THREE-STEP HARMONIC SOLVMANIFOLDS

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Abstract. The Lichnerowicz conjecture asserted that every harmonic Riemannian manifold is locally isometric to a two-point homogeneous space. In 1992, E. Damek and F. Ricci produced a family of counter-examples to this conjecture, which arise as abelian extensions of two-step nilpotent groups of type-H. In this paper we consider a broader class of Riemannian manifolds: solvmanifolds of Iwasawa type with algebraic rank one and two-step nilradical. Our main result shows that the Damek-Ricci spaces are the only harmonic manifolds of this type.

1. Introduction

Harmonic spaces are Riemannian manifolds which exhibit a strong type of radial symmetry. In particular, the volume density expressed in normal coordinates about any point $x_\circ$ depends only on the distance to $x_\circ$. There are many equivalent conditions, most of which concern the associated Laplace-Beltrami operator and harmonic functions. For example, there exists a non-constant harmonic function on $U\backslash\{x_\circ\}$ depending only on $d(\cdot, x_\circ)$ for some neighborhood U of any given point $x_\circ$. We refer the reader to Chapter 6 in [Bes78] for a discussion of these conditions as well as historical background.

It is well known that every harmonic space is also an Einstein manifold. The converse is far from true, as the harmonicity conditions are highly restrictive. Indeed, the Einstein condition is just one of the infinite family of "Ledger Conditions" on the curvature tensor of a harmonic manifold. The most conspicuous harmonic manifolds are the two-point homogeneous spaces, which are the (flat) Euclidean spaces together with the rank one symmetric spaces [Hel78]. For a long time, it was not known whether, conversely, every harmonic manifold was locally isometric to a two-point homogeneous space. This question is known as the Lichnerowicz conjecture [Lic44].

The Lichnerowicz conjecture was proved in [Wal49] for manifolds of dimension four or less and in [Sza90] for compact harmonic spaces with finite fundamental group. Further positive results include Allamigeon's Theorem ([All65]), asserting that every complete connected and simply connected harmonic space is either a Euclidean space or a Blaschke manifold. It is also known that if M is asymptotically harmonic,

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compact, and with negative sectional curvature, then M is locally rank one symmetric ([FL92], [BFL92],[BCG96]). (Asymptotic harmonicity is a property weaker than harmonicity.)

In 1992, E. Damek and F. Ricci produced a new family of non-compact, simply connected harmonic spaces which generalize the rank one symmetric spaces [DR92]. The many non-symmetric Damek-Ricci spaces are counter-examples to the Lichnerowicz conjecture. To date, we know of no other simply connected counter-examples.

The Damek-Ricci spaces arise as abelian extensions of nilpotent Lie groups of type-H. (See [Kap80], [Kor85], [CDKR91], [DR93].) In this paper, we will use the term *solvmanifold* to refer to a connected and simply connected solvable Lie group together with a left-invariant metric. Each Damek-Ricci space is, in particular, a solvmanifold. It is known, moreover, that every connected and simply connected homogeneous space with nonpositive sectional curvature is isometric to a solvmanifold. (See [Hei74], [AW76].)

It seems natural to seek a classification of harmonic solvmanifolds. In this paper we work outward from the class of Damek-Ricci spaces and show that within a larger class of solvmanifolds, the Damek-Ricci spaces are the only harmonic spaces. Our main result is Theorem 1 below. The relevant definitions can be found below in Section 2.

Theorem 1. *Let S be a solvmanifold that is rank one, three-step, and of Iwasawa type. If the first two Ledger conditions hold, then S is a Damek-Ricci space.*

As an immediate corollary, we are able to say

Corollary 2. *Let S be a solvmanifold that is rank one, three-step, and of Iwasawa type. If S is harmonic, then S is a Damek-Ricci space.*

A result of Heber [Heb95] shows that an asymptotically harmonic solvmanifold of Iwasawa type with non-positive sectional curvature must have rank one. Thus we also have

Corollary 3. *Let S be a three-step solvmanifold of Iwasawa type with non-positive sectional curvature. If S is harmonic, then S is a Damek-Ricci space.*

In a particular case, we have the following result. This has been obtained independently by M. Druetta [Dru].

Theorem 4. *Let S be a three-step Carnot solvmanifold. If S is harmonic, then S is a Damek-Ricci space.*

The proof of Theorem 1 is quite technical, so we will also present an independent elementary proof of Theorem 4. If the solvmanifold S in Theorem 1 is in fact *two-step* then the appropriate conclusion is that S is a real hyperbolic space. This is a known result. (See, in particular, Proposition 2.3(b) in [EH96].) Our proof of Theorem 1 encompasses this situation as a "degenerate case".

We conclude this introductory section with a remark concerning work of P. Eberlein and J. Heber. In a detailed study of the geometric and algebraic properties of three-step solvmanifolds, they showed that if S is an Einstein solvmanifold with quarter-pinched negative curvature [EH96], then S is locally rank one symmetric. Since harmonic spaces are Einstein, it follows from this description that harmonic three-step solvmanifolds with quarter-pinched negative curvature are locally rank one symmetric.

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2. Algebraic Properties

A left-invariant metric g on a connected, simply connected solvable group S is determined by its restriction Q to $T_eS \cong \mathfrak{s}$, so we will identify the pair (S, g) with the pair $(\mathfrak{s}, Q)$. For $X, Y \in \mathfrak{s}$, we write $\langle X, Y\rangle$ in place of $Q(X, Y)$.

Definition 1. *A solvmanifold* $(\mathfrak{s}, Q)$ *is said to be of* Iwasawa type *if it has the following properties:*

(1) *The orthogonal complement* $\mathfrak{a}$ *of* $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ *is an abelian subalgebra of* $\mathfrak{s}$*.*
(2) *For all nonzero* A *in* $\mathfrak{a}$, ad_A *is symmetric and nonzero.*
(3) *For some* A *in* $\mathfrak{a}$, $ad_A|_\mathfrak{n}$ *is symmetric and positive definite.*

Our study of harmonic solvmanifolds is motivated by the observation that all known examples of Einstein solvmanifolds are of Iwasawa type (see [Heb98]). Throughout this paper, we consider solvmanifolds $(\mathfrak{s}, Q)$ of Iwasawa type, subject to two additional conditions:

(4) The subalgebra $\mathfrak{a}$ is one dimensional. We say that $(\mathfrak{s}, Q)$ is of (algebraic) *rank one* and let A denote a unit vector in $\mathfrak{a}$ with ad_A positive definite on $\mathfrak{n}$.
(5) The nilradical $\mathfrak{n}$ is either two-step nilpotent, or abelian. This means that $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$. If $[\mathfrak{n}, \mathfrak{n}] \neq 0$ then $\mathfrak{n}$ is a two-step nilpotent Lie algebra. In this case we say that $(\mathfrak{s}, Q)$ is a *three-step* solvmanifold of Iwasawa type.

Moreover, we adopt the following notational conventions:

- $\mathfrak{z}$ denotes the center of $\mathfrak{n}$, and $\mathfrak{v}$ is the orthogonal complement to $\mathfrak{z}$ in $\mathfrak{n}$, so that $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$. We let $n = \dim(\mathfrak{v})$ and $m = \dim(\mathfrak{z})$, with $\dim(\mathfrak{n}) = m + n$.
- For $Z \in \mathfrak{z}$, the linear transformation $J_Z : \mathfrak{v} \to \mathfrak{v}$ is defined by
$$\langle J_Z X, Y\rangle = \langle Z, [X, Y]\rangle$$
for all X and Y in $\mathfrak{v}$. Note that J_Z is skew-symmetric for all Z.
- In the degenerate case where $\mathfrak{n}$ is abelian, $\mathfrak{n} = \mathfrak{z}$, $\mathfrak{v} = 0$, and $J_Z = 0$ for all $Z \in \mathfrak{z}$. In this case, many of the identities derived below become trivial.

All rank one symmetric spaces of noncompact type may be represented as solvmanifolds of Iwasawa type with an abelian or two-step nilradical $\mathfrak{n}$. The latter type form a small subclass of the *Damek-Ricci* spaces.

Definition 2. *A* three-step Carnot solvmanifold $(\mathfrak{s}, Q)$ *is a rank one solvmanifold of Iwasawa type with two-step nilradical* $\mathfrak{n}$ *such that*

$$ad_A|_{\mathfrak{v}} = \tfrac{c}{2} Id_{\mathfrak{v}}, \quad ad_A|_{\mathfrak{z}} = c Id_{\mathfrak{z}}$$

for some constant $c > 0$*. If, in addition, we have*

$$J_Z^2 = -c^2 ||Z||^2 Id_{\mathfrak{v}}$$

for all Z *in* $\mathfrak{z}$*, then* $(\mathfrak{s}, Q)$ *is a* Damek-Ricci space*.*

By rescaling the metric, one can always take $c = 1$ in this definition. More precisely, replacing the inner product Q in Definition 2 by $c^2 Q$, one obtains a three-step solvmanifold with $\mathrm{ad}_A|_{\mathfrak{v}} = (1/2) Id_{\mathfrak{v}}$, $\mathrm{ad}_A|_{\mathfrak{z}} = Id_{\mathfrak{z}}$ and, in addition, $J_Z^2 = -||Z||^2 Id_{\mathfrak{v}}$ in the Damek-Ricci case. A two-step Lie algebra $\mathfrak{n}$ with an inner product is said to be of *type-H* (or Heisenberg type) when $J_Z^2 = -||Z||^2 Id_{\mathfrak{v}}$ for all $Z \in \mathfrak{z}$.

As ad_A is a derivation, it preserves the center $\mathfrak{z}$ and, by symmetry, it also preserves $\mathfrak{v} = \mathfrak{z}^{\perp}$. Hence, again by symmetry, both $\mathfrak{v}$ and $\mathfrak{z}$ are the direct sum of ad_A-eigenspaces. For λ in $\mathbb{R}$, we use $\mathfrak{v}_\lambda$ to denote the λ-eigenspace of ad_A on $\mathfrak{v}$. That is, $\mathfrak{v}_\lambda = \{X \in \mathfrak{v} \mid \mathrm{ad}_A X = \lambda X\}$. Similarly, $\mathfrak{z}_\lambda$ is the λ-eigenspace of ad_A on $\mathfrak{z}$. For ad_A-eigenvectors $V \in \mathfrak{v}$, $Z \in \mathfrak{z}$ we will use λ_V, λ_Z to denote the eigenvalues for V, Z respectively.

We use $\mathcal{B}_{\mathfrak{v}}$ and $\mathcal{B}_{\mathfrak{z}}$ to denote orthonormal bases for $\mathfrak{v}$ and $\mathfrak{z}$ consisting of eigenvectors for ad_A. The following lemma relates the behavior of the operators J_Z to ad_A and its eigenspaces.

