SPACES OF BOUNDED SPHERICAL FUNCTIONS FOR IRREDUCIBLE NILPOTENT GELFAND PAIRS: PART II

CHAL BENSON AND GAIL RATCLIFF

Abstract. In prior work an orbit method, due to Pukanszky and Lipsman, was used to produce an injective mapping $\Psi : \Delta(K,N) \to \mathfrak{n}^*/K$ from the space of bounded K-spherical functions for a nilpotent Gelfand pair (K,N) into the space of K-orbits in the dual for the Lie algebra $\mathfrak{n}$ of N. We have conjectured that Ψ is a topological embedding. In this paper we complete the proof of this conjecture under the hypothesis that (K,N) is an *irreducible* nilpotent Gelfand pair. Following Part I of this work it remains to verify the conjecture in six exceptional cases from Vinberg's classification of irreducible nilpotent Gelfand pairs.

1. Introduction

This paper is a continuation of [BR20] to which we refer the reader for background material and motivation. The context is as follows. N will denote a connected and simply connected nilpotent Lie group and K a compact Lie group acting smoothly on N via automorphisms to yield a *nilpotent Gelfand pair* (abbr. n.G.p.) (K,N). That is, the convolution algebra $L^1_K(N)$ of integrable K-invariant functions on N is commutative as is $\mathbb{D}_K(N)$, the algebra of left-N and K-invariant differential operators on N. It then follows that the group N is necessarily two-step or abelian [BJR90]. The *spherical functions* for such a n.G.p. are the smooth K-invariant joint eigenfunctions $\phi : N \to \mathbb{C}$ for the operators $\mathbb{D}_K(N)$ satisfying $\phi(e) = 1$. We let $\Delta(K,N)$ denote the space of all *bounded* K-spherical functions on N with the topology of uniform convergence on compact sets.

In [BR08] we applied an *orbit method* due to Pukanszky [Puk78] and Lipsman [Lip80, Lip82] to produce an injective map $\Psi : \Delta(K,N) \to \mathfrak{n}^*/K$ from $\Delta(K,N)$ to the set of K-orbits in $\mathfrak{n}^*$ ($\mathfrak{n} := \mathrm{Lie}(N)$). We let $\mathcal{A}(K,N)$ denote the image of Ψ and call this the set of K*-spherical orbits* in $\mathfrak{n}^*$. Endowing $\mathcal{A}(K,N)$ with the quotient topology inherited from $\mathfrak{n}^*/K$ we conjectured that:

$$\text{(O)} : \qquad \text{the bijection } \Psi : \Delta(K,N) \to \mathcal{A}(K,N) \text{ is a homeomorphism.}$$

This paper completes the proof of the following theorem, announced in [BR20].

Theorem 1.1. *Every irreducible n.G.p. satisfies conjecture* (O).

1991 *Mathematics Subject Classification.* Primary 22E30, 43A90.

Key words and phrases. Gelfand Pairs, Spherical Functions, Nilpotent Lie Groups, Orbit Method.

Here *irreducibility* for (K, N) means that K acts irreducibly on $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$. It follows that $[\mathfrak{n},\mathfrak{n}] = \mathfrak{z}$, with $\mathfrak{z} = \mathfrak{z}(\mathfrak{n})$ the center of $\mathfrak{n}$. Fix a K-invariant inner product on $\mathfrak{n}$ and write

$$\mathfrak{n} = V \oplus \mathfrak{z}$$

where $V = \mathfrak{z}^\perp$. The subspaces $V, \mathfrak{z} \subset \mathfrak{n}$ are K-invariant and the Lie bracket amounts to an anti-symmetric bilinear mapping $V \times V \to \mathfrak{z}$. Our conjecture is proved for Heisenberg n.G.p.'s in [BR13, BR15a, BR15b]. From [BR20] the proof of Theorem 1.1 boils down to verifying (O) for the irreducible n.G.p.'s from Vinberg's classification [Vin03] with $\mathfrak{n}$ not a Heisenberg Lie algebra. These are listed in Table 1. The $\mathfrak{z}$ entries in the table follow Vinberg's notational conventions. The actions of K on V and $\mathfrak{z}$ as well as the Lie bracket $V \times V \to \mathfrak{z}$ (which is in fact determined by the K-actions) can be found in [Wol07, Chapter 13] and in [FRY12, FRY13, FRY, FRY18].

	K	V	$\mathfrak{z}$	condition
1	$SO(d)$	$\mathbb{R}^d$	$\Lambda^2(\mathbb{R}^d)$	$d \geq 3$
2	$SU(d)$	$\mathbb{C}^d$	$\Lambda^2(\mathbb{C}^d) \oplus \mathbb{R}$	$d \geq 2$ even
3	$U(d)$	$\mathbb{C}^d$	$\Lambda^2(\mathbb{C}^d) \oplus \mathbb{R}$	$d \geq 3$ odd
4	$SU(d)$	$\mathbb{C}^d$	$\Lambda^2(\mathbb{C}^d)$	$d \geq 3$ odd
5	$U(d)$	$\mathbb{C}^d$	$H\Lambda^2(\mathbb{C}^d) = \mathfrak{u}(d)$	$d \geq 2$
6	$Sp(d)$	$\mathbb{H}^d$	$HS^2(\mathbb{H}^d) \oplus \mathbb{C}$	$d \geq 1$
7	$Sp(1) \times Sp(d)$	$\mathbb{H}^d$	$\mathbb{H}_0 = \mathfrak{sp}(1)$	$d \geq 2$
8	$Spin(7)$	$\mathbb{R}^8$	$\mathbb{R}^7$	
9	$SU(2) \times SU(d)$	$\mathbb{C}^2 \otimes \mathbb{C}^d$	$H\Lambda^2(\mathbb{C}^2) = \mathfrak{u}(2)$	$d \geq 3$
10	$U(2) \times SU(2)$	$\mathbb{C}^2 \otimes \mathbb{C}^2$	$H\Lambda^2(\mathbb{C}^2) = \mathfrak{u}(2)$	
11	$U(2) \times Sp(d)$	$\mathbb{C}^2 \otimes \mathbb{H}^d$	$H\Lambda^2(\mathbb{C}^2) = \mathfrak{u}(2)$	$d \geq 1$
12	$Sp(2) \times Sp(d)$	$\mathbb{H}^2 \otimes \mathbb{H}^d$	$H\Lambda^2(\mathbb{H}^2) = \mathfrak{sp}(2)$	$d \geq 1$
13	G_2	$\mathbb{R}^7$	$\mathbb{R}^7$	
14	$U(1) \times Spin(7)$	$\mathbb{C}^8$	$\mathbb{R}^7 \oplus \mathbb{R}$	

TABLE 1

The first six entries in Table 1 contain families of examples in which $\dim(\mathfrak{z})$ increases without bound. The verification of conjecture (O) in each of these cases was carried out in [BR20]. The remaining table entries are exceptional cases involving a fixed center $\mathfrak{z}$. A result from [FGJ+19] establishes (O) for the pairs in lines 7 and 8 of the table. Here, in the case where the K-orbits in the center are spheres and the form $(u, z) \mapsto ([u, v], z_\circ)_\mathfrak{n}$ is non-degenerate on V for non-zero $z_\circ \in \mathfrak{z}$, the conjecture is proved without the assumption of irreducibility. It remains to verify (O) for entries 9-14 in Table 1. This is done below following a discussion of preliminary material and notation.

2. Notation and preliminaries

This section summarizes required results and notation from [BR20], to which we refer the reader for details. Throughout we assume (K, N) to be a n.G.p. with N two-step and $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{z}$. As noted in Section 1 this is the case for irreducible n.G.p.'s. Fix a K-invariant inner product $(\cdot, \cdot)_{\mathfrak{n}}$ on $\mathfrak{n}$ and form the orthogonal decomposition $\mathfrak{n} = V \oplus \mathfrak{z}$ as in Section 1.

2.1. **$J_{z_\circ}$ mappings and decomposition of V.** Let $z_\circ \in \mathfrak{z}$ be fixed with $z_\circ \neq 0$ and $J_{z_\circ} : V \to V$ denote the operator satisfying

$$(J_{z_\circ}(u), v)_{\mathfrak{n}} = ([u, v], z_\circ)_{\mathfrak{n}} \text{ for all } u, v \in V.$$

The operator $J_{z_\circ}$ is skew-symmetric with respect to the inner product $(\cdot, \cdot)_{\mathfrak{n}}$. Letting

$$\mathfrak{a}_{z_\circ} := Ker(J_{z_\circ}), \qquad \mathfrak{w}_{z_\circ} := Image(J_{z_\circ}),$$

one has $\mathfrak{w}_{z_\circ} \neq \{0\}$ as $z_\circ \neq 0$ and $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{z}$. The operator $J_{z_\circ}$ preserves $\mathfrak{w}_{z_\circ}$ and is non-degenerate and skew-symmetric on $\mathfrak{w}_{z_\circ}$. In particular $\mathfrak{w}_{z_\circ}$ is even dimensional and $J_{z_\circ}^2 : \mathfrak{w}_\circ \to \mathfrak{w}_{z_\circ}$ is negative definite symmetric. Letting

$$\sigma^+(z_\circ) := \{\lambda > 0 \ : \ -\lambda^2 \text{ is an eigenvalue for } J_{z_\circ}^2\}. \tag{2.1}$$

we have an orthogonal direct sum decomposition for V into eigenspaces for $J_{z_\circ}^2$,

$$V = \mathfrak{a}_{z_\circ} \oplus \bigoplus_{\lambda \in \sigma^+(z_\circ)} \mathfrak{w}_{z_\circ, \lambda}, \tag{2.2}$$

where $J_{z_\circ}^2 = -\lambda^2$ on $\mathfrak{w}_{z_\circ,\lambda}$.

For $w \in \mathfrak{w}_{z_\circ}$, we write $w = \sum w_\lambda$ with $w_\lambda \in \mathfrak{w}_{z_\circ,\lambda}$. The operator $J_{z_\circ}$ produces a complex structure on $\mathfrak{w}_{z_\circ}$, namely

$$\widetilde{J}_{z_\circ} : \mathfrak{w}_{z_\circ} \to \mathfrak{w}_{z_\circ}, \quad \widetilde{J}_{z_\circ}\left(\sum w_\lambda\right) = \sum \frac{1}{\lambda} J_{z_\circ}(w_\lambda). \tag{2.3}$$

Let $\widetilde{\mathfrak{w}}_{z_\circ}$ denote the complex vector space $(\mathfrak{w}_{z_\circ}, \widetilde{J}_{z_\circ})$ and equip this with the hermitian inner product $\langle u, v \rangle_{z_\circ} := (u, v)_{\mathfrak{n}} + i(u, \widetilde{J}_{z_\circ}(v))_{\mathfrak{n}}$.

The stabilizer $K_{z_\circ}$ of $z_\circ$ in K acts unitarily on $(\widetilde{\mathfrak{w}}_{z_\circ}, \langle \cdot, \cdot \rangle_{z_\circ})$. Moreover for each $a_\circ \in \mathfrak{a}_{z_\circ}$ the action $K_{z_\circ,a_\circ} : \widetilde{\mathfrak{w}}_{z_\circ}$ of the stabilizer for $a_\circ$ in $K_{z_\circ}$, namely $K_{z_\circ,a_\circ} = K_{z_\circ} \cap K_{a_\circ}$, is a *multiplicity free action* of the compact group $K_{z_\circ,a_\circ}$ on the complex vector space $\widetilde{\mathfrak{w}}_{z_\circ}$. That is, the associated representation of $K_{z_\circ,a_\circ}$ on the polynomial ring $\mathbb{C}[\widetilde{w}_{z_\circ}]$ is multiplicity free.

2.2. **The unitary dual $\widehat{N}$.** Each irreducible unitary representation $\pi \in \widehat{N}$ corresponds to a coadjoint orbit $\mathcal{O}(\pi) \subset \mathfrak{n}^*$. We use the inner product $(\cdot, \cdot)_{\mathfrak{n}}$ to identify $\mathfrak{n}^*$ with $\mathfrak{n}$ and regard $\mathcal{O}(\pi)$ as lying in $\mathfrak{n}$. As N is two-step, $\mathcal{O}(\pi) \subset \mathfrak{n}$ is an affine set that projects to a single point in $\mathfrak{z}$. We say that π is of *type I* (resp. *type II*) when the projection to $\mathfrak{z}$ is non-zero (resp. zero). The type I representations are non-trivial on the center Z whereas the type II representations have Z in their kernel, and act as characters on N/Z. We write $\widehat{N} = \widehat{N}^I \cup \widehat{N}^{II}$ to distinguish the two types of representation.

The coadjoint orbit $\mathcal{O}(\pi)$ for a representation $\pi \in \widehat{N}^I$ of type I has the form

$$\mathcal{O}(\pi) = a_\circ + \mathfrak{w}_{z_\circ} + z_\circ$$

for some unique $z_\circ \in \mathfrak{z} - \{0\}$ and $a_\circ \in \mathfrak{a}_{z_\circ}$. The point $\ell_\pi = a_\circ + z_\circ$ in $\mathcal{O}(\pi)$ is said to be *aligned*. Conversely given any $z_\circ \in \mathfrak{z}$ with $z_\circ \neq 0$ and any $a_\circ \in \mathfrak{a}_{z_\circ}$ the afffine set $a_\circ + \mathfrak{w}_{z_\circ} + z_\circ$ is a coadjoint orbit corresponding to a representation $\pi_{z_\circ,a_\circ}$ of type I. Thus

$$\widehat{N}^I = \{\pi_{z_\circ,a_\circ} \ : \ z_\circ \in \mathfrak{z} - \{0\}, a_\circ \in \mathfrak{a}_{z_\circ}\}.$$

The representation $\pi_{z_\circ,a_\circ}$ can be realized in a Fock space completion of $\mathbb{C}[\widetilde{\mathfrak{w}}_{z_\circ}]$.

The representations $\chi \in \widehat{N}^{II}$ are unitary characters that correspond to single point coadjoint orbits. For each $w \in V$ one has the unitary character

$$\chi_w(\exp(X)) = e^{i(w,X)_\mathfrak{n}}$$

with corresponding coadjoint orbit $\mathcal{O}(\chi_w) = \{w\}$.

2.3. **The space $\Delta(K,N)$.** Bounded K-spherical functions on N are associated with representations $\pi \in \widehat{N}$. A spherical function associated with a representation of type I or II is said to be of that same type and we write $\Delta(K,N) = \Delta^I(K,N) \cup \Delta^{II}(K,N)$.

