# GEOMETRIC MODELS FOR THE SPECTRA OF CERTAIN GELFAND PAIRS ASSOCIATED WITH HEISENBERG GROUPS

CHAL BENSON AND GAIL RATCLIFF

Abstract. Let $K$ be a compact Lie group acting on a finite dimensional Hermitian vector space $V$ via some unitary representation. Now $K$ acts by automorphisms on the associated Heisenberg group $H_V = V \times \mathbb{R}$ and we say that $(K, H_V)$ is a Gelfand pair when the algebra $L^1_K(H_V)$ of integrable $K$-invariant functions on $H_V$ commutes under convolution. In this situation an application of the Orbit Method yields a injective mapping $\Psi$ from the space $\Delta(K, H_V)$ of bounded $K$-spherical functions on $H_V$ to the space $\mathfrak{h}^*_V/K$ of $K$-orbits in the dual of the Lie algebra for $H_V$. We prove that $\Psi$ is a homeomorphism onto its image provided that the action of $K$ on $V$ is "well-behaved" in a sense made precise in this work. Our result encompasses a widely studied class of examples arising in connection with Hermitian symmetric spaces.

## 1. Introduction and overview of results

Let $N$ be a nilpotent Lie group and $K$ a compact Lie group acting smoothly on $N$ via automorphisms to yield a *Gelfand pair*. That is, we assume that the algebra $L^1_K(N)$ of integrable $K$-invariant functions on $N$ is abelian. It is known that $N$ is necessarily at most 2-step nilpotent [BJR90]. Now the spectrum, or Gelfand space, for the commutative Banach algebra $L^1_K(N)$ coincides, via integration, with the set $\Delta(K, N)$ of bounded $K$-spherical functions on $N$ endowed with the compact-open topology.

In [BR08] we established a one-to-one correspondence between spherical functions $\varphi \in \Delta(K, N)$ and certain $K$-orbits in the dual of the Lie algebra for $N$. In outline this works as follows: Form the semi-direct product $G = K \ltimes N$, let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{n}$ denote the Lie algebras for $G$, $K$, $N$, respectively, and identify $\mathfrak{n}^*$ with the annihilator of $\mathfrak{k}$ in $\mathfrak{g}^*$. Each bounded spherical function $\varphi$ on $N$ has positive type [BJR90] and hence is a matrix coefficient for some irreducible unitary representation $\rho_\varphi$ of $G$. An orbit method, due to Pukanszky [Puk78] and Lipsman [Lip80, Lip82], associates an $Ad^*(G)$-orbit $\mathcal{O}(\rho_\varphi) \subset \mathfrak{g}^*$ to $\rho_\varphi$. Now $\mathcal{O}(\rho_\varphi)$ meets $\mathfrak{n}^*$, the intersection $\mathcal{O}(\rho_\varphi) \cap \mathfrak{n}^*$ is a $K$-orbit in $\mathfrak{n}^*$, and the resulting map

$$\Psi : \Delta(K, N) \to \mathfrak{n}^*/K, \qquad \Psi(\phi) = \mathcal{O}(\rho_\varphi) \cap \mathfrak{n}^* \tag{1.1}$$

1991 *Mathematics Subject Classification.* Primary 43A90; Secondary 43A80, 22E25.
*Key words and phrases.* Gelfand Pairs, Spherical Functions, Nilpotent Lie Groups, Orbit Method.
We dedicate this work in fond memory of Joe Jenkins (1941-2010), our friend and collaborator.

is injective. We let $\mathcal{A}(K,N)$ denote the image of $\Psi$ viewed as a subspace of $\mathfrak{n}^*/K$ under the quotient topology.

**Conjecture 1.1.** [BR08] $\Psi$ *is a homeomorphism* $\Delta(K,N) \cong \mathcal{A}(K,N)$.

This is shown in [BR08] for pairs with $N$ abelian, for the action of the unitary group on the Heisenberg group, and for the action of the orthogonal group on the free 2-step group.

The current paper concerns Conjecture 1.1 for Gelfand pairs associated with Heisenberg groups. Throughout, $V \cong \mathbb{C}^n$ will be an $n$-dimensional complex vector space with Hermitian inner product $\langle \cdot, \cdot \rangle$. The associated Heisenberg group is

$$H_V = V \times \mathbb{R} \quad \text{with product} \quad (z,t)(z',t') = \left( z + z',\ t + t' - \frac{1}{2} Im\langle z, z' \rangle \right).$$

The unitary group $U(V)$ acts by automorphisms on $H_V$ via

$$k \cdot (z,t) = (kz, t),$$

as a maximal compact connected subgroup of $Aut(H_V)$. We assume that $K$ is a compact Lie group acting on $(V, \langle \cdot, \cdot \rangle)$ by some unitary representation to yield a Gelfand pair $(K, H_V)$. Equivalently $K : V$ is a linear *multiplicity free action* in the sense that the associated representation of $K$ in the space $\mathbb{C}[V]$ of holomorphic polynomial functions on $V$, namely

$$(k \cdot p)(z) = p(k^{-1} z),$$

is multiplicity free [Car87]. The possibilities for $(K, H_V)$ are known, as the linear multiplicity free actions have been completely classified ([Kac80], [BR96], [Lea98]).