Lemma 5. *Let* λ *be an eigenvalue for* ad_A *on* $\mathfrak{z}$*, and let* μ*,* ν *be eigenvalues for* ad_A *on* $\mathfrak{v}$*. Then we have:*

(1) $[\mathfrak{v}_\mu, \mathfrak{v}_\nu] \subset \mathfrak{z}_{\mu+\nu}$
(2) $J_Z ad_A + ad_A J_Z = J_{ad_A Z}$ *for all* Z *in* $\mathfrak{z}$*.*
(3) *If* Z *is in* $\mathfrak{z}_\lambda$*, then* J_Z *maps* $\mathfrak{v}_\mu$ *to* $\mathfrak{v}_{\lambda-\mu}$
(4) *For* Z *in* $\mathfrak{z}_\lambda$*,* $J_Z^2 ad_A = ad_A J_Z^2$ *.*

Proof. Part 1 holds because ad_A acts on $\mathfrak{n}$ as a derivation.

For 2, take V, W in $\mathfrak{v}$ and use the symmetry of ad_A:

$$\begin{aligned}
\langle J_{\mathrm{ad}_A Z} V, W \rangle &= \langle \mathrm{ad}_A Z, [V, W] \rangle \\
&= \langle Z, \mathrm{ad}_A [V, W] \rangle \\
&= \langle Z, [\mathrm{ad}_A V, W] \rangle + \langle Z, [V, \mathrm{ad}_A W] \rangle \\
&= \langle J_Z(\mathrm{ad}_A V), W \rangle + \langle J_Z V, \mathrm{ad}_A W \rangle \\
&= \langle J_Z(\mathrm{ad}_A V), W \rangle + \langle \mathrm{ad}_A(J_Z V), W \rangle \\
&= \langle (J_Z \mathrm{ad}_A + \mathrm{ad}_A J_Z) V, W \rangle
\end{aligned}$$

Thus $J_{\mathrm{ad}_A Z} = J_Z \mathrm{ad}_A + \mathrm{ad}_A J_Z$.

For V in $\mathfrak{v}_\mu$ and Z in $\mathfrak{z}_\lambda$ we now have, by 2,

$$\mathrm{ad}_A(J_Z V) = J_{\lambda Z} V - J_Z(\mu V) = (\lambda - \mu)(J_Z V),$$

which shows that $J_Z V$ is in $\mathfrak{v}_{\lambda-\mu}$.

For $Z \in \mathfrak{z}_\lambda$, 3 shows that J_Z^2 preserves each eigenspace $\mathfrak{v}_\mu$ of the diagonalizable operator ad_A. Thus J_Z^2 and ad_A commute on $\mathfrak{v}$. □

We also consider, for $V \in \mathfrak{v}$, the operators

$$\mathrm{ad}_V^* \mathrm{ad}_V : \mathfrak{v} \to \mathfrak{v} \quad \text{and} \quad \mathrm{ad}_V \mathrm{ad}_V^* : \mathfrak{z} \to \mathfrak{z}$$

where $\mathrm{ad}_V^* : \mathfrak{z} \to \mathfrak{v}$ is the adjoint of $\mathrm{ad}_V : \mathfrak{v} \to \mathfrak{z}$ with respect to the inner product Q. Note that, for a linear operator T and its adjoint T^*, the operators TT^* and T^*T have the same trace. This fact will be used repeatedly for the operators ad_V and ad_V^*.

Lemma 6. *For all V in $\mathfrak{v}$ and Z in $\mathfrak{z}$:*
1. $ad_V^* Z = J_Z V$
2. $\langle ad_V\, ad_V^* Z, Z\rangle = -\langle J_Z^2 V, V\rangle$
3. $\sum_{V \in \mathcal{B}_\mathfrak{v}} tr(ad_V\, ad_V^*) = -\sum_{Z \in \mathcal{B}_\mathfrak{z}} tr(J_Z^2)$.

Proof. Given $W \in \mathfrak{v}$,

$$\langle J_Z V, W\rangle = \langle Z, [V, W]\rangle = \langle Z, \mathrm{ad}_V(W)\rangle = \langle \mathrm{ad}_V^*(Z), W\rangle.$$

Property 2 follows from

$$\langle \mathrm{ad}_V ad_V^* Z, Z\rangle = \langle \mathrm{ad}_V^* Z, \mathrm{ad}_V^* Z\rangle = \langle J_Z V, J_Z V\rangle = -\langle J_Z^2 V, V\rangle.$$

This last identity, summed over orthonormal bases of $\mathfrak{v}$, $\mathfrak{z}$, gives 3. □

We note that Lemmas 5 and 6 extend easily to higher rank and higher step solvmanifolds of Iwasawa type. In the more general context, one defines J_Z, $\mathrm{ad}_V^* \mathrm{ad}_V$ and $\mathrm{ad}_V \mathrm{ad}_V^*$ as above for *arbitrary* elements Z, V in $\mathfrak{n}$ as operators *on all of* $\mathfrak{n}$. Now Lemma 5 remains true if A denotes any element in $\mathfrak{a}$ with eigenvalues λ and corresponding eigenspaces $\mathfrak{n}_\lambda$. Lemma 6 remains true for arbitrary elements V, Z in $\mathfrak{n}$.

3. Geometry of Solvmanifolds

We continue to employ the notation and hypotheses from the previous section. In particular, $(\mathfrak{s}, Q)$ will always denote a solvmanifold of Iwasawa type having rank one with two-step (or abelian) nilradical $\mathfrak{n}$. We identify a vector X in $\mathfrak{s}$ with the left-invariant vector field that it determines. The Levi-Civita connection is given by

$$\nabla_X Y = \tfrac{1}{2}([X, Y] - \mathrm{ad}_X^* Y - \mathrm{ad}_Y^* X) \tag{1}$$

for X and Y in $\mathfrak{s}$ (see Chapter 7 of [Bes87]). Using this definition one may easily derive explicit formulas for the connection on $(\mathfrak{s}, Q)$.

Recall that for u in the tangent space $T_m M$ to a Riemannian manifold M at m, the curvature transformation R_u is the endomorphism of $T_m M$ defined by $R_u : v \mapsto R(u, v)u$. We use the convention that $R(u, v) = [\nabla_u, \nabla_v] - \nabla_{[u,v]}$. With this sign convention, the sectional curvature of the 2-plane with orthonormal basis $\{u, v\}$ is given by $K(u,v) = -\langle R_u(v), v\rangle$. The operator R_u is homogeneous of degree 2 in $||u||$ and R_u^2 is homogeneous of degree 4 in $||u||$.

Proposition 7. *For ad_A-eigenvectors V in $\mathfrak{v}$ and Z in $\mathfrak{z}$, the curvature transformations R_A, R_V, R_Z, R_{A+V} and R_{A+Z} are given by*

$$
\begin{aligned}
R_X(X) &= 0 && \textit{for all } X \textit{ in } \mathfrak{s}\\
R_A(X) &= ad_A^2 X && \textit{for all } X \textit{ in } \mathfrak{s}\\
R_V(A) &= \lambda_V^2 A\\
R_V(U) &= \lambda_V\, ad_A U + \tfrac{3}{4} ad_V^* ad_V U && \textit{for all } U \textit{ in } \mathfrak{v} \textit{ with } U \perp V\\
R_V(W) &= \lambda_V\, ad_A W - \tfrac{1}{4} ad_V ad_V^* W && \textit{for all } W \textit{ in } \mathfrak{z}\\
R_Z(A) &= \lambda_Z^2 A\\
R_Z(U) &= \lambda_Z\, ad_A U + \tfrac{1}{4} J_Z^2 U && \textit{for all } U \textit{ in } \mathfrak{v}\\
R_Z(W) &= \lambda_Z\, ad_A W && \textit{for all } W \textit{ in } \mathfrak{z} \textit{ with } W \perp Z\\
R_{A+V}(A) &= \lambda_V^2 (A - V)\\
R_{A+V}(U) &= \lambda_V\, ad_A U + ad_A^2 U + [V, ad_A U] + \tfrac{1}{2}\lambda_V [V, U]\\
&\quad + \tfrac{3}{4} ad_V^* ad_V U - \lambda_V^2 \langle V, U\rangle (A + V) && \textit{for all } U \textit{ in } \mathfrak{v}\\
R_{A+V}(W) &= \lambda_V\, ad_A W - \tfrac{1}{4} ad_V ad_V^* W + ad_A^2 W\\
&\quad + ad_A J_W V + \tfrac{1}{2}\lambda_V J_W V && \textit{for all } W \textit{ in } \mathfrak{z}\\
R_{A+Z}(A) &= \lambda_Z^2 (A - Z)\\
R_{A+Z}(U) &= \lambda_Z\, ad_A U + \tfrac{1}{4} J_Z^2 U + ad_A^2 U\\
&\quad + \tfrac{1}{2}(J_Z ad_A - ad_A J_Z) U && \textit{for all } U \textit{ in } \mathfrak{v}\\
R_{A+Z}(W) &= \lambda_Z\, ad_A W + ad_A^2 W && \textit{for all } W \textit{ in } \mathfrak{z} \textit{ with } W \perp Z
\end{aligned}
$$

Proof. To prove this, one uses the definition of R_X and the formulas for the connection in Equation 1. As similar formulas appear elsewhere in the literature (see for example [BTV95]) we omit the calculations. These are straightforward but somewhat tedious. □

Proposition 8. *Let V and Z be unit ad_A-eigenvectors in $\mathfrak{v}$ and $\mathfrak{z}$ respectively. The traces of the curvature transformations R_A, R_V, and R_Z are given by*

$$
\begin{aligned}
tr(R_A) &= tr(ad_A^2)\\
tr(R_V) &= \lambda_V\, tr(ad_A) + \tfrac{1}{2} tr(ad_V ad_V^*)\\
tr(R_Z) &= \lambda_Z\, tr(ad_A) + \tfrac{1}{4} tr(J_Z^2)
\end{aligned}
$$

Proof. That $\mathrm{tr}(R_A) = \mathrm{tr}(\mathrm{ad}_A^2)$ is immediate from $R_A = \mathrm{ad}_A^2$. We find $\mathrm{tr}(R_V)$ by finding the sums of the traces of R_V restricted to $\mathfrak{a}, \mathfrak{v}$, and $\mathfrak{z}$ (noting that $\mathfrak{a}, \mathfrak{v}$, and $\mathfrak{z}$ are orthogonal)

$$\begin{aligned}
\mathrm{tr}(R_V) &= \mathrm{tr}(R_V|_{\mathfrak{a}}) + \mathrm{tr}(R_V|_{\mathfrak{v}}) + \mathrm{tr}(R_V|_{\mathfrak{z}}) \\
&= \lambda_V^2 + \lambda_V \mathrm{tr}(\mathrm{ad}_A|_{\mathfrak{v}\cap V^\perp}) + \tfrac{3}{4}\mathrm{tr}(\mathrm{ad}_V^*\mathrm{ad}_V) + \lambda_V \mathrm{tr}(\mathrm{ad}_A|_{\mathfrak{z}}) - \tfrac{1}{4}\mathrm{tr}(\mathrm{ad}_V\mathrm{ad}_V^*) \\
&= \lambda_V \mathrm{tr}(\mathrm{ad}_A) + \tfrac{3}{4}\mathrm{tr}(\mathrm{ad}_V^*\mathrm{ad}_V) - \tfrac{1}{4}\mathrm{tr}(\mathrm{ad}_V\mathrm{ad}_V^*) \\
&= \lambda_V \mathrm{tr}(\mathrm{ad}_A) + \tfrac{1}{2}\mathrm{tr}(\mathrm{ad}_V^*\mathrm{ad}_V).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{tr}(R_Z) &= \mathrm{tr}(R_Z|_{\mathfrak{a}}) + \mathrm{tr}(R_Z|_{\mathfrak{v}}) + \mathrm{tr}(R_Z|_{\mathfrak{z}}) \\
&= \lambda_Z^2 + \lambda_Z \mathrm{tr}(\mathrm{ad}_A|_{\mathfrak{v}}) + \tfrac{1}{4}\mathrm{tr}(J_Z^2) + \lambda_Z \mathrm{tr}(\mathrm{ad}_A|_{\mathfrak{z}\cap Z^\perp}) \\
&= \lambda_Z \mathrm{tr}(\mathrm{ad}_A) + \tfrac{1}{4}\mathrm{tr}(J_Z^2)
\end{aligned}$$