A spherical function of type I is associated with some $\pi_{z_\circ,a_\circ} \in \widehat{N}^I$. As $K_{z_\circ,a_\circ} : \widetilde{w}_{z_\circ}$ is a multiplicity free action the polynomial ring $\mathbb{C}[\widetilde{\mathfrak{w}}_{z_\circ}]$ has a canonical decomposition into $K_{z_\circ,a_\circ}$-irreducible components. We write this decomposition as

$$\mathbb{C}[\widetilde{\mathfrak{w}}_{z_\circ}] = \bigoplus_{\alpha \in \Lambda_{z_\circ,a_\circ}} P_{z_\circ,a_\circ,\alpha} \tag{2.4}$$

where $\Lambda_{z_\circ,a_\circ}$ is a countably infinite index set that depends on $(z_\circ, a_\circ)$. Proposition 5.1 in [BR20] uses the set $\Lambda_{z_\circ,a_\circ}$ to index the spherical functions associated with $\pi_{z_\circ,a_\circ}$. We write

$$\Delta^I(K,N) = \left\{\phi_{z_\circ,a_\circ,\alpha} \ : \ z_\circ \in \mathfrak{z}, z_\circ \neq 0, a_\circ \in \mathfrak{a}_{z_\circ}, \alpha \in \Lambda_{z_\circ,a_\circ}\right\}.$$

Moreover one has $\phi_{z_\circ,a_\circ,\alpha} = \phi_{z'_\circ,a'_\circ,\alpha'}$ whenever the data $(z'_\circ, a'_\circ, \alpha')$, $(z_\circ, a_\circ, \alpha)$ differ by the action of K. Thus if $\Gamma \subset \mathfrak{z}$ is a cross section to the K-orbits in $\mathfrak{z}$ and for each $z_\circ \in \Gamma$ the set $\Sigma_{z_\circ} \subset \mathfrak{a}_{z_\circ}$ is a cross section to the $K_{z_\circ}$-orbits in $\mathfrak{a}_{z_\circ}$ then $\Delta(K,N)$ is parameterized by

$$\{(z_\circ, a_\circ, \alpha) \ : \ z_\circ \in \Gamma, a_\circ \in \Sigma_{z_\circ}, \alpha \in \Lambda_{z_\circ,a_\circ}\}.$$

The spherical functions of type II are straightforward. For each $w \in V$ the K-average of χ_w, namely

$$\phi_w(\exp(X)) = \int_K e^{i(w,k\cdot X)_\mathfrak{n}}\, dk$$

is a spherical function of type II. Thus $\Delta^{II}(K,N) = \{\phi_w : w \in V\}$. One has $\Delta^{II}(K,N) \cong V/K$ since $\phi_{k\cdot w} = \phi_w$ for $k \in K$, $w \in V$.

2.4. **The map $\Psi : \Delta(K,N) \to \mathcal{A}(K,N)$.** The initial definition for Ψ, given in [BR08], involved working with coadjoint orbits for the semidirect product $K \ltimes N$. An alternate description, also from [BR08], is less conceptual but more suited to calculation in examples. We summarize this approach here. See also [BR20, §2.3, 5] for more detail.

Identifying $\mathfrak{n}^*$ with $\mathfrak{n}$ via $(\cdot,\cdot)_{\mathfrak{n}}$ the map Ψ takes spherical functions $\phi \in \Delta(K,N)$ to K-orbits in $\mathfrak{n}$. For a bounded spherical functions ϕ_w of type II we have simply

$$\Psi(\phi_w) = K \cdot w.$$

The calculation of $\Psi(\phi_{z_\circ,a_\circ,\alpha})$ for $\phi_{z_\circ,a_\circ,\alpha} \in \Delta^I(K,N)$ involves the (unnormalized) moment map for the multiplicity free action $K_{z_\circ,a_\circ} : \widetilde{\mathfrak{w}}_{z_\circ}$, defined as

$$\eta : \widetilde{\mathfrak{w}}_{z_\circ} \to \mathfrak{k}^*_{z_\circ,a_\circ}, \quad \eta(w)(A) := Im\, \langle w, A\cdot w\rangle_{z_\circ},$$

where $\mathfrak{k}_{z_\circ,a_\circ} = \mathrm{Lie}(K_{z_\circ,a_\circ})$. Take as index set $\Lambda_{z_\circ,a_\circ}$ in (2.4) the set of highest weights occurring in $\mathbb{C}[\widetilde{\mathfrak{w}}_{z_\circ}]$. Thus $\Lambda_{z_\circ,a_\circ} \subset \mathfrak{k}^*_{z_\circ,a_\circ}$ and $\alpha \in \Lambda_{z_\circ,a_\circ}$ is the highest weight for the irreducible representation of $K_{z_\circ,a_\circ}$ on $P_{z_\circ,a_\circ,\alpha}$. It is a crucial fact that each such highest weight lies in the image of the moment map η.

Proposition 2.1. [BR20, Proposition 5.4] *Let $\pi \in \widehat{N}^I$ with aligned point $\ell_\pi = a_\circ + z_\circ, z_\circ \neq 0$. Decompose V with respect to $J_{z_\circ}$ as $V = \mathfrak{a}_{z_\circ} \oplus \sum \mathfrak{w}_{z_\circ,\lambda}$. Given $\alpha \in \Lambda_\pi = \Lambda_{z_\circ,a_\circ}$ let $w_\alpha \in \mathfrak{w}_{z_\circ}$ be any point for which $\eta(w_\alpha) = \alpha$. Write $w_\alpha = \sum w_\lambda$ with $w_\lambda \in \mathfrak{w}_{z_\circ,\lambda}$ and let $w'_\alpha := \sum (2\lambda)^{1/2} w_\lambda$. Then*

$$\Psi(\phi_{z_\circ,a_\circ,\alpha}) = K\cdot(a_\circ, w'_\alpha, z_\circ).$$

We call a point $w_\alpha \in \mathfrak{w}_{z_\circ}$ satisfying $\eta(w_\alpha) = \alpha$, as in Proposition 2.1, a *spherical point*. Spherical points for all multiplicity free actions arising in connection with the irreducible n.G.p.'s in Table 1 can be found in [BR13, BR15a, BR15b]. We will make use of these in our subsequent calculations.

2.5. **Fundamental highest weights.** An element $\alpha \in \Lambda_{z_\circ,a_\circ}$ is said to be a *fundamental* highest weight for $K_{z_\circ,a_\circ} : \widetilde{\mathfrak{w}}_{z_\circ}$ if an α-highest weight vector $h_\alpha \in P_{z_\circ,a_\circ,\alpha}$ is an irreducible polynomial. The fundamental highest weights form a finite $\mathbb{Q}$-linearly independent set

$$\{\alpha_1, \ldots, \alpha_r\}$$

which freely generates $\Lambda_{z_\circ,a_\circ}$ as an additive semigroup [HU91]. The number of fundamental highest weights is called the *rank* of the multiplicity free action. With the fundamental highest weights in hand we can identify $\Lambda_{z_\circ,a_\circ}$ with $(\mathbb{Z}^+)^r$. That is an r-tuple $\mathbf{m} = (m_1, \ldots, m_r)$ of non-negative integers represents the weight $m_1\alpha_1 + \cdots + m_r\alpha_r$.

2.6. **Differential operators** $D_p \in \mathbb{D}_K(N)$**.** Each K-invariant polynomial $p \in \mathbb{R}[\mathfrak{n}]^K$ yields a left-N and K-invariant differential operator D_p on the group N. One can apply the symmetrization mapping, or modified symmetrization [FRY12], to obtain D_p from p. We used this approach in [BR20] to study the first six entries in Table 1. In the current paper, however, we prefer an alternate procedure following [FGJ$^+$19] and described below, yielding a non-symmetrized operator $D_p \in \mathbb{D}_K(N)$. This has the advantage that the expressions for the eigenvalues $\widehat{D}_p(\phi)$ for D_p on spherical functions $\phi \in \Delta(K,N)$ are somewhat simpler than those obtained using symmetrized operators. This in turn simplifies the verification of Condition (O) for the pairs in lines 9-14 of Table 1. More details on this process will follow in the examples.

3. The pair $((S)U(2) \times SU(d), (\mathbb{C}^2 \otimes \mathbb{C}^d) \oplus H\Lambda^2(\mathbb{C}^2))$

Here $\mathfrak{n} = \mathfrak{n}_d = V \oplus \mathfrak{z}$ where $V = \mathbb{C}^2 \otimes \mathbb{C}^d$ (tensor over $\mathbb{C}$) and $\mathfrak{z} = H\Lambda^2(\mathbb{C}^2) = \mathfrak{u}(2)$. We realize V as $V = M_{2,d}(\mathbb{C})$, matrices of size $2 \times d$ with complex entries. The bracket is then

$$V \times V \to \mathfrak{z}, \qquad [u,v] = uv^* - vu^*.$$

We have the compact group $K = SU(2) \times SU(d)$ for $d \geq 3$ or $K = U(2) \times SU(2)$ when $d = 2$ acting on $\mathfrak{n}$ via

$$(k_1,k_2) \cdot (v,A) = (k_1 v k_2^*, k_1 A k_1^*).$$

This is case (21) in §13.4 of [Wol07] and lines 9,10 in our table.

3.1. J_B **maps.** Equipping $\mathfrak{n}$ with the K-invariant inner product

$$\big((u,A),(v,B)\big)_{\mathfrak{n}} := Re(tr(uv^*)) + \frac{1}{2}Re(tr(AB^*)) = Re(tr(uv^*)) - \frac{1}{2}Re(tr(AB)).$$

one has the mapping $J_B : V \to V$ for each $B \in \mathfrak{z}$ given by

$$J_B(v) = -Bv.$$

3.2. K**-orbits in** $\mathfrak{z}$**.** Every K-orbit in $\mathfrak{z} = \mathfrak{u}(2)$ contains a unique diagonal point of the form

$$B_{\lambda_1,\lambda_2} = -diag(i\lambda_1, i\lambda_2)$$

with $\lambda_1 \leq \lambda_2$ real. The sign convention here ensures that $J_{\lambda_1,\lambda_2} := J_{B_{\lambda_1,\lambda_2}}$ acts on $V = M_{2,d}(\mathbb{C})$ by multiplying the two rows by $i\lambda_1$ and $i\lambda_2$ respectively.

3.3. **The space** $\Delta(K,N)$**.** Values (λ_1,λ_2) with $\lambda_1 \leq \lambda_2$ as above serve as "$\mathfrak{z}$-parameters" for $\Delta(K,N)$. There are eight possibilities regarding (λ_1,λ_2), namely

$$\left\{ \begin{array}{c|c} 0 < \lambda_1 < \lambda_2 & \lambda_1 < 0 < \lambda_2 \\ 0 < \lambda_1 = \lambda_2 & \lambda_1 < \lambda_2 < 0 \\ 0 = \lambda_1 < \lambda_2 & \lambda_1 = \lambda_2 < 0 \\ 0 = \lambda_1 = \lambda_2 & \lambda_1 < \lambda_2 = 0 \end{array} \right\}. \tag{3.1}$$

These cases determine a "layering" of the space $\Delta(K,N)$ into spherical functions of different sorts. Here we will focus on the four cases in the first column of (3.1), where $0 \leq \lambda_1 \leq \lambda_2$. The other four cases can be handled in a similar manner. In case $0 = \lambda_1 = \lambda_2$ one obtains the spherical functions of Type II. The other three cases produce spherical functions of Type I. The fundamental highest weight vectors for these cases are among the following polynomials $h_j \in \mathbb{C}[V]$,

$$h_1(v) := v_{11}, \quad h_2(v) := v_{21}, \quad h_3(v) := v_{22}, \quad h_4(v) := \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix}. \tag{3.2}$$

Layer 1: $0 < \lambda_1 < \lambda_2$. This is the generic situation for λ_1, λ_2 non-negative. In this case J_{λ_1,λ_2} has trivial kernel and $\widetilde{\mathfrak{w}}_{\lambda_1,\lambda_2} = V$ with its standard complex and hermitian structures. The stabilizer of B_{λ_1,λ_2} in K is $K_{\lambda_1,\lambda_2} = S(U(1) \times U(1)) \times SU(d)$ when $d \geq 3$ and $K_{\lambda_1,\lambda_2} = (U(1) \times U(1)) \times SU(2)$ when $d = 2$. The multiplicity free action of K_{λ_1,λ_2} on V has rank 3 with fundamental highest weight vectors h_1, h_2 and h_4. This is an indecomposable but non-irreducible multiplicity free action. We refer the reader to section 5.2 in [BR15b] regarding this action. The space $\mathbb{C}[V]$ decomposes under K_{λ_1,λ_2} as

$$\mathbb{C}[V] = \bigoplus_{\mathbf{m}\in(\mathbb{Z}^+)^3} P_{\mathbf{m}}$$

where $P_{\mathbf{m}} = P_{m_1,m_2,m_4}$ is the irreducible subspace containing $h_1^{m_1}h_2^{m_2}h_4^{m_4}$. Thus spherical functions in Layer 1 are determined by parameters $(\lambda_1, \lambda_2; m_1, m_2, m_4)$ with $0 < \lambda_1 < \lambda_2$ and m_1, m_2, m_4 non-negative integers. Let $\Delta^1 \subset \Delta(K,N)$ denote the set of all such spherical functions.

Layer 2: $0 < (\lambda_1 = \lambda_2)$. Say $\lambda_1 = \lambda_2 = \lambda$ where $\lambda > 0$. As in Layer 1, $\widetilde{\mathfrak{w}}_{\lambda,\lambda} = V$ with its standard complex and hermitian structures, but now the stabilizer of $B_{\lambda,\lambda}$ in K is $K_{\lambda,\lambda} = K = (S)U(2) \times SU(d)$. The multiplicity free action $K : V$ is irreducible with rank 2 and fundamental highest weight vectors h_1, h_4. Thus spherical functions in Layer 2 are determined by parameters $(\lambda, \lambda; m_1, m_4)$ with $0 < \lambda$ and m_1, m_4 non-negative integers. Let $\Delta^2 \subset \Delta(K,N)$ denote the set of all such spherical functions.

Layer 3: $0 = \lambda_1 < \lambda_2$. Let V_1 and V_2 denote the row spaces in $V = M_{2,d}(\mathbb{C})$. We have $\mathfrak{a}_{0,\lambda_2} = Ker(J_{0,\lambda_2}) = V_1$ (as a real vector space) and $\widetilde{\mathfrak{w}}_{0,\lambda_2} = V_2$ with its usual complex structure. The stabilizer of B_{0,λ_2} in K is $K_{0,\lambda_2} = S(U(1) \times U(1)) \times SU(d)$ in case $d \geq 3$ and $K_{0,\lambda_2} = (U(1) \times U(1)) \times SU(2)$ when $d = 2$. Moreover let $a_\circ \in \mathfrak{a}_{0,\lambda_2}$ be given. Using the action of $SU(d) \subset K_{0,\lambda_2}$ we may suppose that

$$a_\circ = re_1 = (r, 0, \ldots, 0)$$

for some real number $r \geq 0$. There are 2 sub-cases to consider here.

Layer 3,0: $r = 0$. The stabilizer of $a_\circ = 0$ in K_{0,λ_2} is $K_{(0,\lambda_2),0} = K_{0,\lambda_2}$, acting on $\widetilde{\mathfrak{w}}_{0,\lambda_2} = V_2$ as $U(1) \times SU(d) : \mathbb{C}^d$. This is an irreducible multiplicity free action of rank 1 with fundamental highest weight vector h_2. Thus spherical functions in Layer 3,0 are determined by parameters $(0, \lambda_2; 0, m_2)$ with $0 < \lambda_2$ and $m_2 \in \mathbb{Z}^+$. Let $\Delta^{3,0} \subset \Delta(K, N)$ denote the set of all such spherical functions.