The Lie algebra of $H_V$ is $\mathfrak{h}_V = V \times \mathbb{R}$ with bracket $[(z,t),(z',t')] = -Im\langle z, z' \rangle$. The group $K$ acts on $\mathfrak{h}_V^* = \hom(\mathfrak{h}_V, \mathbb{R})$ via

$$(k \cdot \ell)(z,t) = \ell\big(k^{-1} \cdot z, t\big)$$

and we obtain an injective mapping $\Psi : \Delta(K, H_V) \to \mathfrak{h}_V^*/K$ as outlined above.

Letting $\mathcal{A}(K,H_V) := \Psi\big(\Delta(K,H_V)\big)$, we establish Conjecture 1.1 for Gelfand pairs $(K, H_V)$ subject to the hypothesis that the action $K : V$ be *well-behaved*. This condition is made precise below in Definition 2.4. Under this hypothesis $\mathcal{A}(K,H_V)$ provides a canonical geometric model for the space $\Delta(K,H_V)$ in the spirt of the Orbit Method.

Our main result is the following:

**Theorem 1.2.** *If the multiplicity free action* $K : V$ *is well-behaved then*

$$\Psi : \Delta(K, H_V) \to \mathcal{A}(K, H_V)$$

*is a homeomorphism.*

The literature contains an "eigenvalue picture" for the spectrum of a Gelfand pair $(K, H_V)$: The spherical functions $\varphi \in \Delta(K, H_V)$ are joint eigenfunctions for the algebra $\mathbb{D}_K(H_V)$ of differential operators on $H_V$ invariant under $K$ and the left action of $H_V$. Choosing a finite set $\mathcal{L}$ of formally self-adjoint generators for $\mathbb{D}_K(H_V)$ produces a map

$$\widehat{\mathcal{L}} : \Delta(K, H_V) \to \mathbb{R}^{|\mathcal{L}|}$$

sending each spherical function to its eigenvalues with respect to the operators $\mathcal{L}$. This is a homeomorphism onto its image, the *Heisenberg fan* $\mathcal{F}_{\mathcal{L}}(K, H_V) \subset \mathbb{R}^{|\mathcal{L}|}$. Equivalent fan models can be obtained by choosing different sets $\mathcal{L}$ of generators for $\mathbb{D}_K(H_V)$. The papers [Bou81, FH87, Str91, BJRW96, ADBR07, ADBR09] each concern, in part, the Heisenberg fan and its applications. But the fact that $\widehat{\mathcal{L}}$ is a homeomorphism now follows from a more general result of Ferrari Ruffino [FR07].

A byproduct of our proof of Theorem 1.2 relates the orbital model $\mathcal{A}(K, H_V)$ to the fan $\mathcal{F}_{\mathcal{L}}(K, H_V)$, for chosen generators $\mathcal{L} \subset \mathbb{D}_K(H_V)$. (See Corollary 4.4.) Proposition 4.2 below is the key. This asserts that for each operator $L \in \mathbb{D}_K(H_V)$ there is a continuous $K$-invariant function $\varepsilon_L : \mathfrak{h}_V^* \to \mathbb{C}$ whose value on the $K$-orbit $\Psi(\varphi)$ is the eigenvalue $\widehat{L}(\varphi)$ of $L$ on each $\varphi \in \Delta(K, H_V)$. Our proof of Proposition 4.2 provides an explicit description of the map $\varepsilon_L$.

The hypothesis that the multiplicity free action $K : V$ be well-behaved intervenes in our construction of the eigenvalue mappings $\varepsilon_L$. Definition 4.1 below provides an alternate description of the map $\Psi$ which does not require explicit use of the semi-direct product $K \ltimes N$. Key to this formulation is the un-normalized moment map, $\tau : V \to \mathfrak{k}^*$, for the action $K : V$. (See Subsection 2.5 below). The requirement that $K : V$ be well-behaved relates the moment map to highest weight vectors occurring in $\mathbb{C}[V]$. The upshot is that the function $\varepsilon_L \in C(\mathfrak{h}_V^*)^K$ reproduces the eigenvalues of $L \in \mathbb{D}_K(H_V)$ via evaluation on $\mathcal{A}(K, H_V)$.