□

Proposition 9. *For the unit ad_A-eigenvectors V in $\mathfrak{v}$ and Z in $\mathfrak{z}$, the traces of the squares of the curvature transformations R_A, R_V, R_Z, R_{A+V}, and R_{A+Z} are given by:*

$$\begin{aligned}
tr(R_A^2) &= tr(ad_A^4) \\
tr(R_V^2) &= \tfrac{5}{8} tr((ad_V^* ad_V)^2) + \tfrac{3}{2}\lambda_V tr(ad_V^* ad_V ad_A) - \tfrac{1}{2}\lambda_V tr(ad_V ad_V^* ad_A) + \lambda_V^2 tr(ad_A^2) \\
tr(R_Z^2) &= \tfrac{1}{16} tr(J_Z^4) + \tfrac{1}{2}\lambda_Z tr(J_Z^2 ad_A) + \lambda_Z^2 tr(ad_A^2) \\
tr(R_{A+V}^2) &= tr(ad_A^4) + 2\lambda_V tr(ad_A^3) + \lambda_V^2 tr(ad_A^2) + \tfrac{5}{8} tr((ad_V^* ad_V)^2) \\
&\quad + \tfrac{1}{2}\lambda_V^2 tr(ad_V^* ad_V) + \tfrac{5}{2}\lambda_V tr(ad_V^* ad_V ad_A) + \tfrac{5}{2} tr(ad_V^* ad_V ad_A^2) \\
&\quad - \tfrac{3}{2}\lambda_V tr(ad_V ad_V^* ad_A) + \tfrac{1}{2} tr(ad_V ad_V^* ad_A^2) \\
tr(R_{A+Z}^2) &= \tfrac{1}{16} tr(J_Z^4) - \tfrac{1}{2} tr(J_Z^2 ad_A^2) + \tfrac{3}{2}\lambda_Z tr(J_Z^2 ad_A) \\
&\quad - \tfrac{1}{4}\lambda_Z^2 tr(J_Z^2) + tr(ad_A^4) + 2\lambda_Z tr(ad_A^3) + \lambda_Z^2 tr(ad_A^2)
\end{aligned}$$

Proof. This is another set of long calculations, which we again omit. □

4. Ledger's equations

A real analytic Riemannian manifold M is harmonic if and only if it is "infinitesimally harmonic." The latter condition is equivalent to an infinite number of *Ledger conditions*. (See [Lic44], [CR40], [Led54], [Bes78].) Since Lie groups are analytic, a solvmanifold is harmonic if and only if it satisfies each of Ledger's equations. In the sequel, we only use the first two Ledger conditions, and employ the terminology of [CGW82]:

Definition 3. *M is* 2-stein *when both the first and second Ledger conditions hold. That is, there are constants K, H such that, for all $X \in TM$,*

$$\begin{aligned} tr(R_X) =& K||X||^2 \text{ and} \\ tr(R_X^2) =& H||X||^4. \end{aligned}$$

Note that the manifold M is *Einstein* if and only if it satisfies the first Ledger condition.

5. Einstein Constraints

Continuing with the hypotheses and notation used above, we assume that $(\mathfrak{s}, Q)$ is a rank one solvmanifold of Iwasawa type with two-step (or abelian) nilradical $\mathfrak{n}$, and A is a unit vector in $\mathfrak{a}$ with $\mathrm{ad}_A : \mathfrak{n} \to \mathfrak{n}$ positive definite. In addition, we assume that $(\mathfrak{s}, Q)$ satisfies Ledger's first equation, the Einstein condition. This last assumption will impose additional constraints on the traces of powers of ad_A.

We employ the notation:

$$\begin{aligned} T_r &= \mathrm{tr}(\mathrm{ad}_A^r), \\ a_r &= \mathrm{tr}(\mathrm{ad}_A^r|_{\mathfrak{v}}), \\ b_r &= \mathrm{tr}(\mathrm{ad}_A^r|_{\mathfrak{z}}). \end{aligned}$$

Note that $T_r = a_r + b_r$ for each $r \geq 0$. (Here it is understood that $a_r = 0$ in the degenerate case.) Recall that $\mathcal{B}_{\mathfrak{v}}$ and $\mathcal{B}_{\mathfrak{z}}$ denote orthonormal bases for $\mathfrak{v}$ and $\mathfrak{z}$ consisting of eigenvectors for ad_A.

Proposition 10. *For unit ad_A-eigenvectors V in $\mathfrak{v}$ and Z in $\mathfrak{z}$, one has:*

1. $tr(J_Z^2) = 4(T_2 - \lambda_Z T_1)$
2. $tr(J_Z^2 ad_A) = 2\lambda_Z(T_2 - \lambda_Z T_1)$
3. $tr(ad_V ad_V^*) = 2(T_2 - \lambda_V T_1)$

Proof. Equating $\mathrm{tr}(R_V)$ and $\mathrm{tr}(R_Z)$ to $\mathrm{tr}(R_A)$ in Proposition 8, we find that

$$\begin{aligned} \mathrm{tr}(\mathrm{ad}_A^2) &= \lambda_V \mathrm{tr}(\mathrm{ad}_A) + \tfrac{1}{2}\mathrm{tr}(\mathrm{ad}_V \mathrm{ad}_V^*) \\ &= \lambda_Z \mathrm{tr}(\mathrm{ad}_A) + \tfrac{1}{4}\mathrm{tr}(J_Z^2). \end{aligned}$$

Using notation defined just above, we obtain Equations 1 and 3.

Now we compute $\mathrm{tr}(J_Z^2 \mathrm{ad}_A)$ in two ways:

$$\begin{aligned}
\mathrm{tr}(J_Z^2\mathrm{ad}_A) &= \sum_{V\in\mathcal{B}_\mathfrak{v}} \langle J_Z^2\mathrm{ad}_A V, V\rangle \\
&= \sum_{V\in\mathcal{B}_\mathfrak{v}} \lambda_V \langle J_Z^2 V, V\rangle \\
\mathrm{tr}(J_Z\mathrm{ad}_A J_Z) &= \sum_{V\in\mathcal{B}_\mathfrak{v}} \langle J_Z\mathrm{ad}_A J_Z V, V\rangle \\
&= \sum_{V\in\mathcal{B}_\mathfrak{v}} (\lambda_Z - \lambda_V)\langle J_Z^2 V, V\rangle
\end{aligned}$$

In the last step, we use part 3 of Lemma 5, the fact that, for an eigenvector V, $J_Z V$ has eigenvalue $\lambda_Z - \lambda_V$. Finally, we obtain

$$2\sum_{V\in\mathcal{B}_\mathfrak{v}} \lambda_V\langle J_Z^2 V, V\rangle = \sum_{V\in\mathcal{B}_\mathfrak{v}} \lambda_Z\langle J_Z^2 V, V\rangle = \lambda_Z \sum_{V\in\mathcal{B}_\mathfrak{v}} \langle J_Z^2 V, V\rangle,$$

so that

$$\mathrm{tr}(J_Z^2\mathrm{ad}_A) = \sum_{V\in\mathcal{B}_\mathfrak{v}} \langle J_Z^2\lambda_V V, V\rangle = \tfrac{1}{2}\lambda_Z \textstyle\sum_{V\in\mathcal{B}_\mathfrak{v}} \langle J_Z^2 V, V\rangle = \tfrac{1}{2}\lambda_Z \mathrm{tr}(J_Z^2) = 2\lambda_Z(T_2 - \lambda_Z T_1).$$

$\square$

Proposition 11. *Let $(\mathfrak{s}, Q)$ satisfy the Einstein condition. Then*

$$(n+2m)T_2 - (a_1+2b_1)T_1 = 0$$

Proof. We use Lemma 6, together with identities from Proposition 10, to obtain:

$$\begin{aligned}
\sum_{V\in\mathcal{B}_\mathfrak{v}} \mathrm{tr}(\mathrm{ad}_V\mathrm{ad}_V^*) &= 2\sum_{V\in\mathcal{B}_\mathfrak{v}} (T_2 - \lambda_V T_1) \\
&= 2(nT_2 - a_1T_1) \\
= -\sum_{Z\in\mathcal{B}_\mathfrak{z}} trace(J_Z^2) &= -4\sum_{Z\in\mathcal{B}_\mathfrak{z}} (T_2 - \lambda_Z T_1) \\
&= -4(mT_2 - b_1T_1)
\end{aligned}$$

In other words, $(2n+4m)T_2 = 2a_1T_1 + 4b_1T_1$.

$\square$

Proposition 12. *Let $(\mathfrak{s}, Q)$ satisfy the Einstein condition. Then one has:*

(1) $\sum_{V\in\mathcal{B}_\mathfrak{v}} \lambda_V\, tr(ad_V\, ad_V^*\, ad_A) = -2(b_2T_2 - b_3T_1)$
(2) $\sum_{V\in\mathcal{B}_\mathfrak{v}} \lambda_V\, tr(ad_V^*\, ad_V\, ad_A) = -2(T_2^2 - T_1T_3)$
(3) $\sum_{V\in\mathcal{B}_\mathfrak{v}} tr(ad_V\, ad_V^*\, ad_A^2) = -4(b_2T_2 - b_3T_1)$
(4) $\sum_{V\in\mathcal{B}_\mathfrak{v}} tr(ad_V^*\, ad_V\, ad_A^2) = 2(a_2T_2 - a_3T_1)$