Layer 3,1: $r > 0$. The stabilizer of $a_0 = re_1$ in K_{0,λ_2} is a copy of $(S)(U(1) \times U(1)) \times SU(d-1)$. More precisely we have stabilizer

$$\left\{ \left(\begin{bmatrix} \overline{\gamma} & 0 \\ 0 & \gamma \end{bmatrix}, \left[\begin{array}{c|c} \overline{\gamma} & 0 \\ \hline 0 & k \end{array} \right] \right) \; : \; \gamma \in \mathbb{T}, k \in U(d-1), \det(k) = \gamma \right\}$$

when $d \geq 3$ and

$$\left\{ \left(\begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \begin{bmatrix} \overline{\gamma_2} & 0 \\ 0 & \gamma_2 \end{bmatrix} \right) \; : \; \gamma_2, \gamma_2 \in \mathbb{T} \right\}$$

when $d = 2$. The space V_2 decomposes under the action of $K_{(0,\lambda_2),re_1}$ as $V_2 = \mathbb{C}^d = \mathbb{C} \oplus \mathbb{C}^{d-1}$. This is essentially a product multiplicity free action on V_2. The fundamental highest weight vectors are $h_2, h_3 \in \mathbb{C}[V_2]$. Thus spherical functions in Layer 3,1 are determined by parameters $(0, \lambda_2; r, m_2, m_3)$ with $0 < \lambda_2$, $r > 0$ and $m_2, m_3 \in \mathbb{Z}^+$. Let $\Delta^{3,1} \subset \Delta(K, N)$ denote the set of all such spherical functions.

Layer 4: $\lambda_1 = \lambda_2 = 0$. That is $B_{\lambda_1,\lambda_2} = B_{0,0} = 0$. Now $\mathfrak{a}_{0,0} = V$, $\mathfrak{w}_{0,0} = 0$ and we have a spherical function for each $a_\circ \in V$, namely the K-average of

$$(v, A) \mapsto e^{i(v,a_\circ)_\mathfrak{n}}.$$

These are the spherical functions of type II. Taking the action of K on V into account we can suppose that $a_\circ = \begin{bmatrix} re_1 \\ 0 \end{bmatrix}$ with $r \geq 0$ real. Thus spherical functions in Layer 4 are determined by parameters $(0, 0; r)$ with $r \geq 0$. Let $\Delta^4 \subset \Delta(K, N)$ denote the set of all such spherical functions.

3.4. **The space $\mathcal{A}(K, N)$.** We wish to determine the spherical orbits $\mathbf{O} = \Psi(\phi)$ in $\mathfrak{n}^* \cong \mathfrak{n}$ for each $\phi \in \Delta(K, N)$.

Layer 1: Let $\mathbf{O}(\lambda_1, \lambda_2; m_1, m_2, m_4)$ denote the spherical orbit for the Layer 1 spherical function with parameters $(\lambda_1, \lambda_2; m_1, m_2, m_4)$. In view of Proposition 2.1 this has the form

$$\mathbf{O}(\lambda_1, \lambda_2; m_1, m_2, m_4) = K \cdot \left(\begin{bmatrix} (2\lambda_1)^{1/2} v_1(\mathbf{m}) \\ (2\lambda_2)^{1/2} v_2(\mathbf{m}) \end{bmatrix}, B_{\lambda_1,\lambda_2} \right) \tag{3.3}$$

where

$$v(\mathbf{m}) = \begin{bmatrix} v_1(\mathbf{m}) \\ v_2(\mathbf{m}) \end{bmatrix},$$

is a spherical point in V for the parameters $\mathbf{m} = (m_1, m_2, m_4)$. An explicit formula for such a point $v(\mathbf{m})$ is given in Section 5.2 of [BR15b]. For our purposes later on we require only the following facts concerning inner products of the rows in $v(\mathbf{m})$, namely

$$|v_1(\mathbf{m})|^2 = m_1 + m_4,\ |v_2(\mathbf{m})|^2 = m_2 + m_4,\ \langle v_1(\mathbf{m}), v_2(\mathbf{m})\rangle = \sqrt{m_1 m_2}. \tag{3.4}$$

Layer 2: The spherical orbit for the Layer 2 spherical function with parameters $(\lambda, \lambda; m_1, m_4)$ is

$$\mathbf{O}(\lambda, \lambda; m_1, m_4) = K \cdot \left((2\lambda)^{1/2} v(\mathbf{m}), B_{\lambda,\lambda}\right) \tag{3.5}$$

where

$$v(\mathbf{m}) = \begin{bmatrix} \sqrt{m_1 + m_4} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{m_4} & 0 & \cdots & 0 \end{bmatrix}. \tag{3.6}$$

Here we have applied Proposition 2.1 with the spherical point for the relevant multiplicity free action on V, given in [BR13, Proposition 5.4].

Layer 3,0: The spherical orbit for the Layer 3,0 spherical function with parameters $(0, \lambda_2; 0, m_2)$ is

$$\mathbf{O}(0, \lambda_2, 0, m_2) = K \cdot \left(\begin{bmatrix} 0 \\ (2\lambda_2 m_2)^{1/2} e_1 \end{bmatrix}, B_{0,\lambda_2} \right). \tag{3.7}$$

Layer 3,1: For Layer 3,1 parameters $(0, \lambda_2; r, m_2, m_3)$ we obtain spherical orbit

$$\mathbf{O}(0, \lambda_2; r, m_2, m_3) = K \cdot \left(\begin{bmatrix} r e_1 \\ (2\lambda_2 m_2)^{1/2} e_1 + (2\lambda_2 m_3)^{1/2} e_2 \end{bmatrix}, B_{0,\lambda_2} \right). \tag{3.8}$$

Layer 4: For Layer 4 parameters $(0, 0; r)$ we have simply

$$\mathbf{O}(0, 0; r) = K \cdot \left(\begin{bmatrix} r e_1 \\ 0 \end{bmatrix}, 0 \right). \tag{3.9}$$

3.5. **Generators for $\mathbb{R}[\mathfrak{n}]^K$.** [FRY12, Theorem 7.5] provides a set of five generators for $\mathbb{R}[\mathfrak{n}]^K$, namely

$$\left\{ \begin{array}{c} p_1(v, A) = i\, tr(A), \quad p_2(v, A) = -tr(A^2) = 2\|A\|_{\mathfrak{n}}^2 \\ p_3(v, A) = tr(vv^*) = \|v\|_{\mathfrak{n}}^2, \quad p_4(v, A) = \det(vv^*) = \sum_{i<j} |\det_{i,j}(v)|^2 \\ p_5(v, A) = (iAv, v)_{\mathfrak{n}} = i\, tr(Avv^*) \end{array} \right\} \tag{3.10}$$

where $\det_{i,j}(v) = \begin{vmatrix} v_{1,i} & v_{1,j} \\ v_{2,i} & v_{2,j} \end{vmatrix}$. Here $\{p_1, p_2\}$ generates $\mathbb{R}[\mathfrak{z}]^K$ and $\{p_3, p_4\}$ generates $\mathbb{R}[V]^K$. We remark that in [FRY12] one finds the polynomial $F_4(v, A) = tr((vv^*)^2)$ in place of p_4. One can check that p_3, p_4 and F_4 are related via $p_3^2 = F_4 + 2p_4$.

3.6. **Differential operators $D_p \in \mathbb{D}_K(N)$.** In layer 1, for each K-invariant polynomial $p(v,A) \in \mathbb{R}[\mathfrak{n}]^K$ we first replace A with $B = -\ \mathrm{diag}(i\lambda_1, i\lambda_2)$, to obtain a K_B -invariant polynomial $p_B(v)$. Using the complex structure on $\widetilde{\mathfrak{w}}_{\lambda_1,\lambda_2} = V$, we use (2.3) to transfer to a standard Heisenberg group, where we can write the polynomial as $\widetilde{p}_B(v,\overline{v})$. We construct an orthonormal real basis $\{V_1,\ldots,V_{2d},\overline{V}_1,\ldots,\overline{V}_{2d}\}$ and obtain a K_B - invariant differential operator $D_{p_B} = p_B(V,\overline{V})$. Applying the representation, we get $\pi_B(D_{\widetilde{p}_B}) = \widetilde{p}_B(-v, 2\partial/\partial v)$, where derivatives are to the right of multiplication. We get eigenvalues for $D_{\widetilde{p}_B}$ by applying $\widetilde{p}_B(-v,2\partial/\partial v)$ to highest weight vectors in the representation space.

For example, we have $p_4(v,A) = i\ tr(Avv^*)$ becomes, after substituting $A = -\mathrm{diag}(i\lambda_1, i\lambda_2)$, the K_B-invariant $\lambda_1|v_1|^2 + \lambda_2|v_2|^2$. Applying the representation, we get $-2\lambda_1 \sum v_{1,j}\partial/\partial v_{1,j} - 2\lambda_2 \sum v_{2,j}\partial/\partial v_{2,j}$. This operator acts on monomials in the highest weight vectors to obtain the eigenvalues given below. Comparing the eigenvalues to the invariants, we see that in some cases, they differ by a sign. Sign differences are consistent across layers, they depend only on the degree of homogeneity of p in v.

By not symmetrizing, we can more directly compute these eigenvalues. They will differ from eigenvalues for symmetrized operators by lower order terms, which in turn are eigenvalues for lower degree invariant operators.

3.7. **Values $p_j(\mathbf{O}_\phi)$ and $\widehat{D}_{p_j}(\phi)$.** For given $\phi \in \Delta(K,N)$ we require

- the values $p_1(\mathbf{O}_\phi),\ldots,p_5(\mathbf{O}_\phi)$ which the generators $p_j \in \mathbb{R}[\mathfrak{n}]^K$ take on the spherical orbit $\mathbf{O}_\phi = \Psi(\phi)$ associated to ϕ,
- and the eigenvalues $\widehat{D}_{p_1}(\phi),\ldots\widehat{D}_{p_5}(\phi)$ for the differential operators $D_{p_j} \in \mathbb{D}_K(N)$ on ϕ. We will write $\widehat{D}_j(\phi)$ for $\widehat{D}_{p_j}(\phi)$.

Layer 1: For the spherical function ϕ in Layer 1 with parameters $(\lambda_1,\lambda_2;m_1,m_2,m_4)$ values for $p_j(\mathbf{O}_\phi)$ and $\widehat{D}_j(\phi)$ $(j = 1,\ldots,5)$ are listed below. Note that $\widehat{D}_j(\phi) = \pm p_j(\mathbf{O}_\phi)$ for $j = 1,2,3,5$ but $\widehat{D}_4(\phi)$ differs from $p_4(\mathbf{O}_\phi)$ by a lower order term.

$$\left\{\begin{array}{lll} p_1(\mathbf{O}_\phi) = & \lambda_1+\lambda_2 & = \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & \lambda_1^2+\lambda_2^2 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & 2\lambda_1(m_1+m_4)+2\lambda_2(m_2+m_4) & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & 4\lambda_1\lambda_2 m_4(m_1+m_2+m_4), & \\ & p_4(\mathbf{O}_\phi)+4\lambda_1\lambda_2 m_4 & = \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & 2\lambda_1^2(m_1+m_4)+2\lambda_2^2(m_2+m_4) & = -\widehat{D}_5(\phi) \end{array}\right\} \tag{3.11}$$

In outline these values are obtained as follows. Similar, and mostly easier, calculations were used to derive the data given below for subsequent layers.

The values $p_j(\mathbf{O}_\phi)$ are computed by applying each p_j to the spherical orbit given in (3.3). Let $w = \begin{bmatrix} (2\lambda_1)^{1/2}v_1(\mathbf{m}) \\ (2\lambda_2)^{1/2}v_2(\mathbf{m}) \end{bmatrix}$ from (3.3) and let $B = B_{\lambda_1,\lambda_2} = -diag(i\lambda_1, i\lambda_2)$.

Using (3.4) one has

$$ww^* = \begin{bmatrix} \langle w_1, w_1\rangle & \langle w_1, w_2\rangle \\ \langle w_2, w_1\rangle & \langle w_2, w_2\rangle \end{bmatrix} = \begin{bmatrix} 2\lambda_1(m_1+m_4) & 2(\lambda_1\lambda_2 m_1 m_2)^{1/2} \\ 2(\lambda_1\lambda_2 m_1 m_2)^{1/2} & 2\lambda_2(m_2+m_4) \end{bmatrix},$$

and

$$Bww^* = -i\begin{bmatrix} 2\lambda_1^2(m_1+m_4) & 2\lambda_1(\lambda_1\lambda_2 m_1 m_2)^{1/2} \\ 2\lambda_2(\lambda_1\lambda_2 m_1 m_2)^{1/2} & 2\lambda_2^2(m_2+m_4) \end{bmatrix}.$$

We have $p_1(\lambda_1, \lambda_2; \mathbf{m}) = i\,tr(B)$, $p_2(\lambda_1, \lambda_2; \mathbf{m}) = -tr(B^2)$, $p_3(\lambda_1, \lambda_2; \mathbf{m}) = tr(ww^*)$, $p_4(\lambda_1, \lambda_2; \mathbf{m}) = \det(ww^*)$ and $p_5(\lambda_1, \lambda_2; \mathbf{m}) = i\,tr(Bww^*)$.

Layer 2: For the Layer 2 spherical function ϕ with parameters $(\lambda, \lambda; m_1, m_4)$ one computes values

$$(3.12)\qquad \left\{\begin{array}{lll} p_1(\mathbf{O}_\phi) = & 2\lambda & = \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & 2\lambda^2 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & 2\lambda(m_1+2m_4) & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & 4\lambda^2 m_4(m_1+m_4), & \\ & p_4(\mathbf{O}_\phi) + 4\lambda^2 m_4 & = \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & 2\lambda^2(m_1+2m_4) & = -\widehat{D}_5(\phi) \end{array}\right\}.$$

Again one has $\widehat{D}_j(\phi) = \pm p_j(\mathbf{O}_\phi)$ except when $j = 4$, in which case the two differ by a lower order term. In the remaining layers one finds $\widehat{D}_j(\phi) = \pm p_j(\mathbf{O}_\phi)$ for all j.

Layer 3,0: For layer 3, we have $\lambda_1 = 0$ and $\lambda_2 > 0$. The J-map is degenerate, and $V = V_1 \oplus V_2$, the representation acting by a character on V_1. To compute eigenvalues, we write $p(v_1, v_2, A)$ and substitute $v_1 = re_1$, $A = \mathrm{diag}(0, i\lambda_2)$ to get a K_{0,λ_2} - invariant polynomial on V_2. We then proceed, on V_2, as described in section 3.6. For Layer 3,0 parameters $(0, \lambda_2; 0, m_2)$ one has

$$(3.13)\qquad \left\{\begin{array}{lll} p_1(\mathbf{O}_\phi) = & \lambda_2 & = \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & \lambda_2^2 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & 2\lambda_2 m_2 & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & 0 & = \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & 2\lambda_2^2 m_2 & = -\widehat{D}_5(\phi) \end{array}\right\}.$$

Layer 3,1: For Layer 3,1 parameters $(0, \lambda_2; r, m_2, m_3)$ one has

$$(3.14)\qquad \left\{\begin{array}{lll} p_1(\mathbf{O}_\phi) = & \lambda_2 & = \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & \lambda_2^2 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & r^2 + 2\lambda_2(m_2+m_3) & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & 2r^2\lambda_2 m_3 & = \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & 2\lambda_2^2(m_2+m_3) & = -\widehat{D}_5(\phi) \end{array}\right\}.$$

Layer 4: For Layer 4 parameters $(0,0;r)$ one has

$$\left\{\begin{array}{lcl} p_1(\mathbf{O}_\phi) = & 0 & = \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & 0 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & r^2 & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & 0 & = \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & 0 & = \widehat{D}_5(\phi) \end{array}\right\}. \tag{3.15}$$

3.8. **Condition (O) for** (K,N)**.** Let $(\phi_n)_{n=1}^\infty$ be a sequence in $\Delta(K,N)$, $\phi \in \Delta(K,N)$ and let $\mathbf{O}_n, \mathbf{O} \in \mathcal{A}(K,N)$ be the associated spherical orbits, $\mathbf{O}_n = \Psi(\phi_n)$, $\mathbf{O} = \Psi(\phi)$. We must show that $(\phi_n)_{n=1}^\infty$ converges to ϕ in $\Delta(K,N)$ if and only if $(\mathbf{O}_n)_{n=1}^\infty$ converges to $\mathbf{O}$ in $\mathcal{A}(K,N)$. As $\{p_1,\dots,p_5\}$ generates $\mathbb{R}[\mathfrak{n}]^K$ we know, by [FR07], that $(\phi_n)_{n=1}^\infty$ converges to ϕ in $\Delta(K,N)$ if and only if $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for $j=1,\dots,5$. Likewise $(\mathbf{O}_n)_{n=1}^\infty$ converges to $\mathbf{O}$ in $\mathcal{A}(K,N)$ if and only if $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for each j. This is the case as the invariants for a compact linear action on a finite dimensional real vector space separate orbits.