In Section 5 of this paper we prove that Theorem 1.2 encompasses examples arising in the context of Hermitian symmetric spaces. Let $G/K$ be a Hermitian symmetric space of non-compact type and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denote the Cartan decomposition for the Lie algebra of $G$. The Hermitian structure on the manifold $G/K$ makes $\mathfrak{p} \cong T_{eK}(G/K)$ a complex Hermitian vector space and the isotropy representation of $K$ on $T_{eK}(G/K)$ corresponds to the adjoint action of $K$ on $\mathfrak{p}$. It is well known that this action $K : \mathfrak{p}$ is multiplicity free [Joh80]. We will prove the following.

**Theorem 1.3.** *The multiplicity free action $K : \mathfrak{p}$ is well-behaved.*

In view of Theorem 1.2 the associated Gelfand pair $(K, H_{\mathfrak{p}})$ has $\Delta(K, H_{\mathfrak{p}})$ homeomorphic to $\mathcal{A}(K, H_{\mathfrak{p}})$ via $\Psi$. Our proof of Theorem 1.3 uses the structure theory for the Hermitian symmetric space $G/K$ and provides an explicit description of $\mathcal{A}(K, H_{\mathfrak{p}})$.

The extent to which the hypothesis that $K : V$ be well-behaved limits the scope of Theorem 1.2 is unclear. At present we do not, however, know of any multiplicity free actions which fail to be well-behaved. Should this be true in general, Theorem 1.2 would in fact establish Conjecture 1.1 for all Gelfand pairs $(K, H_V)$ associated with Heisenberg groups. We hope to address this problem in a subsequent paper.[1]

## 2. Preliminaries on the multiplicity free action $K : V$

As in the preceding discussion we assume that $K$ is a compact Lie group acting unitarily on $\big(V, \langle\cdot,\cdot\rangle\big)$. We write $k \cdot v$ and $A \cdot v$ for the result of applying elements $k \in K$ and $A \in \mathfrak{k} := \mathrm{Lie}(K)$ to $v \in V$. Throughout $(K, H_V)$ is assumed to be a Gelfand pair and hence $K : V$ a linear multiplicity free action. Fixing notation, let

- $T \subset K$ denote a maximal torus in $K$ with Lie algebra $\mathfrak{t} \subset \mathfrak{k}$,
- $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}}$ the resulting Cartan subalgebra in $\mathfrak{k}_{\mathbb{C}}$,
- $H$ the corresponding subgroup in the complexified group $K_{\mathbb{C}}$,
- $B := HN$ a fixed Borel subgroup in $K_{\mathbb{C}}$ with Lie algebra $\mathfrak{b} \subset \mathfrak{k}_{\mathbb{C}}$ and
- $\Lambda \subset \mathfrak{h}^*$ the set of $B$-highest weights for irreducible representations of $K_{\mathbb{C}}$ (or equivalently of $K$) occurring in $\mathbb{C}[V]$. Moreover, we write
- $P_\alpha \subset \mathbb{C}[V]$ for the unique irreducible subspace with highest weight $\alpha \in \Lambda$. So
$$\mathbb{C}[V] = \bigoplus_{\alpha \in \Lambda} P_\alpha$$
is the canonical decomposition of $\mathbb{C}[V]$ into irreducible subspaces for the actions of $K_{\mathbb{C}}$ and $K$. We let
- $\mathfrak{a}^* \subset \mathfrak{h}^*$ denote the set $\mathfrak{a}^* := \mathbb{C}\text{-Span}(\Lambda)$.

Finally, for each $\alpha \in \Lambda$:

- Choose $h_\alpha \in P_\alpha$, a $B$-highest weight vector (unique modulo $\mathbb{C}^\times$);
- Let $d_\alpha = \dim(P_\alpha)$;
- Let $|\alpha|$ denote the degree of homogeneity of the polynomials in $P_\alpha$.

Then $P_\alpha \subset \mathcal{P}_{|\alpha|}(V)$ where $\mathcal{P}_m(V)$ is the space of holomorphic polynomials on $V$ which are homogeneous of degree $m$.

2.1. **Fundamental highest weights and rank.** An element $\alpha \in \Lambda$ is said to be a *fundamental* highest weight for $K : V$ when $h_\alpha$ is an irreducible polynomial. The fundamental highest weights form a finite $\mathbb{Q}$-linearly independent set
$$\{\alpha_1, \ldots, \alpha_r\}$$
which freely generates $\Lambda$ as an additive semigroup [HU91]. The set $\{\alpha_1, \ldots, \alpha_r\}$ is, in particular, a basis for $\mathfrak{a}^*$. The value $r$ is called the *rank* of the action $K : V$.