Proof. To prove Equation 1, we use Equation 2 of Lemma 6, and Equation 2 of Proposition 10:

$$\begin{aligned}
\sum_{V\in\mathcal{B}_{\mathfrak{v}}} \lambda_V \mathrm{tr}(\mathrm{ad}_V \mathrm{ad}_V^* \mathrm{ad}_A) &= \sum_{V\in\mathcal{B}_{\mathfrak{v}}} \sum_{Z\in\mathcal{B}_{\mathfrak{z}}} \lambda_V \langle \mathrm{ad}_V \mathrm{ad}_V^* \mathrm{ad}_A Z, Z\rangle \\
&= \sum_{V\in\mathcal{B}_{\mathfrak{v}}} \sum_{Z\in\mathcal{B}_{\mathfrak{z}}} \lambda_V \lambda_Z \langle \mathrm{ad}_V \mathrm{ad}_V^* Z, Z\rangle \\
&= -\sum_{V\in\mathcal{B}_{\mathfrak{v}}} \sum_{Z\in\mathcal{B}_{\mathfrak{z}}} \lambda_V \lambda_Z \langle J_Z^2 V, V\rangle \qquad \text{by Lemma 6,} \\
&= -\sum_{Z\in\mathcal{B}_{\mathfrak{z}}} \sum_{V\in\mathcal{B}_{\mathfrak{v}}} \lambda_Z \langle J_Z^2 \mathrm{ad}_A V, V\rangle \\
&= -\sum_{Z\in\mathcal{B}_{\mathfrak{z}}} \lambda_Z \mathrm{tr}(J_Z^2 \mathrm{ad}_A) \\
&= -2\sum_{Z\in\mathcal{B}_{\mathfrak{z}}} \lambda_Z^2 (T_2 - \lambda_Z T_1) \qquad \text{by Proposition 10,} \\
&= -2\sum_{Z\in\mathcal{B}_{\mathfrak{z}}} (\lambda_Z^2 T_2 - \lambda_Z^3 T_1) \\
&= -2(b_2 T_2 - b_3 T_1)
\end{aligned}$$

We now prove Equation 2 using Equation 1, and Equation 3 of Proposition 10:

$$\begin{aligned}
\sum_{V\in\mathcal{B}_{\mathfrak{v}}} \lambda_V \mathrm{tr}(\mathrm{ad}_V^* \mathrm{ad}_V \mathrm{ad}_A) &= \sum_{V\in\mathcal{B}_{\mathfrak{v}}} \lambda_V \mathrm{tr}(\mathrm{ad}_V \mathrm{ad}_A \mathrm{ad}_V^*) \\
&= \sum_{V\in\mathcal{B}_{\mathfrak{v}}} \sum_{Z\in\mathcal{B}_{\mathfrak{z}}} \lambda_V \langle \mathrm{ad}_V \mathrm{ad}_A (\mathrm{ad}_V^* Z), Z\rangle \\
&= \sum_{V\in\mathcal{B}_{\mathfrak{v}}} \sum_{Z\in\mathcal{B}_{\mathfrak{z}}} \lambda_V (\lambda_Z - \lambda_V) \langle \mathrm{ad}_V \mathrm{ad}_V^* Z, Z\rangle \\
&= \sum_{V\in\mathcal{B}_{\mathfrak{v}}} \big(\lambda_V \mathrm{tr}(\mathrm{ad}_V \mathrm{ad}_V^* \mathrm{ad}_A) - \lambda_V^2 \mathrm{tr}(\mathrm{ad}_V \mathrm{ad}_V^*)\big) \\
&= \sum_{V\in\mathcal{B}_{\mathfrak{v}}} \lambda_V \mathrm{tr}(\mathrm{ad}_V \mathrm{ad}_V^* \mathrm{ad}_A) - 2\sum_{V\in\mathcal{B}_{\mathfrak{v}}} \lambda_V^2 (T_2 - \lambda_V T_1) \\
&= -2(b_2 T_2 - b_3 T_1) - 2(a_2 T_2 - a_3 T_1) \\
&= -2(a_2 + b_2) T_2 + 2(a_3 + b_3) T_1 \\
&= -2T_2^2 + 2T_1 T_3
\end{aligned}$$

For Equation 3:

$$\sum_{V\in\mathcal{B}_{\mathfrak{v}}} \mathrm{tr}(\mathrm{ad}_V \mathrm{ad}_V^* \mathrm{ad}_A^2) = \sum_{V\in\mathcal{B}_{\mathfrak{v}}} \sum_{Z\in\mathcal{B}_{\mathfrak{z}}} \langle \mathrm{ad}_V \mathrm{ad}_V^* \mathrm{ad}_A^2 Z, Z\rangle$$

$$
\begin{aligned}
&= \sum_{V\in\mathcal{B}_\mathfrak{v}} \sum_{Z\in\mathcal{B}_\mathfrak{z}} \lambda_Z^2 \langle \mathrm{ad}_V \mathrm{ad}_V^* Z, Z\rangle \\
&= -\sum_{V\in\mathcal{B}_\mathfrak{v}} \sum_{Z\in\mathcal{B}_\mathfrak{z}} \lambda_Z^2 \langle J_Z^2 V, V\rangle \qquad \text{by Lemma 6,} \\
&= -\sum_{Z\in\mathcal{B}_\mathfrak{z}} \lambda_Z^2 \mathrm{tr}(J_Z^2) \\
&= -4\sum_{Z\in\mathcal{B}_\mathfrak{z}} \lambda_Z^2 (T_2 - \lambda_Z T_1) \qquad \text{by Proposition 10,} \\
&= -4(b_2 T_2 - b_3 T_1)
\end{aligned}
$$

The only equality left to prove is the one in Part 4. We make use of (3) of Lemma 5, Lemma 6 and Proposition 10.

$$
\begin{aligned}
\sum_{V\in\mathcal{B}_\mathfrak{v}} \mathrm{tr}(\mathrm{ad}_V^* \mathrm{ad}_V \mathrm{ad}_A^2) &= \sum_{V\in\mathcal{B}_\mathfrak{v}} \mathrm{tr}(\mathrm{ad}_V \mathrm{ad}_A^2 \mathrm{ad}_V^*) \\
&= \sum_{V\in\mathcal{B}_\mathfrak{v}} \sum_{Z\in\mathcal{B}_\mathfrak{z}} \langle \mathrm{ad}_V \mathrm{ad}_A^2 (\mathrm{ad}_V^* Z), Z\rangle \\
&= \sum_{V\in\mathcal{B}_\mathfrak{v}} \sum_{Z\in\mathcal{B}_\mathfrak{z}} (\lambda_Z - \lambda_V)^2 \langle \mathrm{ad}_V \mathrm{ad}_V^* Z, Z\rangle \\
&= \sum_{V\in\mathcal{B}_\mathfrak{v}} \sum_{Z\in\mathcal{B}_\mathfrak{z}} \big(-\lambda_Z^2 \langle J_Z^2 V, V\rangle + 2\lambda_Z \lambda_V \langle J_Z^2 V, V\rangle + \lambda_V^2 \langle \mathrm{ad}_V \mathrm{ad}_V^* Z, Z\rangle \big) \\
&= -\sum_{Z\in\mathcal{B}_\mathfrak{z}} \lambda_Z^2 \mathrm{tr}(J_Z^2) + 2\sum_{Z\in\mathcal{B}_\mathfrak{z}} \lambda_Z \mathrm{tr}(J_Z^2 \mathrm{ad}_A) + \sum_{V\in\mathcal{B}_\mathfrak{v}} \lambda_V^2 \mathrm{tr}(\mathrm{ad}_V \mathrm{ad}_V^*) \\
&= -4\sum_{Z\in\mathcal{B}_\mathfrak{z}} \lambda_Z^2 (T_2 - \lambda_Z T_1) + 4\sum_{Z\in\mathcal{B}_\mathfrak{z}} \lambda_Z^2 (T_2 - \lambda_Z T_1) + 2\sum_{V\in\mathcal{B}_\mathfrak{v}} \lambda_V^2 (T_2 - \lambda_V T_1) \\
&= -4(b_2 T_2 - b_3 T_1) + 4(b_2 T_2 - b_3 T_1) + 2(a_2 T_2 - a_3 T_1) \\
&= 2a_2 T_2 - 2a_3 T_1
\end{aligned}
$$

□

We will derive some inequalities based on the following observation:

Lemma 13. *Let $\rho : \mathbb{R}^N \to \mathbb{R}^N$ be a symmetric linear map. Then*

$$(tr(\rho))^2 \leq N\, tr(\rho^2),$$

with equality if and only if ρ is a multiple of the identity.

Proof. Equip the vector space $End(\mathbb{R}^N)$ with the positive definite inner product $\langle S, T\rangle = \mathrm{tr}(ST^*)$. Applying the Cauchy-Schwartz Inequality to ρ and the identity map Id, we see that

$$(\mathrm{tr}(\rho))^2 \leq \mathrm{tr}(\rho^2)\mathrm{tr}(Id) = \mathrm{tr}(\rho^2)N,$$

with inequality if and only if ρ is a multiple of the identity. □

Proposition 14. *Let* $(\mathfrak{s}, Q)$ *be Einstein. Then*

(1) $na_2 - a_1^2 \geq 0$, *with equality if and only if* $ad_A|_{\mathfrak{v}}$ *is a multiple of the identity.*
(2) $mb_2 - b_1^2 \geq 0$, *with equality if and only if* $ad_A|_{\mathfrak{z}}$ *is a multiple of the identity.*

Proof. These inequalities follow from applying Lemma 13 to the maps $\mathrm{ad}_A|_{\mathfrak{v}}$ and $\mathrm{ad}_A|_{\mathfrak{z}}$:

$$na_2 - a_1^2 = n\mathrm{tr}(\mathrm{ad}_A|_{\mathfrak{v}}^2) - (\mathrm{tr}(\mathrm{ad}_A|_{\mathfrak{v}}))^2 \geq 0$$
$$mb_2 - b_1^2 = m\mathrm{tr}(\mathrm{ad}_A|_{\mathfrak{z}}^2) - (\mathrm{tr}(\mathrm{ad}_A|_{\mathfrak{z}}))^2 \geq 0$$

□

6. 2-stein Constraints

In this section, we derive additional trace equalities and inequalities that hold when $(\mathfrak{s}, Q)$ is 2-stein. We continue to employ the hypotheses and notation from above, with the additional assumption that $(\mathfrak{s}, Q)$ satisfies the first two Ledger conditions. In particular, we make extensive use of the formulas from Proposition 9.

