sqrt{a+c+f}, \quad h_3 = -(c+f)\sqrt{a+c+f}, \quad h_6 = (c+f)\sqrt{f(a+c+f)}.$$

4.7. **Case (k).** Here V is the exterior algebra $V = \Lambda(\mathbb{C}^4) = \sum_{j=1}^4 \Lambda^j(\mathbb{C}^4)$ equipped with its usual Hermitian inner product. This is 16 dimensional with orthonormal basis

$$\mathcal{B} = \{1,\, e_1,\, e_2,\, e_3,\, e_4,\, e_{12},\, e_{13},\, e_{14},\, e_{23},\, e_{24},\, e_{34},\, e_{234},\, e_{134},\, e_{124},\, e_{123},\, e_{1234}\}$$

where $e_{j_1\cdots j_k} = e_{j_1}\wedge\cdots\wedge e_{j_k}$ denotes a wedge product of standard basis vectors in $\mathbb{C}^4$. We write $(z_\emptyset,\, z_1,\, z_2,\, z_3,\, z_4,\, z_{12},\, z_{13},\, z_{14},\, z_{23},\, z_{24},\, z_{34},\, z_{234},\, z_{134},\, z_{124},\, z_{123},\, z_{1234})$ for coordinates with respect to $\mathcal{B}$. The compact group $Spin(9)$ acts unitarily on V via its half spin representation. The derived representation of $so(9)$ on V can be obtained by embedding $so(9)$ in the real Clifford algebra C_9 and realizing V as a module over C_9. The image of $so(9)$ in $u(V)$ is given explicitly e.g. in [8, Chapter 3]. Complexifying yields a copy of $so(9,\mathbb{C})$ inside $gl(V)$. This is the $\mathbb{C}$-span of the 36 operators

$$\left\{ \begin{array}{c} H_k = \frac{1}{2}(D_kW_k - W_kD_k)\ (1\le k\le 4), \\ W_kD_\ell\ (1\le k\ne \ell\le 4),\quad W_kW_\ell\ (1\le k<\ell\le 4),\quad D_kD_\ell\ (1\le k<\ell\le 4), \\ \mathcal{S}W_k\ (1\le k\le 4),\quad \mathcal{S}D_k\ (1\le k\le 4), \end{array} \right\}$$

where W_k is the operator $W_k(v) = e_k\wedge v$, D_k its adjoint, namely contraction by e_k, and $\mathcal{S}$ acts on $\Lambda^j(\mathbb{C}^4)$ via multiplication by $(-1)^j$ for each $0\le j\le 4$. As Cartan subalgebra $\mathfrak{h}_9$ and Borel subalgebra $\mathfrak{b}_9 = \mathfrak{h}_9\oplus\mathfrak{n}_9$ we take $\mathfrak{h}_9 = \mathbb{C}\text{-Span}\{H_1,\dots,H_4\}$,

$$\mathfrak{n}_9 = \mathbb{C}\text{-Span}\Big(\{W_kD_\ell\ :\ 1\le k<\ell\le 4\}\cup\{W_kW_\ell\ :\ 1\le k<\ell\le 4\}\cup\{\mathcal{S}W_1,\dots,\mathcal{S}W_4\}\Big).$$

Now $K = Spin(9)\times\mathbb{T}$, $\mathfrak{k}_{\mathbb{C}} = so(9,\mathbb{C})\times\mathbb{C}$ and $\mathfrak{h} = \mathfrak{h}_9\times\mathbb{C}$. We let $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon'\in\mathfrak{h}^*$ denote the functionals

$$\varepsilon_j\big(a_1H_1+\cdots+a_4H_4,\ b\big) = a_j,\quad \varepsilon'\big(a_1H_1+\cdots+a_4H_4,\ b\big) = b.$$

According to [10, Section 11.11] the multiplicity free action $K : V$ has rank 3 with fundamental highest weights α_j and fundamental highest weight vectors h_j given by

$$\alpha_1 = -\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4) - \varepsilon', \quad \alpha_2 = -2\varepsilon', \quad \alpha_3 = -\varepsilon_1 - 2\varepsilon',$$

$$\left\{ \begin{array}{c} h_1(z) = z_\emptyset, \\ h_2(z) = z_\emptyset z_{1234} + z_1 z_{234} - z_2 z_{134} + z_3 z_{124} - z_4 z_{123} - z_{12} z_{34} + z_{13} z_{24} - z_{14} z_{23}, \\ h_3(z) = z_2 z_{34} - z_3 z_{24} + z_4 z_{23} - z_\emptyset z_{234} \end{array} \right\}.$$

Here $h_2(z)$ is $Spin(9)$-invariant and $h_3(z)$ lies in a 9 dimensional irreducible subspace of $\mathcal{P}_2(V)$ on which $Spin(9)$ acts via a copy of the representation contragredient to the defining representation for $SO(9)$.