[1]Added in proof: The authors have recently verified that all multiplicity free actions $K : V$ with $K$ acting irreducibly on $V$ are well-behaved.

2.2. **Invariant polynomials, differential operators and eigenvalues.** In this subsection we summarize needed results from [Kno98]. Details on this material can also be found in [BR04, §7,9]. We let

- $\mathbb{C}[V_{\mathbb{R}}]^K$ denote the algebra of $K$-invariant polynomials on the underlying real vector space for $V$ and
- $\mathcal{PD}(V)^{K_{\mathbb{C}}}$ the space of $K_{\mathbb{C}}$-invariant polynomial coefficient differential operators on $V$.

Each $D \in \mathcal{PD}(V)^{K_{\mathbb{C}}}$ acts on $P_\alpha$ ($\alpha \in \Lambda$) by a scalar $\widehat{D}(\alpha)$. That is,

$$Dh_\alpha = \widehat{D}(\alpha)h_\alpha.$$

The mapping $\widehat{D}$ on $\Lambda = \mathbb{Z}^+\text{-Span}(\alpha_1, \dots, \alpha_r)$ extends to a polynomial function on $\mathfrak{a}^* = \mathbb{C}\text{-Span}(\alpha_1, \dots, \alpha_r)$ invariant under the *little Weyl group* $W_\circ$, a certain subgroup of the stabilizer of $\mathfrak{a}^*$ in the Weyl group $W(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h})$. The map

$$\rho : \mathcal{PD}(V)^{K_{\mathbb{C}}} \to \mathbb{C}[\mathfrak{a}^*]^{W_\circ}, \qquad \rho(D) = \widehat{D}$$

is an algebra isomorphism.

As explained in [BR04, §7] each irreducible subspace $P_\alpha \subset \mathbb{C}[V]$ ($\alpha \in \Lambda$) produces canonical invariant polynomials and differential operators

$$p_\alpha \in \mathbb{C}[V_{\mathbb{R}}]^K \quad \text{and} \quad \big(D_\alpha := p_\alpha(z, \partial)\big) \in \mathcal{PD}(V)^{K_{\mathbb{C}}},$$

so that

$$\{p_\alpha \ : \ \alpha \in \Lambda\}, \qquad \{D_\alpha \ : \ \alpha \in \Lambda\}, \qquad \{\widehat{D}_\alpha \ : \ \alpha \in \Lambda\}$$

are canonical bases for $\mathbb{C}[V_{\mathbb{R}}]^K$, $\mathcal{PD}(V)^{K_{\mathbb{C}}}$ and $\mathbb{C}[\mathfrak{a}^*]^{W_\circ}$ respectively. Here

$$p_\alpha(z) := \sum_{j=1}^{d_\alpha} v_j(z)\overline{v}_j(\overline{z})$$

where $\{v_j : 1 \le j \le d_\alpha\}$ is an orthonormal basis for $P_\alpha$ with respect to the Fock inner product on $\mathbb{C}[V]$.[2] The polynomial $p_\alpha$ is homogeneous of degree $2|\alpha|$, the differential operator $D_\alpha$ homogeneous of order $|\alpha|$ and the polynomial $\widehat{D}_\alpha$ has degree $|\alpha|$, but is, in general, non-homogeneous.

Letting top$(\cdot)$ denote the homogeneous component of highest degree in a polynomial we now define a mapping

$$\bar{\rho} : \mathbb{C}[V_{\mathbb{R}}]^K \to \mathbb{C}[\mathfrak{a}^*]^{W_\circ} \tag{2.1}$$

as the unique vector space homomorphism for which

$$\bar{\rho}(p_\beta) = \mathrm{top}(\rho(D_\beta)) = \mathrm{top}(\widehat{D}_\beta).$$

As both $p_\beta \mapsto D_\beta$ and $\rho$ are vector space isomorphisms, so is $\bar{\rho}$.

[2]In [BR04] and elsewhere the canonical invariants are $p_\alpha(z) = (1/d_\alpha)\sum_j v_j(z)\overline{v}_j(\overline{z})$. In the current paper we have chosen not to include the normalization factor $1/d_\alpha$.