Proposition 15. *Let* V *and* Z *be unit* ad_A*-eigenvectors in* $\mathfrak{v}$ *and* $\mathfrak{z}$ *respectively. Then*

(1) $tr(J_Z^4) = 16(T_4 - 2\lambda_Z^2 T_2 + \lambda_Z^3 T_1)$
(2) $\sum_{V\in\mathcal{B}_{\mathfrak{v}}} tr((ad_V ad_V^*)^2) = \frac{8}{5}(nT_4 - 3T_1T_3 + 2T_2^2 + b_3T_1)$
(3) $tr(J_Z^2 ad_A^2) = -4T_4 + 4\lambda_Z T_3 + 2\lambda_Z^2 T_2 - 2\lambda_Z^3 T_1$

Proof. To prove Equation 1, we equate $\mathrm{tr}(R_Z^2)$ and $\mathrm{tr}(R_A^2)$ to find that

$$\mathrm{tr}(\mathrm{ad}_A^4) = \tfrac{1}{16}\mathrm{tr}(J_Z^4) + \tfrac{1}{2}\lambda_Z \mathrm{tr}(J_Z^2 \mathrm{ad}_A) + \lambda_Z^2 \mathrm{tr}(\mathrm{ad}_A^2),$$

and hence

$$\tfrac{1}{16}\mathrm{tr}(J_Z^4) = T_4 - \tfrac{1}{2}\lambda_Z \mathrm{tr}(J_Z^2 \mathrm{ad}_A) - \lambda_Z^2 T_2.$$

By Equation 2 of Proposition 10 this is equal to

$$\tfrac{1}{16}\mathrm{tr}(J_Z^4) = T_4 - \lambda_Z^2(T_2 - \lambda_Z T_1) - \lambda_Z^2 T_2 = T_4 - 2\lambda_Z^2 T_2 + \lambda_Z^3 T_1.$$

For Equation 2, we equate $\mathrm{tr}(R_A^2)$ and $\mathrm{tr}(R_V^2)$:

$$\begin{aligned}\mathrm{tr}(\mathrm{ad}_A^4) = {} & \tfrac{5}{8}\mathrm{tr}((\mathrm{ad}_V^* \mathrm{ad}_V)^2) + \tfrac{3}{2}\lambda_V \mathrm{tr}(\mathrm{ad}_V^* \mathrm{ad}_V \mathrm{ad}_A) \\ & - \tfrac{1}{2}\lambda_V \mathrm{tr}(\mathrm{ad}_V \mathrm{ad}_V^* \mathrm{ad}_A) + \lambda_V^2 \mathrm{tr}(\mathrm{ad}_A^2),\end{aligned}$$

and then use Proposition 12 to sum over an orthonormal basis of $\mathfrak{v}$:

$$\begin{aligned}\tfrac{5}{8}\textstyle\sum_{V\in\mathcal{B}_{\mathfrak{v}}} \mathrm{tr}((\mathrm{ad}_V^* \mathrm{ad}_V)^2) &= nT_4 - \tfrac{3}{2}\textstyle\sum_{V\in\mathcal{B}_{\mathfrak{v}}} \lambda_V \mathrm{tr}(\mathrm{ad}_V^* \mathrm{ad}_V \mathrm{ad}_A) \\ &\quad + \tfrac{1}{2}\textstyle\sum_{V\in\mathcal{B}_{\mathfrak{v}}} \lambda_V \mathrm{tr}(\mathrm{ad}_V \mathrm{ad}_V^* \mathrm{ad}_A) - \textstyle\sum_{V\in\mathcal{B}_{\mathfrak{v}}} \lambda_V^2 \mathrm{tr}(\mathrm{ad}_A^2) \\ &= nT_4 + 3(T_2^2 - T_1T_3) - (b_2T_2 - b_3T_1) - a_2T_2, \text{ by Proposition 12} \\ &= nT_4 - 3T_1T_3 + T_2(3T_2 - b_2 - a_2) + b_3T_1 \\ &= nT_4 - 3T_1T_3 + 2T_2^2 + b_3T_1.\end{aligned}$$

To prove the last equation, we equate $\mathrm{tr}(R^2_{A+Z})$ and $||A+Z||^4\mathrm{tr}(R^2_A)$, noting the fact that R^2_u is quartic in $||u||$:

$$4\ \mathrm{tr}(\mathrm{ad}_A^4) = \tfrac{1}{16}\mathrm{tr}(J_Z^4) - \tfrac{1}{2}\mathrm{tr}(J_Z^2\mathrm{ad}_A^2) + \tfrac{3}{2}\lambda_Z\mathrm{tr}(J_Z^2\mathrm{ad}_A) - \tfrac{1}{4}\lambda_Z^2\mathrm{tr}(J_Z^2) + \mathrm{tr}(\mathrm{ad}_A^4) + 2\lambda_Z\mathrm{tr}(\mathrm{ad}_A^3) + \lambda_Z^2\mathrm{tr}(\mathrm{ad}_A^2).$$

Now we solve for $\mathrm{tr}(J_Z^2\mathrm{ad}_A^2)$, applying Proposition 10 and Equation 1.

$$\begin{aligned}\tfrac{1}{2}\mathrm{tr}(J_Z^2\mathrm{ad}_A^2) =& -4T_4 + \tfrac{1}{16}\mathrm{tr}(J_Z^4) + \tfrac{3}{2}\lambda_Z\mathrm{tr}(J_Z^2\mathrm{ad}_A)\\ & - \tfrac{1}{4}\lambda_Z^2\mathrm{tr}(J_Z^2) + T_4 + 2\lambda_Z T_3 + \lambda_Z^2 T_2\\ =& -3T_4 + (T_4 - 2\lambda_Z^2T_2 + \lambda_Z^3T_1) + 3\lambda_Z^2(T_2 - \lambda_Z T_1)\\ & - \lambda_Z^2(T_2-\lambda_Z T_1) + 2\lambda_Z T_3 + \lambda_Z^2 T_2\\ =& -2T_4 + 2\lambda_Z T_3 + \lambda_Z^2T_2 - \lambda_Z^3 T_1.\end{aligned}$$

□

Proposition 16. *Suppose* $(\mathfrak{s}, Q)$ *is 2-stein. Then:*

(1) $nmT_4 + nb_3T_1 - 2nb_2T_2 - mT_2^2 + 2b_1T_1T_2 - b_2T_1^2 \geq 0$,
with equality if and only if, for all Z *in an orthonormal basis of* ad_A*-eigenvectors for* $\mathfrak{z}$*,* J_Z^2 *is a multiple of* $Id_{\mathfrak{v}}$*.*

(2) $2mnT_4 - 2m(3a_3 + 2b_3)T_1 + (4m-5n)T_2^2 + 10a_1T_1T_2 - 5a_2T_1^2 \geq 0$,
with equality if and only if, for all V *in an orthonormal basis of* ad_A*-eigenvectors for* $\mathfrak{v}$*,* $ad_V ad_V^*$ *is a multiple of* $Id_{\mathfrak{z}}$*.*

Proof. First we prove Inequality 1. For all Z in $\mathcal{B}_{\mathfrak{z}}$, we apply Lemma 13 with $\rho = J_Z^2$ and sum over $\mathcal{B}_{\mathfrak{z}}$ to obtain

$$\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}(\mathrm{tr}(J_Z^2))^2 \leq n\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\mathrm{tr}(J_Z^4).$$

Substituting Equation 1 of Proposition 10 on the left and Equation 1 of Proposition 15 on the right gives

$$\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}16(T_2-\lambda_Z T_1)^2 \leq n\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}16(T_4 - 2\lambda_Z^2T_2 + \lambda_Z^3T_1)$$

$$\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}(T_2^2 - 2\lambda_Z T_1T_2 + \lambda_Z^2T_1^2) \leq n(mT_4 - 2b_2T_2 + b_3T_1)$$

$$mT_2^2 - 2b_1T_1T_2 + b_2T_1^2 \leq n(mT_4 - 2b_2T_2 + b_3T_1)$$

This can be rewritten in the form of Inequality 1. Equality holds if and only if equality holds in each of the m summands. By Lemma 13, this happens if and only if J_Z^2 is a scalar multiple of the identity for all Z in $\mathcal{B}_{\mathfrak{z}}$.

To prove Inequality 2, we apply Lemma 13 to the maps $\mathrm{ad}_V \mathrm{ad}_V^* : \mathfrak{z} \to \mathfrak{z}$ for each V in $\mathcal{B}_{\mathfrak{v}}$. We sum over $\mathcal{B}_{\mathfrak{v}}$ to get

$$\sum_{V \in \mathcal{B}_{\mathfrak{v}}} (\mathrm{tr}(\mathrm{ad}_V \mathrm{ad}_V^*))^2 \leq m \sum_{V \in \mathcal{B}_{\mathfrak{v}}} \mathrm{tr}((\mathrm{ad}_V \mathrm{ad}_V^*)^2)$$

Substituting Equation 3 of Proposition 10 on the left and Equation 2 of Proposition 15 on the right gives

$$\begin{aligned}
\sum_{V \in \mathcal{B}_{\mathfrak{v}}} 4(T_2 - \lambda_V T_1)^2 &\leq \tfrac{8}{5} m(nT_4 - 3T_1T_3 + 2T_2^2 + b_3 T_1) \\
5 \sum_{V \in \mathcal{B}_{\mathfrak{v}}} (T_2^2 - 2\lambda_V T_1 T_2 + \lambda_V^2 T_1^2) &\leq 2m(nT_4 - 3T_1T_3 + 2T_2^2 + b_3 T_1) \\
5(nT_2^2 - 2a_1 T_1 T_2 + a_2 T_1^2) &\leq 2m(nT_4 - 3T_1T_3 + 2T_2^2 + b_3 T_1),
\end{aligned}$$

proving the desired inequality. As before, equality holds if and only if equality holds in each summand. This occurs only when $\mathrm{ad}_V \mathrm{ad}_V^*$ is a scalar multiple of the identity for all V in $\mathcal{B}_{\mathfrak{v}}$. □

Proposition 17. *If $(\mathfrak{s}, Q)$ is 2-stein and equality holds in both inequalities of Proposition 16, then $(\mathfrak{s}, Q)$ is a Damek-Ricci space or real hyperbolic space.*

Proof. Suppose that, for each $Z \in \mathcal{B}_{\mathfrak{z}}$, $V \in \mathcal{B}_{\mathfrak{v}}$, there are non-negative real numbers b_Z and c_V such that $J_Z^2 = -b_Z I_{\mathfrak{z}}$ and $\mathrm{ad}_V \mathrm{ad}_V^* = c_V I_{\mathfrak{v}}$. (Recall that J_Z is skew-symmetric, while $\mathrm{ad}_V \mathrm{ad}_V^*$ is symmetric.)

We use Equation 2 of Lemma 6 and the usual manipulations:

$$\begin{aligned}
nb_Z &= \mathrm{tr}(b_Z I_{\mathfrak{v}}) \\
&= -\mathrm{tr}(J_Z^2) \\
&= -\sum_{V \in \mathcal{B}_{\mathfrak{v}}} \langle J_Z^2 V, V \rangle \\
&= \sum_{V \in \mathcal{B}_{\mathfrak{v}}} \langle \mathrm{ad}_V \mathrm{ad}_V^* Z, Z \rangle \\
&= \sum_{V \in \mathcal{B}_{\mathfrak{v}}} c_V
\end{aligned}$$

This shows that b_Z is independent of Z. Likewise, we see that c_V is independent of V:

$$\begin{aligned} mc_V &= \mathrm{tr}(c_V I_{\mathfrak{z}}) \\ &= \mathrm{tr}(\mathrm{ad}_V \mathrm{ad}_V^*) \\ &= \sum_{Z \in \mathcal{B}_{\mathfrak{z}}} \langle \mathrm{ad}_V \mathrm{ad}_V^* Z, Z \rangle \\ &= -\sum_{Z \in \mathcal{B}_{\mathfrak{z}}} \langle J_Z^2 V, V \rangle \\ &= \sum_{Z \in \mathcal{B}_{\mathfrak{z}}} b_Z. \end{aligned}$$

If $\mathfrak{v} \neq \{0\}$, then we can set $c_V = c^2$ with $c > 0$. Substituting into the first equation, we find that $nb_Z = nc^2$, so $b_Z = c^2$ for all Z in $\mathcal{B}_{\mathfrak{z}}$. Thus $\mathrm{tr}(J_Z^2) = -nc^2$ for all Z in $\mathcal{B}_{\mathfrak{z}}$ and $\mathrm{tr}(\mathrm{ad}_V \mathrm{ad}_V^*) = mc^2$ for all V in $\mathcal{B}_{\mathfrak{v}}$. Applying (1) from Proposition 10, we see that for any Z in $\mathcal{B}_{\mathfrak{z}}$,

$$-nc^2 = \mathrm{tr}(J_Z^2) = 4(T_2 - \lambda_Z T_1). \tag{2}$$

As T_1, T_2 are constants and $T_1 \neq 0$ (since ad_A is positive definite on $\mathfrak{n}$), we conclude that λ_Z is constant for all Z and ad_A restricted to $\mathfrak{z}$ is a scalar multiple λ of the identity, $\lambda > 0$. Likewise, using (3) from Proposition 10, we find that for any V in $\mathcal{B}_{\mathfrak{v}}$,

$$mc^2 = \mathrm{tr}(\mathrm{ad}_V \mathrm{ad}_V^*) = 2(T_2 - \lambda_V T_1). \tag{3}$$

Hence λ_V is also independent of V. By Lemma 5, ad_A restricted to $\mathfrak{v}$ is $\frac{1}{2}\lambda$. Using equation 2 or 3, we see that $\lambda^2 = c^2$, and hence $\mathrm{ad}_A|_{\mathfrak{v}} = \frac{1}{2}c$, $\mathrm{ad}_A|_{\mathfrak{z}} = c$. Thus $(\mathfrak{s}, Q)$ is a Damek-Ricci space.