Let $(\lambda_1(n),\lambda_2(n))$ and (λ_1',λ_2') be the $\mathfrak{z}$-parameters for ϕ_n and ϕ. We have

$$p_1(\mathbf{O}_n) = \lambda_1(n)+\lambda_2(n) = \widehat{D}_i(\phi_n), \quad p_2(\mathbf{O}_n) = \lambda_1(n)^2+\lambda_2(n)^2 = \widehat{D}_2(\phi_n)$$

and likewise

$$p_1(\mathbf{O}) = \lambda_1'+\lambda_2' = \widehat{D}_1(\phi), \quad p_2(\mathbf{O}) = (\lambda_1')^2+(\lambda_2')^2 = \widehat{D}_2(\phi).$$

Moreover the values $p_1(\mathbf{O}) = \widehat{D}_1(\phi)$ and $p_2(\mathbf{O}) = \widehat{D}_2(\phi)$ completely determine (λ_1',λ_2'). Thus

- if either $\phi_n \to \phi$ or $\mathbf{O}_n \to \mathbf{O}$ we must have $\lambda_1(n) \to \lambda_1'$ and $\lambda_2(n) \to \lambda_2'$.

So we assume henceforth that $\lambda_1(n) \to \lambda_1'$ and $\lambda_2(n) \to \lambda_2'$.

By passing to a subsequence we may assume, moreover, that every ϕ_n belongs to a single layer in $\Delta(K,N)$. We will suppose here that $(\phi_n)_{n=1}^\infty$ is contained in one of the four layers discussed above.[1] Now as $\lambda_1(n) \to \lambda_1'$ and $\lambda_2(n) \to \lambda_2'$ it follows that

- ϕ lies in the same layer as the ϕ_n's or in a higher layer.

For layers 3 and 4 one has $\widehat{D}_j(\phi_n) = \pm p_j(\mathbf{O}_n)$ and $\widehat{D}_j(\phi) = \pm p_j(\mathbf{O})$ for all j. Thus

- if $(\phi_n)_{n=1}^\infty$ is contained in Layer 3 or 4 then $\phi_1 \to \phi$ if and only if $\mathbf{O}_n \to \mathbf{O}$.

Suppose now that $(\phi_n)_{n=1}^\infty$ is contained in Layer 1 or 2. There are a number of cases to consider.

Case 1: $(\phi_n)_{n=1}^\infty \subset \Delta^1$, $\phi \in \Delta^1$: Let $(\lambda_1(n),\lambda_2(n);m_1(n),m_2(n),m_4(n))$ be the parameters for ϕ_n and $(\lambda_1',\lambda_2';m_1',m_2',m_4')$ those for ϕ. We reason with data from (3.11).

[1] $\Delta(K,N)$ contains four additional layers, given by the right column of (3.1). When $(\phi_n)_{n=1}^\infty$ lies in one of these layers the proof is similar, so we omit the details.

First suppose that $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for all j. As the sequence

$$p_3(\mathbf{O}_n) = 2\lambda_1(n)(m_1(n) + m_4(n)) + 2\lambda_2(n)(m_2(n) + m_4(n))$$

converges and $\lim \lambda_1(n) = \lambda_1'$, $\lim \lambda_2(n) = \lambda_2'$ are positive we conclude that each of $m_1(n)$, $m_2(n)$, $m_4(n)$ are eventually constant. Thus we can assume that $m_1(n) = m_1$, $m_2(n) = m_2$, $m_4(n) = m_4$ independent of n. The limiting values for $p_3(\mathbf{O}_n)$, $p_4(\mathbf{O}_n)$ and $p_5(\mathbf{O}_n)$ are thus

$$\left\{ \begin{array}{l} p_3(\mathbf{O}) = \lim p_3(\mathbf{O}_n) = 2\lambda_1'(m_1 + m_4) + 2\lambda_2'(m_2 + m_4) \\ p_4(\mathbf{O}) = \lim p_4(\mathbf{O}_n) = 4\lambda_1'\lambda_2' m_4(m_1 + m_2 + m_4) \\ p_5(\mathbf{O}) = \lim p_5(\mathbf{O}_n) = 2(\lambda_1')^2(m_1 + m_4) + 2(\lambda_2')^2(m_2 + m_4) \end{array} \right\}.$$

We note that each spherical function/orbit is determined by a unique set of parameters, in the case of $\mathbf{O}$ they are $(\lambda_1', \lambda_2'; m_1', m_2', m_4')$. As the invariants $p_1, \ldots, p_5$ separate K-orbits their values, together with λ_1' and λ_2', determine $\mathbf{O}$ and its parameters. So $\mathbf{O}$ has parameters $(\lambda_1', \lambda_2'; m_1, m_2, m_4)$. In particular $m_4' = m_4$. So now $\lim \widehat{D}_4(\phi_n) = p_4(\mathbf{O}) + 4\lambda_1'\lambda_2' m_4 = \widehat{D}_4(\phi)$.

The proof that $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for $j = 3, 4, 5$ implies $p_4(\mathbf{O}_n) \to p_4(\mathbf{O})$ goes the same way, using the fact that the parameters for ϕ are are determined by the eigenvalues $\widehat{D}_1(\phi), \ldots, \widehat{D}_5(\phi)$.

Case 2: $(\phi_n)_{n=1}^\infty \subset \Delta^1$, $\phi \in \Delta^2$: The argument here is similar to Case 1. Let $(\lambda_1(n), \lambda_2(n); m_1(n), m_2(n), m_4(n))$ be the parameters for ϕ_n and $(\lambda', \lambda'; m_1', m_4')$ those for ϕ. Using (3.11) and (3.12) one shows that $m_1(n)$, $m_2(n)$ and $m_4(n)$ are eventually constant with $m_1 + m_2 = m_1'$ and $m_4 = m_4'$ under the hypothesis that $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for each j. This easily implies that $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for each j. The same argument shows $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for all j when $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for all j.

Case 3: $(\phi_n)_{n=1}^\infty \subset \Delta^1$, $\phi \in \Delta^{3,0}$: Let $(\lambda_1(n), \lambda_2(n); m_1(n), m_2(n), m_4(n))$ be the parameters for ϕ_n and $(0, \lambda_2'; 0, m_2')$ those for ϕ, so that $\lambda_1(n) \to 0$ and $\lambda_2(n) \to \lambda_2'$. See (3.11) and (3.13). First suppose that $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for all j. Thus $p_4(\mathbf{O}_n) \to 0$ and $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for $j \neq 4$. As the sequence $p_3(\mathbf{O}_n) = 2\lambda_1(n)(m_1(n) + m_4(n)) + 2\lambda_2(n)(m_2(n) + m_4(n))$ converges to $p_3(\mathbf{O}) = 2\lambda_2' m_2'$ with $\lambda_1(n), \lambda_2(n), \lambda_2' > 0$ we conclude that the sequence $m_4(n)$ is bounded. So $\lambda_1(n)\lambda_2(n)m_4(n) \to 0$ and hence $\widehat{D}_4(\phi_n) \to \lim p_4(\mathbf{O}_n) + 0 = 0 + 0 = 0 = \widehat{p_4}(\phi)$. The same argument shows $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for all j when $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for all j.

Case 4: $(\phi_n)_{n=1}^\infty \subset \Delta^1$, $\phi \in \Delta^{3,1}$: Let $(\lambda_1(n), \lambda_2(n); m_1(n), m_2(n), m_4(n))$ be the parameters for ϕ_n and $(0, \lambda_2'; r, m_2', m_3')$ those for ϕ. See (3.11) and (3.14). First suppose that $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for all j. Thus $p_4(\mathbf{O}_n) \to 2r^2\lambda_2' m_3'$ and $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for $j \neq 4$. Just as in Case 3 it follows that $\big(m_4(n)\big)_{n=1}^\infty$ is bounded, $\lambda_1(n)\lambda_2(n)m_4(n) \to 0$ and hence $\widehat{D}_4(\phi_n) \to r^2\lambda_2' m_3' = \widehat{D}_4(\phi)$. Likewise $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for all j when $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for all j.

Case 5: $(\phi_n)_{n=1}^\infty \subset \Delta^1$, $\phi \in \Delta^4$: Let $(\lambda_1(n), \lambda_2(n); m_1(n), m_2(n), m_4(n))$ be the parameters for ϕ_n and $(0,0;r)$ those for ϕ, so that $\lambda_1(n), \lambda_2(n) \to 0$. See (3.11) and (3.15). First suppose that $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for all j. Thus $p_4(\mathbf{O}_n) \to 0$ and $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for $j \neq 4$. As the sequence $p_3(\mathbf{O}_n) = 2\lambda_1(n)(m_1(n) + m_4(n)) + 2\lambda_2(n)(m_2(n) + m_4(n))$ converges to $p_3(\mathbf{O}) = r^2$ with $\lambda_1(n), \lambda_2(n) > 0$ it follows that $\big(\lambda_2(n)m_4(n)\big)_{n=1}^\infty$ is bounded. As $\lambda_1(n) \to 0$ it now follows that $\lambda_1(n)\lambda_2(n)m_4(n) \to 0$ and hence the sequence $\widehat{D}_4(\phi_n) = p_4(\mathbf{O}_n) + 4\lambda_1(n)\lambda_2(n)m_4(n)$ converges to $p_4(\mathbf{O}) = \widehat{D}_4(\phi)$. Likewise $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for all j when $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for all j.

Case 6: $(\phi_n)_{n=1}^\infty \subset \Delta^2$, $\phi \in \Delta^2$: The argument here is similar to that for Case 1. Let $(\lambda(n), \lambda(n); m_1(n), m_4(n))$ be the parameters for ϕ_n and $(\lambda', \lambda'; m_1', m_4')$ those for ϕ. First suppose that $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for all j. As in Case 1, the convergence of $\big(p_3(\mathbf{O}_n)\big)_{n=1}^\infty$ together with the fact that $\lambda' > 0$ implies $m_1(n)$ and $m_4(n)$ are eventually constant, $m_1(n) = m_1$, $m_4(n) = m_4$ say. Referring to (3.12) the limiting values for $p_3(\mathbf{O}_n)$, $p_4(\mathbf{O}_n)$ and $p_5(\mathbf{O}_n)$ are now

$$\left\{ \begin{array}{l} p_3(\mathbf{O}) = \lim p_3(\mathbf{O}_n) = 2\lambda'(m_1 + 2m_4) \\ p_4(\mathbf{O}) = \lim p_4(\mathbf{O}_n) = 4(\lambda')^2 m_4(m_1 + m_4) \\ p_5(\mathbf{O}) = \lim p_5(\mathbf{O}_n) = 2(\lambda')^2(m_1 + 2m_4) \end{array} \right\}.$$

As the invariants $p_1, \ldots, p_5$ separate K-orbits in $\mathfrak{n}$ these values, together with λ_1' and λ_2', determine $\mathbf{O}$ and its parameters. So $\mathbf{O}$ has parameters $(\lambda', \lambda'; m_1, m_4)$. In particular $m_4' = m_4$. So now $\lim \widehat{D}_4(\phi_n) = p_4(\mathbf{O}) + 4(\lambda')^2 m_4 = \widehat{D}_4(\phi)$.

The proof that $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for $j = 3, 4, 5$ implies $p_4(\mathbf{O}_n) \to p_4(\mathbf{O})$ goes the same way, using the fact that the parameters for ϕ are are determined by the eigenvalues $\widehat{D}_1(\phi), \ldots, \widehat{D}_5(\phi)$.

Case 7: $(\phi_n)_{n=1}^\infty \subset \Delta^2$, $\phi \in \Delta^4$: Let $(\lambda(n), \lambda(n); m_1(n), m_4(n))$ be the parameters for ϕ_n and $(0,0;r)$ those for ϕ. See (3.12) and (3.15). The argument here parallels Case 5. First suppose that $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for all j. Thus $p_4(\mathbf{O}_n) \to 0$ and $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for $j \neq 4$. As in Case 5, convergence of the sequence $p_3(\mathbf{O}_n)$ implies that $\lambda(n)m_4(n)$ is bounded and hence $\lambda(n)^2 m_4(n) \to 0$. Thus $\widehat{D}_4(\phi_n) = p_4(\mathbf{O}_n) + 4\lambda(n)^2 m_4(n)$ converges to $p_4(\mathbf{O}) = \widehat{D}_4(\phi)$. Likewise $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for all j when $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for all j.

This completes the proof for this example. □

4. The pair $(U(2) \times Sp(d), (\mathbb{C}^2 \otimes \mathbb{H}^d) \oplus H\Lambda^2(\mathbb{C}^2))$

Here $\mathfrak{n} = \mathfrak{n}_d = V \oplus \mathfrak{z}$ where $V = \mathbb{C}^2 \otimes \mathbb{H}^d$ (tensor over $\mathbb{C}$) and $\mathfrak{z} = H\Lambda^2(\mathbb{C}^2) = \mathfrak{u}(2)$. We realize V as $V = M_{2,2d}(\mathbb{C})$, matrices of size $2 \times (2d)$ with complex entries. The bracket $V \times V \to \mathfrak{z}$ is $[u, v] = uv^* - vu^*$, as in the previous eample. We have the compact group $K = U(2) \times Sp(d)$ where $Sp(d) = \{k \in U(2d) \ : \ k\mathcal{J}k^t = \mathcal{J}\}$, with

$\mathcal{J} = \left[\begin{array}{c|c} O_d & I_d \\ \hline -I_d & O_d \end{array}\right]$, acting on $\mathfrak{n}$ via $(k_1, k_2) \cdot (v, A) = (k_1 v k_2^*, k_1 A k_1^*)$. This is case (22) in §13.4 of [Wol07] and line 11 in our table.

The pair $(K, \mathfrak{n})$ is obtained from $(U(2) \times SU(2d), M_{2,2d}(\mathbb{C}) \oplus \mathfrak{u}(2))$, treated in the previous section, by replacing $SU(2d)$ with the smaller group $Sp(d)$. Just as before, matrices $B_{\lambda_1,\lambda_2} = -diag(i\lambda_1, i\lambda_2)$ with $\lambda_1 \leq \lambda_2$ form a cross-section to the K-orbits in $\mathfrak{z}$ and the values (λ_1, λ_2) impose a layering on $\Delta(K, N)$ and $\mathcal{A}(K, N)$. For our proof of Condition (O) we focus on the four layers with λ_1, λ_2 non-negative.