Lemma 3.1 gives a system of 21 distinct equations for the coordinates $(z_\emptyset, \ldots, z_{1234})$ of a spherical point for weight $(\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3) \in \Lambda$. These are obtained by letting X in (3.1) range over the above basis for $\mathfrak{b}_9$ together with an infinitesimal generator T for the scalar action. Numerical experimentation with a computer algebra system reveals that this system has generic solutions in which all of the coordinates vanish save $z_\emptyset$, z_1, z_{234} and z_{1234}. Setting the remaining coordinate variables to zero reduces the system to the four equations

$$\left\{ \begin{array}{c} \|z_\emptyset\|^2 - \|z_1\|^2 + \|z_{234}\|^2 - \|z_{1234}\|^2 = a + 2c \\ \|z_\emptyset\|^2 + \|z_1\|^2 - \|z_{234}\|^2 - \|z_{1234}\|^2 = a \\ \|z_\emptyset\|^2 + \|z_1\|^2 + \|z_{234}\|^2 + \|z_{1234}\|^2 = a + 2b + 2c \\ z_\emptyset \overline{z_1} - z_{234}\overline{z_{1234}} = 0 \end{array} \right\}. \tag{4.3}$$

These arise by taking $X = H_1,\ H_2,\ T,\ \mathcal{S}W_1$ in (3.1). One checks that $v = v(a,b,c)$ given by

$$v := \sqrt{\frac{(a+c)\,(a+b+2\,c)}{a+2\,c}}\,1 + \sqrt{\frac{bc}{a+2\,c}}\,e_1 + \sqrt{\frac{c\,(a+b+2\,c)}{a+2\,c}}\,e_{234} + \sqrt{\frac{b\,(a+c)}{a+2\,c}}\,e_{1234}$$

solves Equations 4.3 for all positive real parameters a, b, c with $a+2c \neq 0$. Evaluating the fundamental highest weight vectors at $v(a,b,c)$ yields

$$\left\{ \begin{array}{c} h_1\big(v(a,b,c)\big) = \sqrt{\frac{(a+c)(a+b+2c)}{a+2c}}, \quad h_2\big(v(a,b,c)\big) = \sqrt{b(a+b+2c)}, \\ h_3\big(v(a,b,c)\big) = -\frac{(a+b+2c)\sqrt{c(a+c)}}{a+2c} \end{array} \right\}.$$

As these values are non-zero we see that condition (2) in Lemma 3.5 holds here. Condition (3) from the Lemma also holds because

$$\lim_{c\to 0^+} \lim_{a \to 0^+} v(a,b,c) = \lim_{c \to 0^+} v(0,b,c) = \sqrt{\frac{b}{2}}\big(1 + e_1 + e_{234} + e_{1234}\big)$$

exists in V. Limiting values for $h_j\big(v(a,b,c)\big)$ $(j = 1,2,3)$ as one or two parameters approach zero are

limit	$h_1\big(v(a,b,c)\big)$	$h_2\big(v(a,b,c)\big)$	$h_3\big(v(a,b,c)\big)$
$lim_{a\to 0^+}$	$\sqrt{c+b/2}$	$\sqrt{b(b+2c)}$	$-(b/2+c)$
$lim_{b\to 0^+}$	$\sqrt{a+c}$	0	$-\sqrt{c(a+c)}$
$lim_{c\to 0^+}$	$\sqrt{a+b}$	$\sqrt{b(a+b)}$	0
$lim_{b\to 0^+} lim_{a\to 0^+}$	$\sqrt{c}$	0	$-c$
$lim_{c\to 0^+} lim_{a\to 0^+}$	$\sqrt{b/2}$	$\sqrt{b(a+b)}$	0
$lim_{c\to 0^+} lim_{b\to 0^+}$	$\sqrt{a}$	0	0

showing, in particular, that condition (4) in Lemma 3.5 applies. This completes the verification that $K : V$ is well-behaved and the proof of Theorem 1.5.

5. Remarks on Computer-aided calculations

We made heavy use of Maple in our case-by-case calculations, especially in connection with actions (h), (i) and (k) from Table 1. In each case the main computational obstacle was the determination of a generic generalized spherical point $v(\mathbf{x}) = v(x_1, \ldots, x_r)$ as in the statement of Lemma 3.5. With this in hand we found that the computer algebra system could be coaxed to check conditions (2), (3) and (4) from the Lemma. The worksheets [4] accomplish this for actions (h) and (i). They also show how Maple can be used to verify that $v(\mathbf{x})$ does in fact solve the equations given in condition (1) of the Lemma. That is, that $v(\mathbf{x})$ is indeed a generic generalized spherical point as claimed.

In each case it is not difficult to program the system of quadratic equations from Lemma 3.5(1). Unfortunately Maple was, in most cases, unable to produce any solutions on a general symbolic input $\mathbf{x} = (x_1, \ldots, x_r)$. We found, however, that Maple could often produce solutions on specific numeric inputs. For actions (g) and (k) the numerical evidence suggested a sparse pattern for a generic generalized spherical point. Maple was then able to produce general solutions upon augmentation of system (1) by this guess. In fact the resulting formulas are, in the end, easy to check with pen and paper. In contrast, the numerical data for actions (h) and (i) did not exhibit a sparse pattern. For these actions we conjectured the forms for the matrices $v(a,b,c,d,e,f)$, given above, entry-by-entry via a process of experimental interpolation from a large quantity of computer-generated numerical data. We used the observation that the square of each matrix entry was a quotient of products of linear terms. Having guessed the formulas Maple was able to verify that these do indeed give generic generalized spherical points, with a little coaxing. Thus although the worksheets [4] do complete the justification for these examples they do not reveal the process which led to the spherical point formulas.

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