If $\mathfrak{v} = \{0\}$, then $n = 0$ and $\mathfrak{n}$ is abelian. Equation 2 becomes $\mathrm{tr}(J_Z^2) = 4(T_2 - \lambda_Z T_1) = 0$, and we see that λ_Z is independent of Z. Thus ad_A is constant, so we have real hyperbolic space. □

Proposition 18. *Let $(\mathfrak{s}, Q)$ be 2-stein. Then*

(1) $2mT_4 + T_3(a_1 - b_1) - T_2^2 = 0$
(2) $T_4(a_1 + 4b_1) - (a_2 + 4b_2)(a_3 + b_3) = 0$
(3) $-2nT_4 + T_3(4a_1 + 2b_1) + T_2(4a_2 - 2b_2) + T_1(-6a_3) = 0$

Proof. To prove the first equation, we sum $\mathrm{tr}(J_Z^2 \mathrm{ad}_A^2)$ over an orthonormal basis of $\mathfrak{z}$. On the one hand, by Equation 3 of Proposition 15,

$$\begin{aligned} \sum_{Z \in \mathcal{B}_{\mathfrak{z}}} \mathrm{tr}(J_Z^2 \mathrm{ad}_A^2) &= -4mT_4 + 4 \sum_{Z \in \mathcal{B}_{\mathfrak{z}}} \lambda_Z T_3 + 2 \sum_{Z \in \mathcal{B}_{\mathfrak{z}}} \lambda_Z^2 T_2 - 2 \sum_{Z \in \mathcal{B}_{\mathfrak{z}}} \lambda_Z^3 T_1 \\ &= -4mT_4 + 4b_1 T_3 + 2b_2 T_2 - 2b_3 T_1. \end{aligned} \tag{4}$$

On the other hand,

$$\begin{aligned}
\sum_{Z\in\mathcal{B}_{\mathfrak{z}}} \operatorname{tr}(J_Z^2\operatorname{ad}_A^2) &= \sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\sum_{V\in\mathcal{B}_{\mathfrak{v}}} \langle J_Z^2\operatorname{ad}_A^2 V, V\rangle \\
&= \sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\sum_{V\in\mathcal{B}_{\mathfrak{v}}} \lambda_V^2\langle J_Z^2 V, V\rangle \\
&= -\sum_{V\in\mathcal{B}_{\mathfrak{v}}}\sum_{Z\in\mathcal{B}_{\mathfrak{z}}} \lambda_V^2\langle \operatorname{ad}_V\operatorname{ad}_V^* Z, Z\rangle \text{ by Lemma 6,} \\
&= -\sum_{V\in\mathcal{B}_{\mathfrak{v}}} \lambda_V^2\operatorname{tr}(\operatorname{ad}_V\operatorname{ad}_V^*) \\
&= -2\sum_{V\in\mathcal{B}_{\mathfrak{v}}} \lambda_V^2(T_2-\lambda_V T_1) \qquad \text{by Proposition 10,} \\
&= -2\sum_{V\in\mathcal{B}_{\mathfrak{v}}} (\lambda_V^2 T_2-\lambda_V^3 T_1) \\
&= -2(a_2T_2-a_3T_1). \tag{5}
\end{aligned}$$

Equating (4) and (5) yields the equation we wanted to prove.

For the second equation:

$$\begin{aligned}
\sum_{Z\in\mathcal{B}_{\mathfrak{z}}} \operatorname{tr}(J_Z^2\operatorname{ad}_A^3) &= \sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\sum_{V\in\mathcal{B}_{\mathfrak{v}}} \langle J_Z^2\operatorname{ad}_A^3 V, V\rangle \\
&= \sum_{V\in\mathcal{B}_{\mathfrak{v}}}\sum_{Z\in\mathcal{B}_{\mathfrak{z}}} \lambda_V^3\langle J_Z^2 V, V\rangle
\end{aligned}$$

$$\begin{aligned}
= \sum_{Z\in\mathcal{B}_{\mathfrak{z}}} \operatorname{tr}(J_Z\operatorname{ad}_A^3 J_Z) \qquad & \\
&= \sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\sum_{V\in\mathcal{B}_{\mathfrak{v}}} \langle J_Z\operatorname{ad}_A^3 J_Z V, V\rangle \\
&= \sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\sum_{V\in\mathcal{B}_{\mathfrak{v}}} (\lambda_Z-\lambda_V)^3\langle J_Z^2 V, V\rangle \\
&= \sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\sum_{V\in\mathcal{B}_{\mathfrak{v}}} (\lambda_Z^3-3\lambda_Z^2\lambda_V+3\lambda_Z\lambda_V^2-\lambda_V^3)\langle J_Z^2 V, V\rangle.
\end{aligned}$$

Hence

$$2\sum_{V\in\mathcal{B}_{\mathfrak{v}}}\sum_{Z\in\mathcal{B}_{\mathfrak{z}}} \lambda_V^3\langle J_Z^2 V, V\rangle = \sum_{V\in\mathcal{B}_{\mathfrak{v}}}\sum_{Z\in\mathcal{B}_{\mathfrak{z}}} (\lambda_Z^3-3\lambda_Z^2\lambda_V+3\lambda_Z\lambda_V^2)\langle J_Z^2 V, V\rangle. \tag{6}$$

By Equation 2 of Lemma 6 and Equation 3 of Proposition 10, we can write the left hand side of (6) as

$$
\begin{aligned}
2\sum_{V\in\mathcal{B}_{\mathfrak{v}}}\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\lambda_V^3\langle J_Z^2V,V\rangle =& -2\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\sum_{V\in\mathcal{B}_{\mathfrak{v}}}\lambda_V^3\langle \mathrm{ad}_V\mathrm{ad}_V^*Z,Z\rangle\\
=& -2\sum_{V\in\mathcal{B}_{\mathfrak{v}}}\lambda_V^3\mathrm{tr}(\mathrm{ad}_V\mathrm{ad}_V^*)\\
=& -4\sum_{V\in\mathcal{B}_{\mathfrak{v}}}\lambda_V^3(T_2-\lambda_VT_1)\\
=& -4a_3T_2+4a_4T_1.
\end{aligned}
$$

Now we compute, term by term, the right hand side of (6). Applying Property 1 of Proposition 10, we see that:

$$
\begin{aligned}
\sum_{V\in\mathcal{B}_{\mathfrak{v}}}\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\lambda_Z^3\langle J_Z^2V,V\rangle =& \sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\lambda_Z^3\mathrm{tr}(J_Z^2)\\
=&4\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\lambda_Z^3(T_2-\lambda_ZT_1)\\
=&4b_3T_2-4b_4T_1.
\end{aligned}
$$

Next we use Equation 2 of Proposition 10 to write:

$$
\begin{aligned}
-3\sum_{V\in\mathcal{B}_{\mathfrak{v}}}\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\lambda_Z^2\lambda_V\langle J_Z^2V,V\rangle =& -3\sum_{V\in\mathcal{B}_{\mathfrak{v}}}\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\lambda_Z^2\langle J_Z^2\mathrm{ad}_AV,V\rangle\\
=& -3\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\lambda_Z^2\mathrm{tr}(J_Z^2\mathrm{ad}_A)\\
=& -6\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\lambda_Z^3(T_2-\lambda_ZT_1)\\
=& -6(b_3T_2-b_4T_1),
\end{aligned}
$$

and Equation 3 of Proposition 15 gives:

$$
\begin{aligned}
3\sum_{V\in\mathcal{B}_{\mathfrak{v}}}\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\lambda_Z\lambda_V^2\langle J_Z^2V,V\rangle =& \,3\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\lambda_Z\mathrm{tr}(J_Z^2\mathrm{ad}_A^2)\\
=& \,3\sum_{Z\in\mathcal{B}_{\mathfrak{z}}}\lambda_Z(-4T_4+4\lambda_ZT_3+2\lambda_Z^2T_2-2\lambda_Z^3T_1)\\
=& \,3(-4b_1T_4+4b_2T_3+2b_3T_2-2b_4T_1).
\end{aligned}
$$

Now, for the right hand side of (6), we have a total of

$$(4b_3T_2 - 4b_4T_1) - 6(b_3T_2 - b_4T_1) + 3(-4b_1T_4 + 4b_2T_3 + 2b_3T_2 - 2b_4T_1)$$
$$= -12b_1T_4 + 12b_2T_3 + 4b_3T_2 - 4b_4T_1.$$

Equating the two sides of (6), we get

$$-4a_3T_2 + 4a_4T_1 = -12b_1T_4 + 12b_2T_3 + 4b_3T_2 - 4b_4T_1$$

or

$$3b_1T_4 - 3b_2T_3 - T_2T_3 + T_1T_4 = 0.$$

In other words,

$$T_4(a_1 + 4b_1) - (a_2 + 4b_2)(a_3 + b_3) = 0.$$

Now we prove Equation 3. Let A be our chosen unit vector in $\mathfrak{a}$ and let V be a unit ad_A-eigenvector in $\mathcal{B}_\mathfrak{v}$. The second Ledger condition on $\mathrm{tr}(R_U^2)$ is quadratic in $||U||^2$, so if we apply it to A and $A+V$, we get $\mathrm{tr}(R_{A+V}^2) = ||A+V||^4\mathrm{tr}(R_A^2) = 4\ \mathrm{tr}(\mathrm{ad}_A^4)$. Summing this over V in $\mathcal{B}_\mathfrak{v}$, and using Proposition 9, we obtain:

$$\begin{aligned}4nT_4 = n\mathrm{tr}(\mathrm{ad}_A^4) &+ 2\sum_{V\in\mathcal{B}_\mathfrak{v}} \lambda_V \mathrm{tr}(\mathrm{ad}_A^3) + \sum_{V\in\mathcal{B}_\mathfrak{v}} \lambda_V^2 \mathrm{tr}(\mathrm{ad}_A^2) + \tfrac{5}{8}\textstyle\sum_{V\in\mathcal{B}_\mathfrak{v}} \mathrm{tr}((\mathrm{ad}_V^*\mathrm{ad}_V)^2)\\ &+ \tfrac{1}{2}\textstyle\sum_{V\in\mathcal{B}_\mathfrak{v}} \lambda_V^2 \mathrm{tr}(\mathrm{ad}_V^*\mathrm{ad}_V) + \tfrac{5}{2}\sum_{V\in\mathcal{B}_\mathfrak{v}} \lambda_V \mathrm{tr}(\mathrm{ad}_V^*\mathrm{ad}_V\mathrm{ad}_A) + \tfrac{5}{2}\sum_{V\in\mathcal{B}_\mathfrak{v}} \mathrm{tr}(\mathrm{ad}_V^*\mathrm{ad}_V\mathrm{ad}_A^2)\\ &- \tfrac{3}{2}\textstyle\sum_{V\in\mathcal{B}_\mathfrak{v}} \lambda_V \mathrm{tr}(\mathrm{ad}_V\mathrm{ad}_V^*\mathrm{ad}_A) + \tfrac{1}{2}\sum_{V\in\mathcal{B}_\mathfrak{v}} \mathrm{tr}(\mathrm{ad}_V\mathrm{ad}_V^*\mathrm{ad}_A^2).\end{aligned}$$