As generators for $\mathbb{R}[\mathfrak{n}]^K$ we take (see [FRY12][2])

$$\tag{4.1} \left\{ \begin{array}{c} p_1(v, A) = i\,tr(A), \quad p_2(v, A) = -tr(A^2) = 2\|A\|_{\mathfrak{n}}^2 \\ p_3(v, A) = tr(vv^*) = \|v\|_{\mathfrak{n}}^2,\ p_4(v, A) = \det(vv^*),\ p_5(v, A) = |\omega(v_1, v_2)|^2 \\ p_6(v, A) = (iAv, v)_{\mathfrak{n}} = i\,tr(Avv^*) \end{array} \right\}$$

where $\omega(v_1, v_2) = v_1 \mathcal{J} v_2^t$ is the symplectic inner product of the rows in v. Here $\{p_1, p_2\}$ generates $\mathbb{R}[\mathfrak{z}]^K$ and $\{p_3, p_4, p_5\}$ generates $\mathbb{R}[V]^K$. Note that generator p_5 is a new ingredient in this example. As in the previous example one obtains (unsymmetrized) generators $D_j := D_{p_j}$ $(j = 1, \ldots, 6)$ for $\mathbb{D}_K(N)$.

Next we introduce parameters on layers 1 through 4 and use these to determine spherical orbits $\mathbf{O}_\phi = \Psi(\phi)$ and values $p_j(\mathbf{O}_\phi)$, $\widehat{D}_j(\phi)$ for each layer. The following polynomials $h_j \in \mathbb{C}[V]$ will appear as fundamental highest weight vectors in the first three layers.

$$\tag{4.2} \left\{ \begin{array}{c} h_1(v) := v_{11}, \quad h_2(v) := v_{21}, \quad h_3(v) := v_{22}, \quad h_4(v) = v_{2,d+1}, \\ h_5(v) := \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix}, \quad h_6(v) = \omega(v_1, v_2) \end{array} \right\}.$$

Layer 1: $0 < \lambda_1 < \lambda_2$. Here J_{λ_1,λ_2} has trivial kernel and $\widetilde{\mathfrak{w}}_{\lambda_1,\lambda_2} = V$ with its standard complex and hermitian structures. The stabilizer of B_{λ_1,λ_2} in K is $K_{\lambda_1,\lambda_2} = (U(1) \times U(1)) \times Sp(d)$. The multiplicity free action of K_{λ_1,λ_2} on V is indecomposable but not irreducible. It is a multiplicity free action of rank 4, discussed in Section 5.8 in [BR15b]. Fundamental highest weight vectors in $\mathbb{C}[V]$ are h_1, h_2, h_5 and h_6. So the Layer 1 spherical function parameters are $(\lambda_1, \lambda_2; m_1, m_2, m_5, m_6)$. The function $\phi \in \Delta^1$ with these parameters has spherical orbit $\mathbf{O}_\phi$

$$\mathbf{O}_\phi = \mathbf{O}(\lambda_1, \lambda_2; m_1, m_2, m_5, m_6) = K \cdot \left(\begin{bmatrix} (2\lambda_1)^{1/2} v_1(\mathbf{m}) \\ (2\lambda_2)^{1/2} v_2(\mathbf{m}) \end{bmatrix}, B_{\lambda_1,\lambda_2} \right)$$

where $v(\mathbf{m}) = \begin{bmatrix} v_1(\mathbf{m}) \\ v_2(\mathbf{m}) \end{bmatrix}$, given in [BR15b], satisfies

$$\left\{ \begin{array}{c} |v_1(\mathbf{m})|^2 = m_1 + m_5 + m_6,\ |v_2(\mathbf{m})|^2 = m_2 + m_5 + m_6,\ \langle v_1(\mathbf{m}), v_2(\mathbf{m}) \rangle = \sqrt{m_1 m_2} \\ \omega(v_1(\mathbf{m}), v_2(\mathbf{m})) = -\sqrt{m_6(m_1 + m_2 + 2m_5 + m_6)} \end{array} \right\}.$$

[2]In [FRY12] the invariant $tr((vv^*)^2)$ is used in place of generator p_4.

These facts may be used to compute the values $p_j(\mathbf{O}_\phi)$, listed below along with the eigenvalues $\widehat{D}_j(\phi)$.

$$\text{(4.3)}\quad \left\{\begin{array}{lll} p_1(\mathbf{O}_\phi) = & \lambda_1+\lambda_2 & = \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & \lambda_1^2+\lambda_2^2 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & 2\lambda_1(m_1+m_5+m_6)+2\lambda_2(m_2+m_5+m_6) & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & 4\lambda_1\lambda_2(m_5+m_6)(m_1+m_2+m_5+m_6), & \\ & p_4(\mathbf{O}_\phi)+4\lambda_1\lambda_2(m_5+m_6) & = \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & 4\lambda_1\lambda_2 m_6(m_1+m_2+2m_5+m_6) & \\ & p_5(\mathbf{O}_\phi)+4\lambda_1\lambda_2 m_6(2d-1) & = \widehat{D}_5(\phi) \\ p_6(\mathbf{O}_\phi) = & 2\lambda_1^2(m_1+m_5+m_6)+2\lambda_2^2(m_2+m_5+m_6) & = -\widehat{D}_6(\phi) \end{array}\right\}.$$

Thus $\widehat{D}_j(\phi) = \pm p_j(\mathbf{O}_\phi)$ for $j = 1,2,3,6$ but $\widehat{D}_4(\phi)$ and $\widehat{D}_5(\phi)$ differ from $p_4(\mathbf{O}_\phi)$ and $p_5(\mathbf{O}_\phi)$ by lower order terms.

Layer 2: $0 < \lambda_1 = \lambda_2$. For $\lambda > 0$ we again have $\widetilde{\mathfrak{w}}_{\lambda,\lambda} = V$ but now $B_{\lambda,\lambda}$ has stabilizer $K_{\lambda,\lambda} = K = U(2)\times Sp(d)$. The multiplicity free action $K : V$ is irreducible of rank 3 with fundamental highest weight vectors h_1, h_5, h_6. See Section 4.7 in [BR15a]. Thus parameters for spherical functions $\phi \in \Delta^2$ are $(\lambda_1,\lambda_2;m_1,m_5,m_6)$ and we have associated spherical orbit

$$\mathbf{O}_\phi = \mathbf{O}(\lambda,\lambda;m_1,m_5,m_6) = K\cdot\big((2\lambda)^{1/2}v(\mathbf{m}), B_{\lambda,\lambda}\big)$$

where $v(\mathbf{m})$, given in [BR15a], satisfies

$$\left\{\begin{array}{c} |v_1(\mathbf{m})|^2 = m_1+m_5+m_6,\ |v_2(\mathbf{m})|^2 = m_5+m_6,\ \langle v_1(\mathbf{m}), v_2(\mathbf{m})\rangle = 0 \\ \omega(v_1(\mathbf{m}), v_2(\mathbf{m})) = \sqrt{m_6(m_1+2m_5+m_6)} \end{array}\right\}.$$

One obtains the following values for $p_j(\mathbf{O}_\phi)$ and $\widehat{D}_j(\phi)$.

$$\text{(4.4)}\quad \left\{\begin{array}{lll} p_1(\mathbf{O}_\phi) = & 2\lambda & = \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & 2\lambda^2 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & 2\lambda(m_1+2m_5+2m_6) & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & 4\lambda^2(m_5+m_6)(m_1+m_5+m_6), & \\ & p_4(\mathbf{O}_\phi)+4\lambda^2(m_5+m_6) & = \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & 4\lambda^2 m_6(m_1+2m_5+m_6) & \\ & p_5(\mathbf{O}_\phi)+4\lambda^2 m_6(2d-1) = & \widehat{D}_5(\phi) \\ p_6(\mathbf{O}_\phi) & = 2\lambda^2(m_1+2m_5+2m_6) & = -\widehat{D}_6(\phi) \end{array}\right\}.$$

Here again for Layer 2 the eigenvalues $\widehat{D}_j(\phi)$ differ from $p_i(\mathbf{O}_\phi)$ for $j = 4,5$. In the remaining layers one has $\widehat{D}_j(\phi) = \pm p_j(\mathbf{O}_\phi)$ for all j, as in the previous example.

Layer 3: $0 = \lambda_1 < \lambda_2$. Here $\mathfrak{a}_{0,\lambda_2}$ and $\widetilde{\mathfrak{w}}_{0,\lambda_2}$ are the row spaces V_1 (as a real vector space) and V_2 (with its usual complex structure). The stabilizer of B_{0,λ_2} in K is

$K_{0,\lambda_2} = (U(1)\times U(1))\times Sp(d)$. Each K_{0,λ_2}-orbit in V_1 contains a unique point of the form $a_\circ = re_1$ with $r \geq 0$.

Layer 3,0: $r = 0$. K_{0,λ_2} stabilizes $a_\circ = 0$ and $K_{0,\lambda_2} : V_2$ is an irreducible multiplicity free action of rank 1 with fundamental highest weight vector h_2. Thus a spherical function $\phi \in \Delta^{3,0}$ has parameters $(0,\lambda_2;0,m_2)$ and associated orbit $\mathbf{O}_\phi = \mathbf{O}(0,\lambda_2;0,m_2)$ as in (3.7). One obtains

$$\left\{\begin{array}{lll} p_1(\mathbf{O}_\phi) = & \lambda_2 & = \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & \lambda_2^2 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & 2\lambda_2 m_2 & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & 0 & = \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & 0 & = \widehat{D}_5(\phi) \\ p_6(\mathbf{O}_\phi) = & 2\lambda_2^2 m_2 & = -\widehat{D}_6(\phi) \end{array}\right\}. \tag{4.5}$$

Layer 3,1: $r > 0$. The stabilizer of $a_\circ = re_1$ in K_{0,λ_2} is a copy of $(U(1)\times U(1))\times Sp(d-1)$. Fundamental highest weight vectors for the action $K_{(0,\lambda_2),a_\circ} : V_2$ are h_2, h_3 and h_4. Thus the parameters for a function $\phi \in \Delta^{3,1}$ are $(0,\lambda_2;r,m_2,m_3,m_4)$. The spherical orbit $\mathbf{O}_\phi = \mathbf{O}(0,\lambda_2;r,m_2,m_3,m_4)$ is now

$$\mathbf{O}_\phi = K\cdot\left(\begin{bmatrix} re_1 \\ (2\lambda_2 m_2)^{1/2}e_1 + (2\lambda_2 m_3)^{1/2}e_2 + (2\lambda_2 m_4)^{1/2}e_{d+1}\end{bmatrix}, B_{0,\lambda_2}\right)$$

and one has

$$\left\{\begin{array}{lll} p_1(\mathbf{O}_\phi) = & \lambda_2 & = \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & \lambda_2^2 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & r^2 + 2\lambda_2(m_2+m_3+m_4) & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & 2\lambda_2 r^2(m_3+m_4) & = \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & 2\lambda_2 r^2 m_4 & = \widehat{D}_5(\phi) \\ p_6(\mathbf{O}_\phi) = & 2\lambda_2^2(m_2+m_3+m_4) & = -\widehat{D}_6(\phi) \end{array}\right\}. \tag{4.6}$$

Layer 4: $\lambda_1 = \lambda_2 = 0$. As in the previous example Layer 4 parameters are $(0,0;r)$ with $r \geq 0$. For $\phi \in \Delta^4$ with these parameters we have $\mathbf{O}_\phi = \mathbf{O}(0,0;r)$ given by (3.9) and

$$\left\{\begin{array}{lll} p_1(\mathbf{O}_\phi) = & 0 & = \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & 0 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & r^2 & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & 0 & = \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & 0 & = \widehat{D}_5(\phi) \\ p_6(\mathbf{O}_\phi) = & 0 & = \widehat{D}_6(\phi) \end{array}\right\}. \tag{4.7}$$

4.1. **Condition (O) for** (K, N)**.** The proof that (K, N) satisfies Condition (O) closely parallels that for the previous example. See Section 3.8. Given a sequence $(\phi_n)_{n=1}^{\infty}$ in $\Delta(K, N)$ and $\phi \in \Delta(K, N)$ one argues that $\phi_n)$ converges to ϕ if and only if the sequence $\mathbf{O}_n := \Psi(\phi_n)$ converges to $\mathbf{O} := \Psi(\phi)$. Equivalently one must show that $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for $j = 1, \ldots, 6$ if and only if $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for $j = 1, \ldots, 6$. We can assume that the ϕ_n's all lie in a single layer in $\Delta(K, N)$ and, for simplicity, here suppose that $(\phi_n)_{n=1}^{\infty}$ is contained in one of the four layers discussed above. If either $\phi_n \to \phi$ or $\mathbf{O}_n \to \mathbf{O}$ it follows that ϕ lies in the same layer as the ϕ_n's or in a higher layer. On layers 3 and 4 the values $\widehat{D}_j(\phi)$ and $p_j(\mathbf{O}_\phi)$ agree, up to sign, for all j. Thus it suffices to assume that $(\phi_n)_{n=1}^{\infty}$ is contained in Layers 1 or 2.

There are seven cases to examine, just as in Section 3.8. In each case one needs to check that if the sequence $\widehat{D}_j(\phi_n) = \pm p_j(\mathbf{O}_n)$ converges to $\widehat{D}_j(\phi) = p_j(\mathbf{O})$ for $j = 1, 2, 3, 6$ then both $\widehat{D}_4(\phi_n) \to \widehat{D}_4(\phi)$ and $\widehat{D}_5(\phi_n) \to \widehat{D}_5(\phi)$ if and only if $p_4(\mathbf{O}_n) \to p_4(\mathbf{O})$ and $p_5(\mathbf{O}_n) \to p_5(\mathbf{O})$. An argument for each case can be given that is similar to that in Section 3.8. To illustrate we give the argument for *Case 4* and omit the details for the remaining cases.

Case 4: $(\phi_n)_{n=1}^{\infty} \subset \Delta^1$, $\phi \in \Delta^{3,1}$: Let $(\lambda_1(n), \lambda_2(n); m_1(n), m_2(n), m_5(n), m_6(n))$ be the parameters for ϕ_n and $(0, \lambda_2'; r, m_2', m_3', m_4')$ those for ϕ. See (4.3) and (4.6). First suppose that $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for all j. Thus also $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for $j \neq 4, 5$. As the sequence

$$p_3(\mathbf{O}_n) = 2\lambda_1(n)(m_1(n) + m_5(n) + m_6(n)) + 2\lambda_2(n)(m_2(n) + m_5(n) + m_6(n))$$

converges and $\lambda_1(n) \to 0$, $\lambda_2(n) \to \lambda_2' > 0$ it follows that both $m_5(n)$ and $m_6(n)$ are bounded sequences. Thus the lower order terms in the expressions for $\widehat{D}_4(\phi_n)$ and $\widehat{D}_5(\phi_n)$, namely

$$4\lambda_1(n)\lambda_2(n)(m_5(n) + m_6(n)) \quad \text{and} \quad 4\lambda_1(n)\lambda_2(n)m_6(n)(2d - 1)$$

respectively, both converge to zero as $n \to \infty$. So $\lim \widehat{D}_4(\phi_n) = \lim p_4(\mathbf{O}_n) = p_4(\mathbf{O}) = \widehat{D}_4(\phi)$ and likewise $\lim \widehat{D}_5(\phi_n) = \lim p_5(\mathbf{O}_n) = p_5(\mathbf{O}) = \widehat{D}_5(\phi)$. The same reasoning shows that $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for all j when $\widehat{D}_j(\phi_n) \to \widehat{D}_j(\phi)$ for all j. □

5. The pair $(Sp(2) \times Sp(d), (\mathbb{H}^2 \otimes \mathbb{H}^d) \oplus H\Lambda^2(\mathbb{H}^2))$

Here $\mathfrak{n} = \mathfrak{n}_d = V \oplus \mathfrak{z}$ where $V = \mathbb{H}^2 \otimes \mathbb{H}^d$ (tensor over $\mathbb{H}$) and $\mathfrak{z} = H\Lambda^2(\mathbb{C}^2) = \mathfrak{sp}(2)$. We realize V as $V = M_{2,d}(\mathbb{H})$, matrices of size $2 \times d$ with quaternion entries. As usual the bracket $V \times V \to \mathfrak{z}$ is given by $[u, v] = uv^* - vu^*$ and $K = Sp(2) \times Sd(d)$ acts via $(k_1, k_2) \cdot (v, A) = (k_1 v k_2^*, k_1 A k_1^*)$. Here we are viewing both $Sp(2)$ and $Sp(d)$ as quaternionic matrices satisfying $kk^* = I$. This is case (18) in [Wol07, §13.4] and line 12 in Table 1.