There is also an algebra isomorphism

$$\mathbb{C}[V_{\mathbb{R}}]^K \cong \mathbb{C}[V \oplus V^*]^{K_{\mathbb{C}}}, \quad p \leftrightarrow \widetilde{p}$$

determined by the rule

$$p(z) = \widetilde{p}(z, z^*)$$

where $z^* = \langle \cdot, z \rangle$ for $z \in V$. Using this correspondence one can transplant $\bar{\rho}$ to the domain $\mathbb{C}[V \oplus V^*]^{K_{\mathbb{C}}}$ and provide an alternate description. For $\alpha \in \Lambda$ and any point $z \in V$ with $h_\alpha(z) \neq 0$, let $\eta(z, \alpha)$ denote the differential form $(\partial \log h_\alpha)(z)$ on $V$. That is, $\eta(z, \alpha) \in V^*$ is a linear functional given by

$$\eta(z, \alpha)(w) := \frac{\big(\partial_w h_\alpha\big)(z)}{h_\alpha(z)} \tag{2.2}$$

where $\partial_w h_\alpha$ is the directional derivative

$$\big(\partial_w h_\alpha\big)(z) := \lim_{t \to 0} \frac{h_\alpha(z + tw) - h_\alpha(z)}{t}.$$

Then

$$\bar{\rho}(p_\beta)(\alpha) = \widetilde{p}_\beta\big(z, \eta(z, \alpha)\big) \tag{2.3}$$

for $\alpha, \beta \in \Lambda$, independent of $z \in V$ with $h_\alpha(z) \neq 0$.

2.3. **Eigenvalue polynomials on $V$.** We use the isomorphism $\bar{\rho} : \mathbb{C}[V_{\mathbb{R}}]^K \cong \mathbb{C}[\mathfrak{a}^*]^{W_\circ}$ to pull the polynomials $\widehat{D}_\beta \in \mathbb{C}[\mathfrak{a}^*]^{W_\circ}$ back to the vector space $V$.

**Definition 2.1.** For $\beta \in \Lambda$, $E_\beta \in \mathbb{C}[V_{\mathbb{R}}]^K$ will denote the unique polynomial with $\bar{\rho}(E_\beta) = \widehat{D}_\beta$.

**Lemma 2.2.** *$top(E_\beta) = p_\beta$. In particular, $E_\beta$ is a (non-homogeneous) polynomial of degree $2|\beta|$.*

*Proof.* One has $\bar{\rho}\big(\mathrm{top}(E_\beta)\big) = \mathrm{top}\big(\bar{\rho}(E_\beta)\big) = \bar{\rho}(\widehat{D}_\beta) = \bar{\rho}(p_\beta)$. As $\bar{\rho}$ is an isomorphism it follows that $\mathrm{top}(E_\beta) = p_\beta$. □

The eigenvalue polynomials $E_\beta$ are key to our subsequent analysis and proof of Theorem 1.2. Lemma 2.2 shows, in particular, that $\{E_\beta \ : \ \beta \in \Lambda\}$ is a basis for $\mathbb{C}[V_{\mathbb{R}}]^K$.

2.4. **Orbit method for $K$.** As in [BJLR97, BR08] we use a version of the *Orbit Method* for compact Lie groups to associate a coadjoint orbit $\mathcal{O}_\alpha$ in $\mathfrak{k}^*$ to each irreducible subspace $P_\alpha$ in the decomposition of $\mathbb{C}[V]$. Note that the weight $\alpha \in \Lambda$ takes pure imaginary values on $\mathfrak{t}$. We extend the real valued functional $(1/i)\alpha$ from $\mathfrak{t}$ to all of $\mathfrak{k}$ as follows: Fix an $Ad(K)$-invariant inner product $(\cdot|\cdot)$ on the Lie algebra $\mathfrak{k}$ and let $\mathfrak{t}^\perp$ denote the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{k}$ with respect to $(\cdot|\cdot)$.[3] We let

[3]Replacing $K$ by its image in $U(V)$ one may, for concreteness, use $(A|B) := tr(AB^*) = -tr(AB)$.

- $\alpha_{\mathfrak{k}} \in \mathfrak{k}^*$ be the (real valued) linear functional on $\mathfrak{k}$ satisfying

$$\alpha_{\mathfrak{k}}(A) = \begin{cases} -i\alpha(A) & \text{if } A \in \mathfrak{t} \\ 0 & \text{if } A \in \mathfrak{t}^{\perp} \end{cases},$$

and set

- $\mathcal{O}_\alpha = Ad^*(K)\alpha_{\mathfrak{k}}$.