We may substitute for all the traces using Propositions 10, 12 and 15 to get

$$\begin{aligned}4nT_4 = nT_4 &+ 2a_1T_3 + a_2T_2 + (nT_4 - 3T_1T_3 + 2T_2^2 + b_3T_1) + (a_2T_2 - a_3T_1)\\ &+ (-5T_2^2 + 5T_1T_3) + (5a_2T_2 - 5a_3T_1) + 3(b_2T_2 - b_3T_1) - 2(b_2T_2 - b_3T_1)\end{aligned}$$

Simplification yields the desired equality. $\square$

7. Solutions to the Constraints

In this section, we focus on the ad_A-trace constants T_r, a_r and b_r defined at the beginning of Section 5. We consider the possible values that these constants may assume for a rank one three-step solvmanifold of Iwasawa type which satisfies the first two Ledger conditions. Recall that ad_A is positive definite on $\mathfrak{n}$, and hence the trace constants are all positive (with the exception that $a_r = 0$ in the degenerate case). We will show that the only possible values are those obtained for Damek-Ricci spaces and real hyperbolic spaces (viewed as "degenerate three-step solvmanifolds").

We summarize conditions derived in Sections 5 and 6, which give us a set of constraints on the eigenvalues of ad_A:

$$\begin{aligned}
&\text{(Prop 11)} && (n+2m)T_2 - (a_1+2b_1)T_1 = 0 \\
&\text{(Prop 18)} && 2mT_4 + T_3(a_1-b_1) - T_2^2 = 0 \\
& && T_4(a_1+4b_1) - (a_2+4b_2)(a_3+b_3) = 0 \\
& && -2nT_4 + (4a_1+2b_1)T_3 + (4a_2-2b_2)T_2 - 6a_3T_1 = 0 \\
&\text{(Prop 14)} && na_2 - a_1^2 \geq 0 \\
& && mb_2 - b_1^2 \geq 0 \\
&\text{(Prop 16)} && nmT_4 + nT_1b_3 - 2nT_2b_2 - mT_2^2 + 2T_1T_2b_1 - T_1^2b_2 \geq 0 \\
& && 2mnT_4 - 2mT_1(3a_3+2b_3) + T_2^2(4m-5n) + 10T_1T_2a_1 - 5T_1^2a_2 \geq 0
\end{aligned}$$

The manifold $(\mathfrak{s}, Q)$ remains 2-stein if we rescale the metric Q by a positive scalar factor c. This changes T_r, a_r and b_r by a factor of c^r for $r = 1, 2, 3, 4$. Note that, as we expect, all of our constraints are "homogeneous" with respect to this scaling. Hence we may normalize Q by setting $b_1 = m$. Making this substitution and eliminating some variables using the identities $T_2 = a_2 + b_2$, $a_3 = T_3 - b_3$, we get the constraints:

$$E_1 = (n+2m)(a_2+b_2) - (a_1+2m)(a_1+m) = 0 \tag{7}$$

$$E_2 = 2mT_4 + (a_1-m)T_3 - (a_2+b_2)^2 = 0 \tag{8}$$

$$E_3 = (a_1+4m)T_4 - (a_2+4b_2)T_3 = 0 \tag{9}$$

$$E_4 = -2nT_4 - (2a_1+4m)T_3 + 6(a_1+m)b_3 + (4a_2-2b_2)(a_2+b_2) = 0 \tag{10}$$

$$I_1 = na_2 - a_1^2 \geq 0 \tag{11}$$

$$I_2 = b_2 - m \geq 0 \tag{12}$$

$$\begin{aligned} I_3 = nmT_4 + n(a_1+m)b_3 - (2n+m)b_2^2 - ((2m+2n)a_2 - m^2 + a_1^2)b_2 \\ - ma_2^2 + 2m(a_1+m)a_2 \geq 0 \end{aligned} \tag{13}$$

$$\begin{aligned} I_4 = 2mnT_4 - 6m(a_1+m)T_3 + 2m(a_1+m)b_3 + (4m-5n)(a_2+b_2)^2 \\ + 5(a_1+m)(a_1a_2 - ma_2 + 2a_1b_2) \geq 0 \end{aligned} \tag{14}$$

Theorem 19. *The only simultaneous solutions to the constraints (7) to (14) are* $\{a_r = n/2^r,\ b_r = m\}$ *or* $\{a_r = n,\ b_r = m\}$*, where all inequalities are equalities.*

Proof. We have made extensive use of a computer algebra system (Maple) to perform many of the onerous calculations involved in this proof. For example, equalities (7) to (10) can be used to solve for the remaining variables a_2, b_3, T_3, T_4 in terms of (a_1, b_2). Since the denominators of the solutions reappear in many expressions, we denote:

$$\gamma(a_1, b_2) = 12m^2b_2 - 4m^3 + 6b_2mn - 4m^2n + 12a_1m^2 + 3a_1mn + 4a_1^2m + a_1^2n. \tag{15}$$

Note that $\gamma(a_1, b_2)$ is positive when $b_2 \geq m$.

The solutions are:

(16)

$$\begin{aligned}
a_2 &= (a_1^2 + 3ma_1 + 2m^2 - (2m+n)b_2)/(n+2m) \\
b_3 &= (a_1 + 2m)\big((6n^2m + 24nm^2 + 24m^3)b_2^2 + \big((2n^2 + 4nm)a_1^2 + (6n^2m + 12nm^2)a_1 - 2n^2m^2 \\
&\quad - 16nm^3 - 24m^4\big)b_2 - 2ma_1^4 + (nm - 10m^2)a_1^3 + (9nm^2 - 8m^3)a_1^2 \\
&\quad + (20nm^3 + 16m^4)a_1 + 12nm^4 + 16m^5\big)\big/(n+2m)^2\gamma(a_1, b_2) \\
T_3 &= (a_1 + 4m)(a_1 + 2m)^2(a_1 + m)^2\big/(n+2m)\gamma(a_1, b_2) \\
T_4 &= (a_1 + 2m)^2(a_1 + m)^2((3n + 6m)b_2 + a_1^2 + 3ma_1 + 2m^2)\big/(n+2m)^2\gamma(a_1, b_2)
\end{aligned}$$

Now we see that any solutions to the constraints will depend only on (a_1, b_2). In particular, the inequalities $I_1 \geq 0$, $I_2 \geq 0$ constrain the solutions (a_1, b_2) to a connected region $\mathcal{R}$ bounded by a line $\mathcal{L}$ and a parabola $\mathcal{P}$:

$$\mathcal{R} : \begin{cases} b_2 \geq m, \\ n(n+2m)b_2 \leq -2ma_1^2 + 3nma_1 + 2nm^2, \\ n/2 \leq a_1 \leq n. \end{cases} \tag{17}$$

Now we consider Inequalities 13 and 14 on the region $\mathcal{R}$. In the sequence of lemmas below, we show that the only simultaneous solutions to $I_3 \geq 0$, $I_4 \geq 0$ in the region $\mathcal{R}$ are at the corner points $(n/2, m)$, (n, m) . We show that the equations $I_3 = 0$ and $I_4 = 0$ define two continuous curves which connect the corner points and only intersect at those points. The inequality $I_3 > 0$ holds below the curve $I_3 = 0$, while the inequality $I_4 > 0$ holds above the curve $I_4 = 0$. We then show that the curve $I_4 = 0$ lies above the curve $I_3 = 0$, so that the only points in $\mathcal{R}$ at which all inequalities hold are the two corners.

Lemma 20. *On the line segment $\mathcal{L} = \{(a_1, m) : n/2 < a_1 < n\}$, $I_3 > 0$ and $I_4 < 0$. On the parabolic arc $\mathcal{P} = \{(a_1, b_2) : n(n+2m)b_2 = -2ma_1^2 + 3nma_1 + 2nm^2, n/2 < a_1 < n\}$, $I_3 < 0$ and $I_4 > 0$.*

Proof. Along the line segment $\mathcal{L}$, after substituting the solutions (16), expressions I_3 and I_4 become:

$$I_3 = \frac{2m(a_1 - n/2)(n - a_1)(a_1 + m)^2}{(n+4m)(n+2m)}$$

$$I_4 = \frac{-8m(a_1 - n/2)(n - a_1)(a_1 + m)(4a_1m + na_1 + 6m^2 + nm)}{(n+4m)(n+2m)^2}$$

Using $n/2 < a_1 < n$, it is not hard to check that I_3 is positive and I_4 is negative.