The pair $(K, \mathfrak{n})$ is closely related to the example treated in the previous section, namely $(\widetilde{K}, \widetilde{\mathfrak{n}}) := (U(2) \times Sp(d), M_{2,2d}(\mathbb{C}) \oplus \mathfrak{u}(2))$. We will use the fact that $(\widetilde{K}, \widetilde{N})$ satisfies Condition (O) to establish Condition (O) for (K, N).

Each K-orbit in $\mathfrak{z}$ contains a unique point of the form $B_{\lambda_1,\lambda_2} = -diag(i\lambda_1, i\lambda_2)$ with $0 \le \lambda_1 \le \lambda_2$ real.[3] This fact imposes a decomposition of $\Delta(K, N)$ and $\mathcal{A}(K, N)$ into layers as before. Moreover using the K-invariant inner product on $\mathfrak{n}$, given by

$$\big((u, A), (v, B)\big)_{\mathfrak{n}} := Re(tr(uv^*)) + \frac{1}{2}Re(tr(AB^*)),$$

one again has that the J-mapping, $J_B : V \to V$ for $B \in \mathfrak{z}$ is simply $J_B(v) = -Bv$. Thus $J_{\lambda_1,\lambda_2} = J_{B_{\lambda_1,\lambda_2}}$ is left multiplication by $i\lambda_1$, $i\lambda_2$ on the two rows. For Layer 1 and Layer 2, where $0 < \lambda_1 \le \lambda_2$, the complex vector space $\widetilde{\mathfrak{w}}_{\lambda_1,\lambda_2}$ is thus $V = M_{2,d}(\mathbb{H})$ with its *standard* complex structure, i.e. left multiplication by i. For Layer 3 ($0 = \lambda_1 < \lambda_2$) one has $\widetilde{\mathfrak{w}}_{0,\lambda_2} = V_2 \cong \mathbb{H}^d$, the second row with its standard complex structure.

To connect analysis for (K, N) with that for $(\widetilde{K}, \widetilde{N})$ we apply the *standard isomorphism* $M_{2,d}(\mathbb{H}) \cong M_{2,2d}(\mathbb{C})$, namely

$$(v = u + wj) \mapsto \widetilde{v} := [u|w],$$

for $u, w \in M_{2,d}(\mathbb{C})$. Giving $V = M_{2,d}(\mathbb{H})$ and $\widetilde{V} := M_{2,2d}(\mathbb{C})$ their standard complex structures the map $v \mapsto \widetilde{v}$ is an isomorphism of complex vector spaces. Moreover $v \mapsto \widetilde{v}$ transforms the right action of $Sp(d)$ on V into its right action on $\widetilde{V}$ with $Sp(d)$ realized as $\{k \in U(2d) \ : \ k\mathcal{J}k^t = \mathcal{J}\}$, as in the previous example. The left action of the subgroup $U(2) \subset Sp(2)$ carries over from V to the usual left action of $U(2)$ on $\widetilde{V}$. So now $\widetilde{K} \subset K$ (as $U(2) \subset Sp(2)$) and we regard $\widetilde{\mathfrak{n}}$ as a subspace of $\mathfrak{n}$ by identifying $\widetilde{V} \cong V$ and noting that $(\mathfrak{z}(\widetilde{\mathfrak{n}}) = \mathfrak{u}(2)) \subset (\mathfrak{sp}(2) = \mathfrak{z}(\mathfrak{n}))$.

For the first three layers we have the following stabilizers for B_{λ_1,λ_2}.

- Layer 1: $0 < \lambda_1 < \lambda_2$, $K_{\lambda_1,\lambda_2} = (U(1) \times U(1)) \times Sp(d)$.
- Layer 2: $0 < \lambda_1 = \lambda = \lambda_2$, $K_{\lambda,\lambda} = U(2) \times Sp(d)$.
- Layer 3: $0 = \lambda_1 < \lambda_2$, $K_{0,\lambda_2} = (U(1) \times U(1)) \times Sp(d)$.

These all lie in $\widetilde{K} = U(2) \times Sp(d)$ and agree with the stabilizers from the previous example. As the standard isomorphism $V \cong \widetilde{V}$ respects the standard complex structures and is $\widetilde{K}$-equivariant one has the following facts.

- The relevant multiplicity free actions for Layers 1,2,3 are as in Section 4 and the fundamental highest weight vectors that appear are given in (4.2).
- The parameters for Δ^1, Δ^2, $\Delta^{3,0}$ and $\Delta^{3,1}$ coincide with those in Section 4. Letting $\phi \in \Delta(K, N)$ and $\widetilde{\phi} \in \Delta(\widetilde{K}, \widetilde{N})$ denote spherical functions for the two pairs in these layers with common parameters one has $\phi|_{\widetilde{N}} = \widetilde{\phi}$.
- The spherical orbits $\mathbf{O}_\phi$ for functions ϕ in Layers 1,2,3 are as given in Section 4. That is, in each case we have a base point in $\mathbf{O}_\phi$ of the form $(\widetilde{v}, B_{\lambda_1,\lambda_2})$ with

[3] As the Weyl group for $Sp(2)$ includes sign changes one can ensure $\lambda_1, \lambda_2 \ge 0$ here.

$\widetilde{v} \in \widetilde{V}$ determined by the layer parameters. Regard this as lying in $V \oplus \mathfrak{sp}(2)$ and take the orbit under the group $K = Sp(2) \times Sp(d)$.

The Layer 4 situation, $\lambda_1 = 0 = \lambda_2$, is entirely straightforward. Here the spherical functions ϕ and their orbits $\mathbf{O}_\phi$ coincide with those from Section 4.

Let $\Delta^{\geq 0}(\widetilde{K}, \widetilde{N}) := \big(\Delta^1 \cup \Delta^2 \cup \Delta^{3,0} \cup \Delta^{3,1} \cup \Delta^4\big)(\widetilde{K}, \widetilde{N})$ and $\mathcal{A}^{\geq 0}(\widetilde{K}, \widetilde{N}) := \{\mathbf{O}_{\widetilde{\phi}} \,:\, \widetilde{\phi} \in \Delta^{\geq 0}(\widetilde{K}, \widetilde{N})\}$. The discussion above yields homeomorphisms

$$\Delta(K, N) \cong \Delta^{\geq 0}(\widetilde{K}, \widetilde{N}) \quad \text{and} \quad \mathcal{A}(K, N) \cong \mathcal{A}^{\geq 0}(\widetilde{K}, \widetilde{N})$$

which intertwine with the orbit mappings for the two pairs. The argument given in Section 4 shows that the orbit mapping for $(\widetilde{K}, \widetilde{N})$ restricts to a homeomorphism $\Delta^{\geq 0}(\widetilde{K}, \widetilde{N}) \cong \mathcal{A}^{\geq 0}(\widetilde{K}, \widetilde{N})$. Thus now $\Psi : \Delta(K, N) \to \mathcal{A}(K, N)$ is a homeomorphism and the pair (K, N) satisfies Condition (O) as claimed. □

6. The pair $(G_2, \mathbb{R}^7 \oplus \mathbb{R}^7)$

Here $\mathfrak{n} = V \oplus \mathfrak{z}$ where $V = Im(\mathbb{O}) = \mathfrak{z}$ with Lie bracket $V \times V \to \mathfrak{z}$ given by

$$[v, w] := \frac{1}{2}(vw - wv).$$

The group $K = G_2$ acts on $\mathfrak{n}$ via two copies of its usual representation on $\mathbb{R}^7 = Im(\mathbb{O})$. This is case (3) in [Wol07, §13.4] and line 13 in Table 1.

Equipping V and $\mathfrak{z}$ with their usual inner products, denoted by dot, one has

$$J_z(v) = [z, v]$$

for $z \in \mathfrak{z}$, $v \in V$. The orbits for $K = G_2$ on $\mathfrak{z} = Im(\mathbb{O})$ are spheres. (This example is not handled in [FGJ+19] because the J-map is singular.) For $z \in \mathfrak{z}$ with $z \neq 0$ we find that

$$\mathfrak{a}_z := Ker(J_z) = \mathbb{R}z$$

and that the restriction of J_z to $\mathfrak{w}_z := \mathfrak{a}_z^\perp \cap V$ is given by

$$J_z(w) = zw.$$

The stabilizer of z in K is a copy of $SU(3)$ and this is, of course, also the stabilizer of the point $(a, z) \in \mathfrak{n}$ for any $a \in \mathfrak{a}_z$.

Letting $\{e_1, \ldots, e_7\}$ denote the standard basis for $Im(\mathbb{O})$ the set $\{\lambda e_1 \,:\, \lambda \geq 0\}$ is thus a cross section to the K-orbits in $\mathfrak{z}$, which are parameterized by λ. The situation here is straightforward. There are just two layers in $\Delta(K, N)$, the spherical functions of type I, for which $\lambda > 0$, and those of type II, for which $\lambda = 0$. As generators for $\mathbb{R}[\mathfrak{n}]^K$ one has (see [FRY12])

$$\big\{p_1(v, z) := \|v\|^2, \quad p_2(v, z) := \|z\|^2, \quad p_3(v, z) := v \cdot z\big\}.$$

As usual we let $D_j \in \mathbb{D}_K(N)$ denote the (unsymmetrized) operator obtained from p_j.

Layer 1: $\lambda > 0$. Taking $z = \lambda e_1$ with $\lambda > 0$ we have $\mathfrak{w}_z = \{0\} \oplus \mathbb{R}^6$ with complex structure $\widetilde{J}_z(w) = e_1 w$ and $J_z^2 = -\lambda^2$ on $\mathfrak{w}_z$. The multiplicity free action $K_z : \widetilde{\mathfrak{w}}_z$ is a copy of $SU(3) : \mathbb{C}^3$. For each $t \in \mathbb{R}$ and $m \in \mathbb{Z}^+$ we have a spherical function $\phi = \phi_{\lambda,t,m}$ determined by the data $(z = \lambda e_1, a = te_1, m)$. The associated spherical orbit $\mathbf{O}_\phi = \Psi(\phi)$ is

$$\mathbf{O}_\phi = K \cdot \big(te_1 + (2\lambda m)^{1/2} e_2,\, \lambda e_1\big)$$

and the values $p_j(\mathbf{O}_\phi)$ and $\widehat{D}_j(\phi)$ agree for $j = 1, 2, 3$, namely

$$\left\{ \begin{array}{lll} p_1(\mathbf{O}_\phi) = & t^2 + 2\lambda m & = -\widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & \lambda^2 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & t\lambda & = \widehat{D}_3(\phi) \end{array} \right\}.$$

Layer 2: $\lambda = 0$. For each $r \geq 0$ we have a spherical function $\phi = \phi_r$, the K-average of the character $(v, z) \mapsto e^{ire_1 \cdot v}$. The associated spherical orbit $\mathbf{O}_\phi = \Psi(\phi)$ is

$$\mathbf{O}_\phi = K \cdot (re_1,\, 0)$$

and we have

$$\left\{ \begin{array}{c} p_1(\mathbf{O}_\phi) = r^2 = -\widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = 0 = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = 0 = \widehat{D}_3(\phi) \end{array} \right\}.$$

6.1. **Condition (O) for** (K, N)**.** As $p_j(\mathbf{O}_\phi) = \pm\widehat{D}_j(\phi)$ for $j = 1, 2, 3$ and all $\phi \in \Delta(K, N)$ it follows immediately that (K, N) satisfies Condition (O). □

7. The pair $(U(1) \times Spin(7), \mathbb{C}^8 \oplus (\mathbb{R}^7 \oplus \mathbb{R}))$

In line 14 of Table 1 one has $\mathfrak{n} = V \oplus \mathfrak{z}$ where $V = \mathbb{O}^2 = \left\{ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \;:\; v_1, v_2 \in \mathbb{O} \right\}$ and $\mathfrak{z} = Im(\mathbb{O}) \oplus \mathbb{R}$. The bracket $V \times V \to \mathfrak{z}$ is given by

$$[v, w] = \big(\{v, w\}, \omega(v, w)\big)$$

where $\{v, w\} \in Im(\mathbb{O})$ and $\omega(v, w) \in \mathbb{R}$ are defined as

$$\{v, w\} := -\frac{1}{2}\big(v_1\overline{w}_1 - w_1\overline{v}_1 + v_2\overline{w}_2 - w_2\overline{v}_2\big), \quad \omega(v, w) := v_1 \cdot w_2 - v_2 \cdot w_1.$$

The expression for $\{v, w\}$ involves arithmetic in the octonians $\mathbb{O}$. The expression for $\omega(v, w)$ involves the usual inner product on $\mathbb{O} \cong \mathbb{R}^8$, namely $v \cdot w = Re(v\overline{w})$. The group $K = U(1) \times Spin(7)$ acts on $\mathfrak{n}$ as follows. $Spin(7)$ acts on V via two copies of its spin representation on $\mathbb{O} \cong \mathbb{R}^8$ and acts on $Im(\mathbb{O}) \cong \mathbb{R}^7$ by the vector representation via $SO(Im(\mathbb{O})) \cong SO(7)$. The action of $Spin(7)$ on $\mathbb{R} \subset \mathfrak{z}$ is trivial.

The scalars $U(1)$ act via rotations on V, and act trivially on $\mathfrak{z}$. This example is case (12) in [Wol07, §13.4].[4]

We equip V and $\mathfrak{z}$ with their usual inner products. For $(z,t)\in\mathfrak{z}$ one finds

$$J_{z,t}(v) = \begin{bmatrix} zv_1 - tv_2 \\ zv_2 + tv_1 \end{bmatrix}. \tag{7.1}$$

As the orbits for $Spin(7)$ on $Im(\mathbb{O})$ are spheres we can take for z a non-negative multiple of a chosen unit vector.