2.5. **The moment map and spherical orbits in $V$.** The unnormalized *moment map* $\tau : V \to \mathfrak{k}^*$ for the action $K : V$ is given by the formula [Wil92]

$$\tau(v)(A) := i\langle A \cdot v, v\rangle.$$

Note that $\tau(v)$ takes real values because $\mathfrak{k}$ acts on $(V, \langle\cdot,\cdot\rangle)$ by skew-hemitian operators. The moment map intertwines the action of the group $K$ on $V$ with its coadjoint action on $\mathfrak{k}^*$. Hence $\tau$ maps $K$-orbits in $V$ to $Ad^*(K)$-orbits in $\mathfrak{k}^*$. Moreover as $K : V$ is a multiplicity free action it is known that

- $\tau$ is one-to-one on $K$-orbits ([BJLR97, Theorem 1,3], [DP96]), and
- each coadjoint orbit $\mathcal{O}_\alpha$ ($\alpha \in \Lambda$) lies in the image of $\tau$ ([BJLR97, Proposition 4.1]).

**Definition 2.3.** The *spherical orbit* $\mathcal{K}_\alpha \in V/K$ for $\alpha \in \Lambda$ is the unique $K$-orbit in $V$ satisfying $\tau\big(\mathcal{K}_\alpha\big) = \mathcal{O}_\alpha$.

One has

$$\mathcal{K}_\alpha = K \cdot v_\alpha \text{ for some (possibly non-unique) } v_\alpha \in V \text{ with } \tau(v_\alpha) = \alpha_{\mathfrak{k}}.$$

We call any such point $v_\alpha \in V$ a *spherical point* for $\alpha$.

2.6. **Well-behaved actions.** Our formulation and proof of Theorem 1.2 requires a compatibility condition relating the spherical orbits $\mathcal{K}_\alpha \subset V$ and highest weight vectors $h_\alpha \in P_\alpha$.

**Definition 2.4.** Given $\alpha \in \Lambda$ we say that a spherical point $v_\alpha$ for $\alpha$ is *well-adapted to* $h_\alpha$ when the following conditions hold.

(i) $h_\alpha(v_\alpha) \neq 0$, and
(ii) $\big(\partial_w h_\alpha\big)(v_\alpha) = \langle w, v_\alpha\rangle h_\alpha(v_\alpha)$ for all $w \in V$.

We say that the multiplicity free action $K : V$ is *well-behaved* if for every $\alpha \in \Lambda$ one can choose a spherical point $v_\alpha$ well-adapted to $h_\alpha$.

In Section 5 it is shown that actions arising in connection with Hermitian symmetric spaces are well-behaved. The following proposition provides support for the idea that all multiplicity free actions are well-behaved. As $K : V$ is multiplicity free the Borel subgroup $B = HN$ in $K_{\mathbb{C}}$ has a Zariski-open dense orbit in the vector space $V$ [Vin86].

**Proposition 2.5.** *If $v_\alpha$ lies in the open $B$-orbit then $v_\alpha$ is well-adapted to $h_\alpha$.*

*Proof.* Suppose that $v_\alpha$ lies in the open $B$-orbit. As $h_\alpha$ is a non-zero $B$-semi-invariant we must have $h_\alpha(v_\alpha) \neq 0$. It remains to verify condition (ii) in Definition 2.4.

A suitable ordering on the roots for $\mathfrak{k}_\mathbb{C}$ relative to $\mathfrak{h}$ enables one to decompose the Lie algebra for $B = HN = BN_+$ as $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ where $\mathfrak{n}_+$ is the sum of positive root spaces. Moreover $\mathfrak{k}_\mathbb{C} = \mathfrak{b} \oplus \mathfrak{n}_- = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ where $\mathfrak{n}_-$ is the sum of negative root spaces. The weight $\alpha \in \mathfrak{h}^*$ extends to a linear functional on all of $\mathfrak{k}_\mathbb{C}$ as zero on $\mathfrak{n}_+ \oplus \mathfrak{n}_-$. On the other hand, one can extend the real-valued linear functional $\alpha_\mathfrak{k} \in \mathfrak{k}^*$ to a complex-linear functional on $\mathfrak{k}_\mathbb{C}$. It is not hard to check that these extensions are related via $\alpha = i\alpha_\mathfrak{k}$ on $\mathfrak{k}_\mathbb{C}$. But

$$\alpha_\mathfrak{k}(A) = \tau(v_\alpha)(A) = i\langle A \cdot v_\alpha, v_\alpha\rangle$$

for $A \in \mathfrak{k}$ and the right hand side of this expression has an obvious extension to $\mathfrak{k}_\mathbb{C}$. We conclude that

$$\alpha(X) = -\langle X \cdot v_\alpha, v_\alpha\rangle$$

holds for all $X \in \mathfrak{k}_\mathbb{C}$. As $h_\alpha$ is a $B$-highest weight vector we have $X \cdot h_\alpha = \alpha(X)h_\alpha$ for $X \in \mathfrak{b}$ and hence

$$X \cdot h_\alpha = -\langle X \cdot v_\alpha, v_\alpha\rangle h_\alpha \quad \text{for } X \in \mathfrak{b}.$$