Along the parabolic arc $\mathcal{P}$, we obtain:

$$I_3 = -2m^2(a_1 - n/2)(n - a_1)(a_1 + m)\big(16nm^4 + (-24a_1^2 + 76na_1 + 40n^2)m^3 + (-24a_1^3 + 38na_1^2 + 102n^2a_1)m^2 + (-10na_1^3 + 65n^2a_1^2)m + 9n^2a_1^3\big)/ \big(n(n+2m)^2(a_1^2n^2 + 4nma_1^2 - 12m^2a_1^2 + 3n^2ma_1 + 30nm^2a_1 - 4m^2n^2 + 8nm^3)\big)$$

Since $76na_1 > 24a_1^2$, $38na_1^2 > 24a_1^3$, $65n^2a_1^2 > 10na_1^3$, and $30nm^2a_1 > 12m^2a_1^2 + 4n^2m^2$ for $n/2 < a_1 < n$, we see that I_3 is negative.

$$I_4 = 8m^2(a_1 - n/2)(n - a_1)(a_1 + m)\big(56nm^4 + (-20n^2 + 182na_1 - 42a_1^2)m^3 + (139na_1^2 - 5n^2a_1 - 36a_1^3)m^2 + (20n^2a_1^2 + 16na_1^3)m + 5n^2a_1^3\big)/ n(n+2m)^2(a_1^2n^2 + 4nma_1^2 - 12m^2a_1^2 + 3n^2ma_1 + 30nm^2a_1 - 4m^2n^2 + 8nm^3)$$

Similarly, I_4 is positive because $182na_1 > 20n^2 + 42a_1^2$ and $139na_1^2 > 5n^2a_1 + 36a_1^3$ when $n/2 < a_1 < n$. $\square$

Lemma 21. *I_3 and I_4 are zero at the corners of $\mathcal{R}$.*

Proof. When we substitute $a_1 = n/2, b_2 = m$ into the solutions (16), we obtain $a_2 = n/4$, $b_3 = m$, $T_3 = n/8 + m$, $T_4 = n/16 + m$. With $a_1 = n$, $b_2 = m$, we obtain $a_2 = n$, $b_3 = m$, $T_3 = n + m$, $T_4 = n + m$. In other words, the first point corresponds to $ad_A = Id_{\mathfrak{v}}/2$ on $\mathfrak{v}$, $ad_A = Id_{\mathfrak{z}}$ on $\mathfrak{z}$, while the second point gives us $ad_A = Id_{\mathfrak{n}}$ on $\mathfrak{n}$. $\square$

Lemma 22. *The equations $I_3 = 0$, $I_4 = 0$ define continuous curves inside $\mathcal{R}$ which connect the corner points of $\mathcal{R}$.*

Proof. We can express I_3, I_4 in terms of a_1, b_2:

$$I_3 = -i_3(a_1, b_2)(a_1 + m)/(n + 2m)\gamma(a_1, b_2),$$

where

$$i_3(a_1, b_2) = \Big((24m^3 + 12n^2m + 36nm^2)a_1 + 48nm^3 + 24m^4 + 18m^2n^2\Big)b_2^2 + \Big((11nm + n^2 + 14m^2)a_1^3 + (37nm^2 + 38m^3 + 4n^2m)a_1^2 + (-8m^4 - 2nm^3 - 7m^2n^2)a_1 - 32m^5 - 16m^3n^2 - 40m^4n\Big)b_2 + 2ma_1^5 + (-3nm + 8m^2)a_1^4 + (-4m^3 - 23nm^2)a_1^3 + (-34m^4 - 52nm^3)a_1^2 + (-16m^5 - 40m^4n)a_1 - 8m^5n + 8m^6. \tag{18}$$

$$I_4 = i_4(a_1, b_2)(a_1 + m)/(n + 2m)^2\gamma(a_1, b_2),$$

where

$$
\begin{aligned}
(19)\quad i_4(a_1,b_2) = \Big(&(30n^3m + 288m^4 + 408nm^3 + 192n^2m^2)a_1 \\
&+ 30n^3m^2 + 336m^5 + 204m^3n^2 + 456m^4n\Big)b_2^2 + \Big((50n^2m + 5n^3 + 204m^2n + 248m^3)a_1^3 \\
&+(800m^4+20n^3m+120n^2m^2+560nm^3)a_1^2+(-5n^3m^2+376m^5-182m^3n^2-156m^4n)a_1 \\
&- 224m^6 - 264m^4n^2 - 560m^5n - 20n^3m^3\Big)b_2 + (10mn + 40m^2)a_1^5 \\
&+ (-15n^2m - 20m^2n + 180m^3)a_1^4 + (-90n^2m^2 - 424nm^3 + 12m^4)a_1^3 \\
&+ (-105m^3n^2 - 930m^4n - 680m^5)a_1^2 + (-432m^5n - 664m^6 + 90m^4n^2)a_1 \\
&+ 104m^6n + 120m^5n^2 - 112m^7.
\end{aligned}
$$

While the precise form of these expressions is not important, we see that i_3, i_4 are quadratic in b_2, and that the curves $I_3 = 0$, $I_4 = 0$ correspond to the curves $i_3 = 0$, $i_4 = 0$. Thus, for each a_1 in the interval $(n/2, n)$, there are at most two solutions to each of the equations $I_3 = 0$, $I_4 = 0$. Since we know that each of I_3, I_4 changes sign from the bottom $\mathcal{L}$ to the top $\mathcal{P}$ of our region $\mathcal{R}$, we conclude that for each a_1 in $(n/2, n)$, there is exactly one solution to each equation in $\mathcal{R}$. ☐

Lemma 23. *In the region $\mathcal{R}$, the curves $I_3 = 0$ and $I_4 = 0$ intersect only at the corners.*

Proof. We parametrize the interior of $\mathcal{R}$ by $a_1(s) = n/2 + sn/2$, $b_2(s,t) = (1-t)m + t(-2ma_1(s)^2 + 3mna_1(s) + 2m^2n)/(n(n+2m))$ for $(s,t) \in (0,1) \times (0,1)$. Substituting these values into (18) and (19) gives polynomials $p_3(s,t)$ and $p_4(s,t)$.

We take a combination H of p_3 and p_4 which eliminates the terms in t^0: $H(s,t) = p_3(s,0)p_4(s,t) - p_4(s,0)p_3(s,t)$. Note that if I_3 and I_4 are simultaneously zero in the interior of $\mathcal{R}$, then H is zero for some $(s,t) \in (0,1) \times (0,1)$. We obtain

$$
\begin{aligned}
H(s,t) = \tfrac{1}{256(n+2m)} &s^2tn^3m^3(s-1)^2(2m + n + sn)(n + 4m + sn)^2 \\
&\times (h_1(s)t + h_0(s))
\end{aligned}
$$

where

$$
\begin{aligned}
h_1(s) = (-336n^3m^2 - 192m^3n^2 - 60n^4m)s^3 &+ (-384m^4n - 912m^3n^2 - 120n^3m^2)s^2 \\
&+ (1104m^3n^2 + 456n^3m^2 + 60n^4m + 384m^4n)s
\end{aligned}
$$

and

$$
\begin{aligned}
h_0(s) = (25n^5 + 86n^4m + 32n^3m^2)s^3 &+ (75n^5 + 518n^4m + 884n^3m^2 + 136m^3n^2)s^2 \\
&+ (75n^5 + 898n^4m + 3324n^3m^2 + 3840m^3n^2 + 208m^4n)s \\
&+ 25n^5 + 466n^4m + 2712n^3m^2 + 6648m^3n^2 + 5840m^4n + 128m^5.
\end{aligned}
$$

To show that $H(s,t) \neq 0$, we just need to check that $h_1(s)t + h_0(s)$ is positive when $t = 0$ and $t = 1$. If we let $t = 0$, we obtain $h_0(s)$, which is a sum of positive terms. If we let $t = 1$, we have

$$\begin{aligned}h_1(s)+h_0(s) = 128m^5 &+ (5840+592s-384s^2)nm^4 + (6648+4944s-776s^2-192s^3)n^2m^3 \\ &+ (2712 + 3780s + 764s^2 - 304s^3)n^3m^2 + (466 + 958s + 518s^2 + 26s^3)n^4m \\ &+ (75s^2 + 75s + 25 + 25s^3)n^5.\end{aligned}$$

Since $s < 1$, the negative terms are easily dominated, and hence $h_1(s)t + h_0(s)$ is never zero for (s,t) in $(0,1) \times (0,1)$. Thus H is never zero in the interior of $\mathcal{R}$, so I_3 and I_4 are never simultaneously zero, as claimed. □

Lemma 24. *The curve $I_4 = 0$ is always above the curve $I_3 = 0$ in the region $\mathcal{R}$.*

Proof. We calculate the slopes of the curves $I_3 = 0$ and $I_4 = 0$ at the point of intersection $(n/2, m)$. Recall that $i_3 = 0$ (18) and $i_4 = 0$ (19) describe the same curves. If we differentiate implicitly and evaluate at $a_1 = n/2, b_2 = m$ we find that the slope of I_3 is:

$$m_3 = \frac{nm(n+4m)(n+2m)}{n^4 + 17n^3m + 122m^2n^2 + 272m^3n + 64m^4}.$$

Similarly, the slope of I_4 is:

$$m_4 = \frac{5nm(n+4m)(12m^2 + 6mn + n^2)}{(n+2m)(896m^4 + 760nm^3 + 364n^2m^2 + 70mn^3 + 5n^4)}.$$

Note that both of these slopes are positive, as we would expect. To see which is larger, we compute their difference:

$$\begin{aligned}m_4 - m_3 = nm^2(n+4m)^2(64m^4 + 2888nm^3 + 1880n^2m^2 + 416mn^3 + 25n^4)\big/ \\ (n+2m)(5n^4 + 70mn^3 + 364n^2m^2 + 760nm^3 + 896m^4) \\ (64m^4 + 272nm^3 + 122n^2m^2 + 17mn^3 + n^4),\end{aligned}$$

showing that the curve $I_4 = 0$ lies above the curve $I_3 = 0$ in the interior of $\mathcal{R}$. □

We now combine the results from Lemmas 20 to 24 to conclude that the open regions $I_4 > 0$ and $I_3 > 0$ do not intersect in the interior of the constrained region $\mathcal{R}$. Indeed, we have shown that I_3, I_4 are also non-zero on the boundary of $\mathcal{R}$, except at the corner points $(n/2, m)$ and (n, m). This concludes the proof of Theorem 19. □

8. Harmonic solvmanifolds

First we provide a simple proof of Theorem 4 that does not rely on the inequalities from Proposition 16 or the depth of analysis leading to Theorem 19.

Theorem 25. *Let $(\mathfrak{s}, Q)$ be a three-step Carnot solvmanifold. If $(\mathfrak{s}, Q)$ is 2-stein, then $(\mathfrak{s}, Q)$ is a Damek-Ricci space.*

Proof. By rescaling the metric, we can ensure that $\mathrm{ad}_A|_{\mathfrak{v}} = \frac{1}{2} Id_{\mathfrak{v}}$ and $\mathrm{ad}_A|_{\mathfrak{z}} = Id_{\mathfrak{v}}$. Thus we get the trace constants $a_r = n/2^r, b_r = m$. By Equation 1 of Proposition 10 we have, for each unit vector Z in $\mathfrak{z}$,

$$\mathrm{tr}(J_Z^2) = 4(T_2 - \lambda_Z T_1) = 4(n/4 + m - (n/2 + m)) = -n.$$

By Equation 1 of Proposition 15, we also have

$$\mathrm{tr}(J_Z^4) = 16(T_4 - 2\lambda_Z^2 T_2 + \lambda_Z^3 T_1) = 16(n/16 + m - 2(n/4 + m) + n/2 + m) = n.$$

Applying Schwartz's Inequality (Lemma 13), to J_Z^2, we conclude that J_Z^2 is a multiple of the identity, with trace $-n$. Thus $J_Z^2 = -Id_{\mathfrak{v}}$ for all unit Z, and $(\mathfrak{s}, Q)$ is a Damek-Ricci space. □

Now we prove Theorem 1.

Theorem 26. *Let $(\mathfrak{s}, Q)$ be a three-step solvmanifold of Iwasawa type and of algebraic rank one. If $(\mathfrak{s}, Q)$ is 2-stein, then it is a Damek-Ricci space.*

Proof. By Theorem 19, all inequalities obtained via the Ledger conditions must be equalities. This includes the inequalities derived in Proposition 16. Then by Proposition 17, this is only possible if $(\mathfrak{s}, Q)$ is a Damek-Ricci space or a hyperbolic space. □

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