Let $\{e_0, e_1, \ldots, e_7\}$ be the standard basis for $\mathbb{O}$, where $e_0 = 1$, so that $\{e_1,\ldots,e_7\}$ is the standard basis for $Im(\mathbb{O})$. We may assume that $z = re_1$ for some $r \geq 0$ and let $J_{r,t} := J_{re_1,t}$. Note, in particular, that

$$J_1(v) := J_{1,0}(v) = \begin{bmatrix} e_1 v_1 \\ e_1 v_2 \end{bmatrix}$$

is a complex structure on V. We denote this space, with its complex structure, as $\widetilde{V}$. Letting $V_\pm$ be the subspaces of V defined as

$$V_+ := \left\{ \begin{bmatrix} u \\ -e_1 u \end{bmatrix} \;:\; u\in\mathbb{O}\right\}, \quad V_- := \left\{ \begin{bmatrix} u \\ +e_1 u \end{bmatrix} \;:\; u\in\mathbb{O}\right\},$$

one obtains the following Lemma via routine calculation.

Lemma 7.1. *The subspaces $V_\pm$ are $U(1)$-invariant and $V = V_+\oplus V_-$ is an orthogonal direct sum decomposition. The subspaces $V_\pm$ are invariant under $J_{r,t}$ for all $r\geq 0$, $t\in\mathbb{R}$ with*

$$J_{r,t}|_{V_+} = (r+t)J_1|_{V_+}, \quad J_{r,t}|_{V_-} = (r-t)J_1|_{V_-}.$$

Hence also $J_{r,t}^2|_{V_\pm} = -(r\pm t)^2 I_{V_\pm}$.

Let

- $\mathfrak{a}_{r,t} = Ker(J_{r,t})$, $\mathfrak{w}_{r,t} = Image(J_{r,t})$, $\sigma(r,t) = \{\lambda\geq 0 \;:\; -\lambda^2$ is an eigenvalue for $J_{r,t}\}$,
- $\widetilde{J}_{r,t}$ be the complex structure on $\mathfrak{w}_{r,t}$ obtained from $J_{r,t}$,
- $\widetilde{\mathfrak{w}}_{r,t}$ denote the complex vector space $(\mathfrak{w}_{r,t}, \widetilde{J}_{r,t})$.

Moreover let $\widetilde{V}_\pm$ denote the complex vector space $(V_\pm, J_1)$ and $\widetilde{V}_\pm^-$ denote its conjugate complex space, i.e. $(V_\pm, -J_1)$. In view of Lemma 7.1 we have the following.

(1) If $r>0$ and $|t|<r$ then $\sigma(r,t) = \{r+t, r-t\}$, $\mathfrak{a}_{r,t} = \{0\}$ and $\widetilde{\mathfrak{w}}_{r,t} = \widetilde{V}_+\oplus\widetilde{V}_-$.
(2) If $r>0$ and $r<|t|$ then $\sigma(r,t) = \{|r+t|, |t-r|\}$, $\mathfrak{a}_{r,t} = \{0\}$ and

$$\widetilde{\mathfrak{w}}_{r,t} = \begin{cases} \widetilde{V}_+\oplus\widetilde{V}_-^- & \text{if } r<t \\ \widetilde{V}_+^-\oplus\widetilde{V}_- & \text{if } t<-r \end{cases}.$$

(3) If $r>0$ and $t=\pm r$ then $\sigma(r,t) = \{0, 2r\}$, $\mathfrak{a}_{r,t} = V_\mp$ and $\widetilde{\mathfrak{w}}_{r,t} = \widetilde{V}_\pm$.

[4] The factor of $-1/2$ on $\{v.w\}$ is not, however, used in [Wol07] or elsewhere. This has been introduced to simplify Formula 7.1.

(4) If $r = 0$ and $t \neq 0$ then $\sigma(r,t) = \{|t|\}$, $\mathfrak{a}_{r,t} = \{0\}$ and $\widetilde{\mathfrak{w}}_{r,t} = \widetilde{V}_+^{\pm} \oplus \widetilde{V}_-^{\mp}$. In this case it is best to ignore the $V_+ \oplus V_-$ decomposition and write simply $\widetilde{\mathfrak{w}}_{r,t} = (V, J_{0,1})$. See Equation 7.1.

(5) If $r = 0 = t$ then $\sigma(r,t) = \{0\}$, $\mathfrak{a}_{r,t} = V$ and $\widetilde{\mathfrak{w}}_{r,t} = \{0\}$.

Figure 1 illustrates these cases as regards $\widetilde{\mathfrak{w}}_{r,t}$.

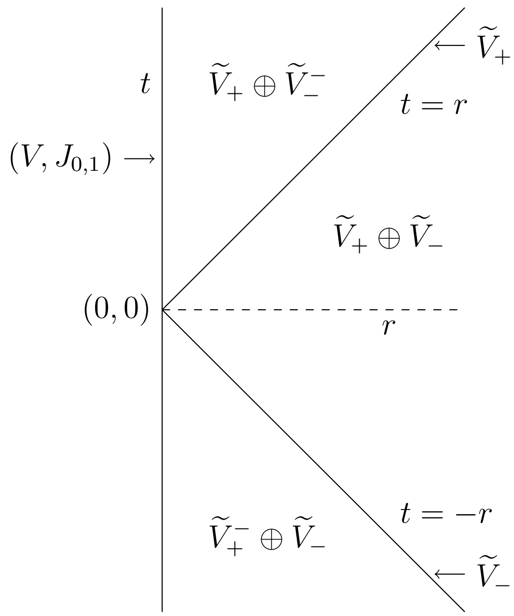

FIGURE 1. The complex spaces $\widetilde{\mathfrak{w}}_{r,t}$

The values $(r,t) \in \mathbb{R}^+ \times \mathbb{R}$ parameterize the K-orbits in $\mathfrak{z}$ and determine a layering on $\Delta(K,N)$ and $\mathcal{A}(K,N)$ corresponding to the five possibilities above. Before discussing these layers in detail we introduce, from [FRY12, Theorem 7.5], generators for $\mathbb{R}[\mathfrak{n}]^K$, namely[5]

$$\left\{ \begin{array}{c} p_1(v,z,t) = t, \quad p_2(v,z,t) = \|z\|^2 \\ p_3(v,z,t) = \|v\|^2, \quad p_4(v,A) = (\|v_1\|^2 - \|v_2\|^2)^2 + 4(v_1 \cdot v_2)^2 \\ p_5(v,z,t) = Re(z(v_1\overline{v}_2)) = -z \cdot (v_1\overline{v}_2). \end{array} \right\}. \tag{7.2}$$

Here $\{p_1, p_2\}$ generates $\mathbb{R}[\mathfrak{z}]^K$ and $\{p_3, p_4\}$ generates $\mathbb{R}[V]^K$. As usual one obtains (unsymmetrized) generators $D_j := D_{p_j}$ $(j = 1, \ldots, 5)$ for $\mathbb{D}_K(N)$.

Next we introduce parameters on each layer and give values $p_j(\mathbf{O}_\phi)$, $\widehat{D}_j(\phi)$ for subsequent use in the proof that (K,N) satisfies Condition (O). As the calculations required to justify the various formulas are similar to those for prior examples we omit

[5]In [FRY12] one finds the invariant $F_4(v,z,t) = |v_1|^2|v_2|^2 - (v_1 \cdot v_2)^2$ in place of p_4. These invariants are related via $p_4 + 4F_4 = p_3^2$. We prefer p_4 here as the values $p_4(\mathbf{O}_\phi)$, $\widehat{D}_4(\phi)$ given below are somewhat simpler than those obtained using F_4.

the details. Letting $K_{r,t}$ denote the stabilizer of $(re_1, t) \in \mathfrak{z}$ in $K = U(1) \times Spin(7)$ one has

$$K_{r,t} = \left\{ \begin{array}{ll} U(1) \times SU(4) & \text{if } r \neq 0 \text{ (i.e. in Layers 1, 2, 3)} \\ U(1) \times Spin(7) & \text{if } r = 0 \text{ (i.e. in Layers 4, 5)} \end{array} \right\}.$$

Layer 1: $r > 0$ **and** $|t| < r$. We have $K_{r,t} = U(1) \times SU(4)$ acting diagonally on the complex vector space $\widetilde{\mathfrak{w}}_{r,t} = \widetilde{V}_+ \oplus \widetilde{V}_-$ with $J_{r,t}^2 = -(r \pm t)^2$ on $\widetilde{V}_\pm$. Identifying $\widetilde{V}_+ \oplus \widetilde{V}_-$ with $M_{2,4}(\mathbb{C})$ we have fundamental highest weight vectors

$$h_1(z) = z_{11}, \quad h_2(z) = z_{21}, \quad h_3(z) = det_2(z)$$

and write $(r, t; m_1, m_2, m_3)$ $(m_j \in \mathbb{Z}^+)$ for the Layer 1 spherical function parameters. The spherical orbit $\mathbf{O}_\phi$ for the function $\phi \in \Delta^1$ with these parameters is obtained by applying Proposition 2.1 to the spherical point given in [BR15b, §5.2]. Further calculation yields the following values.

$$\text{(7.3)} \qquad \left\{ \begin{array}{lll} p_1(\mathbf{O}_\phi) = & t & = \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & r^2 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & 2(r+t)(m_1+m_3) + 2(r-t)(m_2+m_3) & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & 16(r+t)(r-t)m_1m_2, & \\ & p_4(\mathbf{O}_\phi) - 16(r+t)(r-t)m_3 = & \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & -r(r+t)(m_1+m_3) + r(r-t)(m_2+m_3) & = -\widehat{D}_5(\phi) \end{array} \right\}.$$

Layer 2: $r > 0$ **and** $r < |t|$. Suppose here that $0 < r < t$. Now $K_{r,t} = U(1) \times SU(4)$ acts diagonally on $\widetilde{\mathfrak{w}}_{r,t} = \widetilde{V}_+ \oplus \widetilde{V}_-^-$. (In case $t < -r < 0$ just interchange the roles of V_+ and V_-.) This is a *twisted* variant of the multiplicity free action from Layer 1, treated in [BR15b, §5.3]. Identifying $\widetilde{V}_+ \oplus \widetilde{V}_-^-$ with $M_{2,4}(\mathbb{C})$ the fundamental highest weight vectors are as in Layer 1 but with h_3 replaced by

$$h_3(z) = z_{1,-} \cdot z_{2,-},$$

the dot product of the rows. We again write $(r, t; m_1, m_2, m_3)$ for the Layer 2 spherical function parameters. One obtains the following values $p_j(\mathbf{O}_\phi)$ and $\widehat{D}_j(\phi)$ for the function $\phi \in \Delta^2$ with these parameters.

$$\text{(7.4)} \qquad \left\{ \begin{array}{lll} p_1(\mathbf{O}_\phi) = & t & = \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & r^2 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & 2|r+t|(m_1+m_3) + 2|r-t|(m_2+m_3) & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & 16|r+t||r-t|m_3(m_1+m_2+m_3), & \\ & p_4(\mathbf{O}_\phi) + 48|r+t||r-t|m_3 & = \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & -r|r+t|(m_1+m_3) + r|r-t|(m_2+m_3) & = -\widehat{D}_5(\phi) \end{array} \right\}.$$

Layer 3: $r > 0$**,** $t = \pm r$. Here suppose $t = r$. (In case $t = -r$ just interchanges the roles of V_+ and V_-.) We have $K_{r,r} = U(1) \times SU(4)$, $\mathfrak{a}_{r,r} = V_-$, $\widetilde{\mathfrak{w}}_{r,r} = \widetilde{V}_+$ with

$J^2_{r,r} = -(2r)^2 = -4r^2$ on $\widetilde{V}_+$. Let $a \in V_-$ be given. As the action of $K_{r,r}$ on V_- is transitive on spheres we can assume that

$$a = s \begin{bmatrix} e_0 \\ e_1 \end{bmatrix}$$

for some $s \geq 0$. There are two sub-cases to consider.

Layer 3,0: $s = 0$. The stabilizer of $a = 0$ in $K_{r,r}$ is $K_{(r,r),0} = K_{r,r} = U(1) \times SU(4)$. This acts on $\widetilde{V}_+ \cong \mathbb{C}^4$ as $(U(1) \times SU(4)) : \mathbb{C}^4$, a rank 1 multiplicity free action with fundamental highest weight vector

$$h(z) = z_1.$$

We write $(r, \pm r; 0, m)$ for the parameters of a spherical function $\phi \in \Delta^{3,0}$ and compute

$$\left\{ \begin{array}{lll} p_1(\mathbf{O}_\phi) = & \pm r & = \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & r^2 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & 2rm & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & 0 & = \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & \mp r^2 m & = -\widehat{D}_5(\phi) \end{array} \right\}. \tag{7.5}$$

Layer 3,1: $s > 0$. Now $a \neq 0$ has stabilizer $K_{(r,r),a} = S(U(1) \times U(3))$ in $K_{r,r}$ acting on $\widetilde{V}_+ \cong \mathbb{C}^4$ as $S(U(1) \times U(3)) : \mathbb{C} \oplus \mathbb{C}^3$. Our fundamental highest weight vectors are

$$h_1(z) = z_1, \quad h_2(z) = z_2$$

and we write $(r, \pm r; s, m_1, m_2)$ for the parameters of $\phi \in \Delta^{3,1}$. One finds

$$\left\{ \begin{array}{lll} p_1(\mathbf{O}_\phi) = & \pm r = & \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & r^2 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & 2s^2 + 2r(m_1 + m_2) & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & 16rs^2 m_1 & = \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & \pm rs^2 \mp r^2(m_1 + m_2) & = -\widehat{D}_5(\phi) \end{array} \right\}. \tag{7.6}$$

Layer 4: $r = 0$, $t \neq 0$. We have $K_{0,t} = U(1) \times Spin(7) = K$ acting on the complex vector space $\widetilde{\mathfrak{w}}_{0,t} = (V, J_{0,1})$ where

$$J_{0,1}(v) = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}.$$

The multiplicity free action $K_{0,t} : \widetilde{\mathfrak{w}}_{0,t}$ is a copy of $(U(1) \times Spin(7)) : \mathbb{C}^8$, which is irreducible of rank two with fundamental highest weight vectors

$$h_1(z) = z_1 + iz_2, \quad h_2(z) = z_1^2 + \cdots + z_8^2.$$

Thus spherical functions $\phi \in \Delta^4$ are parametrized by $(0, t; m_1, m_2)$ and one computes

$$(7.7)\quad \left\{ \begin{array}{lll} p_1(\mathbf{O}_\phi) = & \pm t & = \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & 0 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & 2|t|(m_1 + 2m_2) & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & 16t^2 m_2(m_1 + m_2), & \\ & p_4(\mathbf{O}_\phi) + 48t^2 m_2 & = \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & 0 & = -\widehat{D}_5(\phi) \end{array} \right\}.$$

Layer 5: $r = 0 = t$. This layer gives the spherical functions of type II. These correspond to $(K = U(1) \times Spin(7))$-orbits in V, which are parameterized as follows.

Lemma 7.2. [Sas09, Lemma 5.9] *Every K-orbit in V contains a unique point of the form* $v(s_0, s_1) := \begin{bmatrix} s_0 e_0 \\ s_1 e_1 \end{bmatrix}$ *with* $0 \le s_0 \le s_1$.