On the other hand

$$(X \cdot h_\alpha)(z) = \frac{d}{dt}\bigg|_{t=0} h_\alpha\big(\exp(-tX)\cdot z\big) = \frac{d}{dt}\bigg|_{t=0} h_\alpha\big(z - tX\cdot z + O(t^2)\big) = \big(\partial_{(-X\cdot z)} h_\alpha\big)(z).$$

So for $X \in \mathfrak{b}$ we obtain

$$\big(\partial_{(-X\cdot v_\alpha)} h_\alpha\big)(v_\alpha) = -\langle X \cdot v_\alpha, v_\alpha\rangle h_\alpha(v_\alpha),$$

or equivalently

$$\big(\partial_w h_\alpha\big)(v_\alpha) = \langle w, v_\alpha\rangle h_\alpha(v_\alpha)$$

for all $w \in \mathfrak{b} \cdot v_\alpha$. But $\mathfrak{b} \cdot v_\alpha = V$ since $v_\alpha$ lies in the open $B$-orbit. □

**Remark 2.6.** For multiplicity free actions $K : V$ of rank greater than one some spherical orbits $\mathcal{K}_\alpha$ typically lie in the complement of the open $B$-orbit. In particular, the generators of $\Lambda$ will not be in the open orbit. The examples given below in Section 5 illustrate this situation. So Proposition 2.5 alone is, in general, insufficient to conclude that a given action $K : V$ is well-behaved.

**Lemma 2.7.** *For a well-behaved multiplicity free action $K : V$ the eigenvalue polynomials $E_\beta$ (see Definition 2.1) have the following properties.*

(a) $E_\beta(\mathcal{K}_\alpha) = \widehat{D}_\beta(\alpha)$ *for all* $\alpha, \beta \in \Lambda$, *and*
(b) $E_\beta(0) = 0$ *for* $\beta \neq 0$.

*Proof.* As $K : V$ is well-behaved we can choose, for each $\alpha \in \Lambda$, a spherical point $v_\alpha$ well-adapted to $h_\alpha$. Note that $E_\beta(\mathcal{K}_\alpha) = E_\beta(v_\alpha)$ by $K$-invariance. Let $t_\beta :=$

$\mathrm{top}(\widehat{D}_\beta) = \bar{\rho}(p_\beta)$ so that $\{t_\beta \ : \ \beta \in \Lambda\}$ is a homogeneous basis for $\mathbb{C}[\mathfrak{a}^*]^{W_\circ}$ and we can write

$$\widehat{D}_\beta = t_\beta + \sum_{|\delta|<|\beta|} c_{\beta,\delta} t_\delta$$

for some coefficients $c_{\beta,\delta} \in \mathbb{C}$. Equivalently

$$\bar{\rho}(E_\beta) = \bar{\rho}\left(p_\beta + \sum_{|\delta|<|\beta|} c_{\beta,\delta} p_\delta\right)$$

and as $\bar{\rho}$ is a vector space isomorphism this gives also

$$E_\beta = p_\beta + \sum_{|\delta|<|\beta|} c_{\beta,\delta} p_\delta.$$

Equation 2.3 yields

$$t_\beta(\alpha) = \bar{\rho}(p_\beta)(\alpha) = \widetilde{p}_\beta\big(z, \eta(z,\alpha)\big)$$

for any $z \in V$ satisfying $h_\alpha(z) \neq 0$. Condition (i) in Definition 2.4 allows us to take $z = v_\alpha$ and condition (ii) gives $\eta(v_\alpha, \alpha) = \langle \cdot, v_\alpha \rangle = v_\alpha^*$. So now

$$t_\beta(\alpha) = \widetilde{p}_\beta(v_\alpha, v_\alpha^*) = p_\beta(v_\alpha)$$

and we obtain

$$\begin{aligned} \widehat{D}_\beta(\alpha) &= t_\beta(\alpha) + \sum_{|\delta|<|\beta|} c_{\beta,\delta} t_\delta(\alpha) \\ &= p_\beta(v_\alpha) + \sum_{|\delta|<|\beta|} c_{\beta,\delta} p_\delta(v_\alpha) \\ &= E_\beta(v_\alpha). \end{aligned}$$

This proves (a) in the statement of Lemma 2.7.