Letting $(0, 0; s_0, s_1)$ denote the parameters for $\phi \in \Delta^5$ one has

$$(7.8)\quad \left\{ \begin{array}{lll} p_1(\mathbf{O}_\phi) = & 0 & = \widehat{D}_1(\phi) \\ p_2(\mathbf{O}_\phi) = & 0 & = \widehat{D}_2(\phi) \\ p_3(\mathbf{O}_\phi) = & s_0^2 + s_1^2 & = -\widehat{D}_3(\phi) \\ p_4(\mathbf{O}_\phi) = & (s_0 - s_1)^2 & = \widehat{D}_4(\phi) \\ p_5(\mathbf{O}_\phi) = & 0 & = -\widehat{D}_5(\phi) \end{array} \right\}.$$

7.1. **Condition (O) for** (K, N)**.** To verify Condition (O) for this example we reason as in Section 3.8. Note that in all layers $p_j(\mathbf{O}_\phi) = \pm\widehat{D}_j(\phi)$ for $j = 1, 2, 3, 5$ but that $\widehat{D}_4(\phi)$ differs from $p_4(\mathbf{O}_\phi)$ by a lower order term if $\phi \in \Delta^1 \cup \Delta^2 \cup \Delta^4$. So given a sequence $(\phi_n)_{n=1}^\infty$ in Layers 1, 2 or 4 and a function $\phi \in \Delta(K, N)$ we need to check that $\big(\widehat{D}_4(\phi_n) \to \widehat{D}_4(\phi)\big) \iff \big(p_4(\mathbf{O}_n) \to p_4(\mathbf{O})\big)$ under the hypothesis that the sequence $\widehat{D}_j(\phi_n) = \pm p_j(\mathbf{O}_n)$ converges to $\widehat{D}_j(\phi) = \pm p_j(\mathbf{O})$ for $j = 1, 2, 3, 5$. The assumption that $p_j(\mathbf{O}_n) \to p_j(\mathbf{O})$ for $j = 1, 2$ implies that the r and t parameters for ϕ_n converge to those for ϕ. In particular

- if $(\phi_n)_{n=1}^\infty \subset \Delta^1$ then $\phi \in \Delta^1 \cup \Delta^{3,0} \cup \Delta^{3,1} \cup \Delta^5$,
- if $(\phi_n)_{n=1}^\infty \subset \Delta^2$ then $\phi \in \Delta^2 \cup \Delta^{3,0} \cup \Delta^{3,1} \cup \Delta^4 \cup \Delta^5$, and
- if $(\phi_n)_{n=1}^\infty \subset \Delta^4$ then $\phi \in \Delta^4 \cup \Delta^5$.

Thus there are eleven cases to examine.

Case 1: $(\phi_n)_{n=1}^\infty \subset \Delta^1$, $\phi \in \Delta^1$: Let $(r(n), t(n); m_1(n), m_2(n), m_3(n))$ be the parameters for ϕ_n and $(r', t'; m_1', m_2', m_3')$ those for ϕ. Using (7.3) the argument here is similar to that given for *Case 1* in Section 3.8. As $p_3(\mathbf{O}_n) \to p_3(\mathbf{O})$ and $r' + t' \neq 0 \neq r' - t'$ it follows that the sequences $m_1(n)$, $m_2(n)$ and $m_3(n)$ are eventually constant, $(m_1(n), m_2(n), m_3(n)) = (m_1, m_2, m_3)$ say. If we assume that $p_4(\mathbf{O}_n) \to p_4(\mathbf{O})$ it then follows that ϕ has parameters (r', t', m_1, m_2, m_3) and hence $(m_1', m_2', m_3') =$

(m_1, m_2, m_3). This is the case as the limiting values for $p_1(\mathbf{O}_n), \ldots, p_5(\mathbf{O}_n)$, namely $p_1(\mathbf{O}) = \lim p_1(\mathbf{O}_n) = t'$, $p_2(\mathbf{O}) = \lim p_2(\mathbf{O}_n) = (r')^2$ and

$$\left\{\begin{array}{l} p_3(\mathbf{O}) = \lim p_3(\mathbf{O}_n) = 2(r'+t')(m_1+m_3) + 2(r'-t')(m_2+m_3) \\ p_4(\mathbf{O}) = \lim p_4(\mathbf{O}_n) = 16(r'+t')(r'-t')m_1m_2 \\ p_5(\mathbf{O}) = \lim p_5(\mathbf{O}_n) = -r'(r'-t')(m_1+m_3) + r'(r'-t')(m_2+m_3) \end{array}\right\},$$

completely determine $\mathbf{O}$ and its parameters. So now

$$\lim \widehat{D}_4(\phi) = p_4(\mathbf{O}) - 16(r'+t')(r'-t')m_3 = \widehat{D}_4(\phi)$$

as desired. The proof that $p_4(\mathbf{O}_n) \to p_4(\mathbf{O})$ when $\widehat{D}_4(\phi_n) \to \widehat{D}_4(\phi)$ goes the same way, using the fact that the eigenvalues $\widehat{D}_1(\phi), \ldots, \widehat{D}_5(\phi)$ determine ϕ and its parameters.

Case 2: $(\phi_n)_{n=1}^{\infty} \subset \Delta^1$, $\phi \in \Delta^{3,0}$: Let $(r(n), t(n); m_1(n), m_2(n), m_3(n))$ be the parameters for ϕ_n and $(r', \pm r'; 0, m')$ those for ϕ. See (7.3) and (7.5). As $p_3(\mathbf{O}_n) \to p_3(\mathbf{O})$ and one of $r' + (\pm r')$, $r' - (\pm r')$ is non-zero it follows that the sequence $m_3(n)$ is eventually constant, $m_3(n) = m_3$ say. So $\lim(r(n)+t(n))(r(n)-t(n))m_3(n) = 0$ and hence $\lim \widehat{D}_4(\phi_n) = \widehat{D}_4(\phi)$ if and only if $\lim p_4(\mathbf{O}_n) = p_4(\mathbf{O})$.

Case 3: $(\phi_n)_{n=1}^{\infty} \subset \Delta^1$, $\phi \in \Delta^{3,1}$: The reasoning from *Case 2* applies equally here.

Case 4: $(\phi_n)_{n=1}^{\infty} \subset \Delta^1$, $\phi \in \Delta^5$: Let $(r(n), t(n); m_1(n), m_2(n), m_3(n))$ be the parameters for ϕ_n and $(0, 0; s_0', s_1')$ those for ϕ. See (7.3) and (7.8). As $p_3(\mathcal{O}_n) \to p_3(\mathbf{O}) = 0$, we have $\lim(r(n) \pm t(n))m_3(n) = 0$ and hence also $\lim(r(n)+t(n))(r(n)-t(n))m_3(n) = 0$. Thus $\lim \widehat{D}_4(\phi_n) = \widehat{D}_4(\phi)$ if and only if $\lim p_4(\mathbf{O}_n) = p_4(\mathbf{O})$.

Case 5: $(\phi_n)_{n=1}^{\infty} \subset \Delta^2$, $\phi \in \Delta^2$: The argument here is identical to that for *Case 1.*

Cases 6,7: $(\phi_n)_{n=1}^{\infty} \subset \Delta^2$, $\phi \in \Delta^{3,0} \cup \Delta^{3,1}$: The proofs here are as in *Case 2.*

Case 8: $(\phi_n)_{n=1}^{\infty} \subset \Delta^2$, $\phi \in \Delta^4$: Let $(r(n), t(n); m_1(n), m_2(n), m_3(n))$ be the parameters for ϕ_n and $(0, t'; m_1', m_2')$ those for ϕ. See (7.4) and (7.7). As $p_3(\mathbf{O}_n) \to p_3(\mathbf{O})$ and $t' \neq 0$ the sequences $m_1(n)$, $m_2(n)$ and $m_3(n)$ are eventually constant, $(m_1(n), m_2(n), m_3(n)) = (m_1, m_2, m_3)$ say. So now $\lim p_3(\mathbf{O}_n) = p_3(\mathbf{O})$ implies

$$m_1 + m_2 + 2m_3 = m_1' + 2m_2'. \tag{7.9}$$

Suppose that $\lim p_4(\mathbf{O}_n) = p_4(\mathbf{O})$. This gives

$$m_3(m_1 + m_2 + m_3) = m_2'(m_1' + m_2'). \tag{7.10}$$

Together (7.9) and (7.10) imply that

$$m_3 = m_2' \quad \text{and} \quad m_1 + m_2 + m_3 = m_1' + m_2'.$$

Thus now

$$\lim \widehat{D}_4(\phi_n) = p_4(\mathbf{O}) + 48(t')^2 m_3 = p_4(\mathbf{O}) + 48(t')^2 m_2' = \widehat{D}_4(\phi)$$

as desired. Likewise assuming $\lim \widehat{D}_4(\phi) = \widehat{D}_4(\phi)$ gives

$$m_3(m_1 + m_2 + m_3 + 3) = m_2'(m_1' + m_2' + 3), \tag{7.11}$$

which together with (7.9) again implies that both $m_3 = m_2'$ and $m_1 + m_2 + m_3 = m_1' + m_2'$. So

$$\lim p_4(\mathbf{O}_n) = \widehat{D}_4(\phi) - 48(t')^2 m_3 = \widehat{D}_4(\phi) - 48(t')^2 m_2' = p_4(\mathbf{O})$$

Case 9: $(\phi_n)_{n=1}^\infty \subset \Delta^2$, $\phi \in \Delta^5$: The argument here is identical to that for *Case 4*.

Case 10: $(\phi_n)_{n=1}^\infty \subset \Delta^4$, $\phi \in \Delta^4$: Let $(0, t(n); m_1(n), m_2(n))$ be the parameters for ϕ_n and $(0, t'; m_1', m_2')$ those for ϕ. See (7.7). As $p_3(\mathbf{O}_n) \to p_3(\mathbf{O})$ and $t' \neq 0$ the sequences $m_1(n)$, $m_2(n)$ are eventually constant, $(m_1(n), m_2(n)) = (m_1, m_2)$ say. Thus now $\lim p_3(\mathbf{O}_n) = p_3(\mathbf{O})$ impies $m_1 + 2m_2 = m_1' + 2m_2'$. If we assume, in addition, that either $\lim p_4(\mathbf{O}_n) = p_4(\mathbf{O})$ or $\lim \widehat{D}_4(\phi_n) = \widehat{D}_4(\phi)$ then it follows that $(m_1', m_2') = (m_1, m_2)$. This is the case as the values $p_1(\mathbf{O}), \ldots, p_5(\mathbf{O})$ and $\widehat{D}_1(\phi), \ldots, \widehat{D}_5(\phi)$ each determine the parameters for ϕ. Thus

$$\lim p_4(\mathbf{O}_n) = p_4(\mathbf{O}) \iff \lim \widehat{D}_4(\phi_n) = p_4(\mathbf{O}) + 48(t')^2 m_2 = \widehat{D}_4(\phi).$$

Case 11: $(\phi_n)_{n=1}^\infty \subset \Delta^4$, $\phi \in \Delta^5$: Let $(0, t(n); m_1(n), m_2(n))$ be the parameters for ϕ_n and $(0, 0; s_0', s_1')$ those for ϕ. See (7.7) and (7.8). As $p_3(\mathbf{O}_n) \to p_3(\mathbf{O})$ it follows that the sequence $|t(n)|m_2(n)$ is convergent, hence bounded. As $t(n) \to 0$ this implies that $t(n)^2 m_2(n) \to 0$ hence $\lim p_4(\mathbf{O}_n) = p_4(\mathbf{O})$ if and only if $\lim \widehat{D}_4(\phi_n) = \widehat{D}_4(\phi)$. □

References

[BJR90] Chal Benson, Joe Jenkins, and Gail Ratcliff. On Gel′fand pairs associated with solvable Lie groups. *Trans. Amer. Math. Soc.*, 321(1):85–116, 1990.

[BR08] Chal Benson and Gail Ratcliff. The space of bounded spherical functions on the free 2-step nilpotent Lie group. *Transform. Groups*, 13(2):243–281, 2008.

[BR13] Chal Benson and Gail Ratcliff. Geometric models for the spectra of certain Gelfand pairs associated with Heisenberg groups. *Ann. Mat. Pura Appl. (4)*, 192(4):719–740, 2013. (with *Erratum* on pages 741–743).

[BR15a] Chal Benson and Gail Ratcliff. Spaces of bounded spherical functions on Heisenberg groups: part I. *Ann. Mat. Pura Appl. (4)*, 194(2):321–342, 2015.

[BR15b] Chal Benson and Gail Ratcliff. Spaces of bounded spherical functions on Heisenberg groups: part II. *Ann. Mat. Pura Appl. (4)*, 194(2):533–561, 2015.

[BR20] Chal Benson and Gail Ratcliff. Spaces of bounded spherical functions for irreducible nilpotent Gelfand pairs: Part I. *J. Lie Theory*, 30(3):779–810, 2020.

[FGJ+19] Holley Friedlander, William Grodzicki, Wayne Johnson, Gail Ratcliff, Anna Romanov, Benjamin Strasser, and Brent Wessel. An orbit model for the spectra of nilpotent Gelfand pairs. *Transform. Groups*, 2019. to appear.

[FR07] Fabio Ferrari Ruffino. The topology of the spectrum for Gelfand pairs on Lie groups. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, 10(3):569–579, 2007.

[FRY] Véronique Fischer, Fulvio Ricci, and Oksana Yakimova. Nilpotent Gelfand pairs and spherical transforms of Schwartz functions III: Isomorphisms between Schwartz spaces under Vinberg's condition. available on arXiv: 1210.7962.

[FRY12] Véronique Fischer, Fulvio Ricci, and Oksana Yakimova. Nilpotent Gelfand pairs and spherical transforms of Schwartz functions I: rank-one actions on the centre. *Math. Z.*, 271(1-2):221–255, 2012.

[FRY13] Véronique Fischer, Fulvio Ricci, and Oksana Yakimova. Nilpotent Gelfand pairs and spherical transforms of Schwartz functions II: Taylor expansions on singular sets. In *Lie groups: structure, actions, and representations*, volume 306 of *Progr. Math.*, pages 81–112. Birkhäuser/Springer, New York, 2013.

[FRY18] Véronique Fischer, Fulvio Ricci, and Oksana Yakimova. Nilpotent Gelfand pairs and Schwartz extensions of spherical transforms via quotient pairs. *J. Funct. Anal.*, 274(4):1076–1128, 2018.

[HU91] Roger Howe and Tōru Umeda. The Capelli identity, the double commutant theorem, and multiplicity-free actions. *Math. Ann.*, 290(3):565–619, 1991.

[Lip80] Ronald L. Lipsman. Orbit theory and harmonic analysis on Lie groups with co-compact nilradical. *J. Math. Pures Appl. (9)*, 59(3):337–374, 1980.

[Lip82] Ronald L. Lipsman. Orbit theory and representations of Lie groups with co-compact radical. *J. Math. Pures Appl. (9)*, 61(1):17–39, 1982.

[Puk78] L. Pukanszky. Unitary representations of Lie groups with cocompact radical and applications. *Trans. Amer. Math. Soc.*, 236:1–49, 1978.

[Sas09] Atsumu Sasaki. Visible actions on irreducible multiplicity-free spaces. *Int. Math. Res. Not. IMRN*, (18):3445–3466, 2009.

[Vin01] E. B. Vinberg. Commutative homogeneous spaces and co-isotropic symplectic actions. *Russian Math. Surveys*, 56:1–60, 2001.

[Vin03] E. B. Vinberg. Commutative homogeneous spaces of Heisenberg type. *Trans. Moscow Math. Soc.*, 64:45–78, 2003.

[Wol07] Joseph A. Wolf. *Harmonic analysis on commutative spaces*, volume 142 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

Dept of Mathematics, East Carolina University, Greenville, NC 27858
E-mail address: bensonf@ecu.edu, ratcliffg@ecu.edu