The trivial representation of $K$ occurs in $\mathbb{C}[V]$ on the constant polynomials $\mathcal{P}_0(V) = \mathbb{C}$. This corresponds to the coadjoint orbit $\{0\} \subset \mathfrak{k}^*$ and $v_0 = 0$ is a spherical point in $V$ with $\tau(v_0) = 0$. For $(\beta \neq 0) \in \Lambda$ the differential operator $D_\beta$ is homogeneous of order $|\beta| > 0$ and hence annihilates constants. So $\widehat{D}_\beta(0) = 0$ and (a) implies

$$E_\beta(0) = E_\beta(v_0) = \widehat{D}_\beta(0) = 0,$$

completing the proof. □

## 3. Spherical functions on the Heisenberg group

**3.1. The algebra $\mathbb{D}_K(H_V)$.** Using an orthonormal basis to identify $V$ with $\mathbb{C}^n$ the Lie algebra $\mathfrak{h}_V$ for $H_V$ has basis

$$\{Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n, T\}$$

where

$$Z_j = 2\frac{\partial}{\partial \overline{z}_j} + i\frac{z_j}{2}\frac{\partial}{\partial t}, \qquad \overline{Z}_j = 2\frac{\partial}{\partial z_j} - i\frac{\overline{z}_j}{2}\frac{\partial}{\partial t}, \qquad T = -i\frac{\partial}{\partial t},$$

satisfy $[Z_j, \overline{Z}_j] = 2T$. Given a polynomial $p \in \mathbb{C}[V_{\mathbb{R}}]$ we let $p(Z, \overline{Z})$ denote the left-invariant differential operator

$$p(Z, \overline{Z}) = \sum c_{\mathbf{a},\mathbf{b}} Z_1^{a_1} \cdots Z_n^{a_n} \overline{Z}_1^{b_1} \cdots \overline{Z}_n^{b_n},$$

where $p(z, \overline{z}) = \sum c_{\mathbf{a},\mathbf{b}} z^{\mathbf{a}} \overline{z}^{\mathbf{b}} = \sum c_{\mathbf{a},\mathbf{b}} z_1^{a_1} \cdots z_n^{a_n} \overline{z}_1^{b_1} \cdots \overline{z}_n^{b_n}$ is an expression for $p$ in coordinates with respect to the chosen basis.

As $(K, H_V)$ is a Gelfand pair the algebra $\mathbb{D}_K(H_V)$ of left-$H_V$ and $K$-invariant differential operators on $H_V$ is abelian with generating set

$$\left\{ L_\beta := (-1)^{|\beta|} p_\beta(Z, \overline{Z}) \ : \ \beta \in \Lambda \right\} \cup \{T\}.$$

3.2. **The space $\Delta(K, H_V)$.** A smooth $K$-invariant function $\varphi : H_V \to \mathbb{C}$ is said to be $K$-*spherical* if it is a joint eigenfunction for the operators $L \in \mathbb{D}_K(H_V)$,

$$(L\varphi)(z, t) = \widehat{L}(\varphi)\varphi(z, t),$$

with $\varphi(0, 0) = 1$. The space $\Delta(K, H_V)$ is the set of *bounded* $K$-spherical functions endowed with the compact-open topology.

The bounded $K$-spherical functions are of two types [BJR92].

- **Type 1**: For each $\lambda \in \mathbb{R}^\times$ and $\alpha \in \Lambda$ one has $\phi_{\lambda,\alpha} \in \Delta(K, H_V)$ given by

$$\phi_{\lambda,\alpha}(z, t) = q_\alpha \left( \frac{|\lambda||z|^2}{2} \right) e^{-|\lambda||z|^2/4} e^{i\lambda t}$$

  where $q_\alpha \in \mathbb{C}[V_{\mathbb{R}}]^K$ is a certain $K$-invariant polynomial on $V_{\mathbb{R}}$ with $\mathrm{top}(q_\alpha) = (1/d_\alpha) p_\alpha$.
- **Type 2**: For each $K$-orbit $K \cdot w \in V/K$ one has $\eta_{K\cdot w} \in \Delta(K, H_V)$ given by

$$\eta_{K\cdot w}(z, t) = \int_K e^{iRe\langle w, kz \rangle} \, dk.$$

The differential operators $L_\beta = (-1)^{|\beta|} p_\beta(Z, \overline{Z})$ and $T = -i\partial/\partial t$ take the following eigenvalues on $\Delta(K, H_V)$:

$$\widehat{L}_\beta(\phi_{\lambda,\alpha}) = (2|\lambda|)^{|\beta|} \widehat{D}_\beta(\alpha), \qquad \widehat{T}(\phi_{\lambda,\alpha}) = \lambda, \tag{3.1}$$

$$\widehat{L}_\beta(\eta_{K\cdot w}) = p_\beta(w), \qquad \widehat{T}(\eta_{K\cdot w}) = 0. \tag{3.2}$$

(See [BR98, §4] and [BJRW96, Lemma 3.2] for the calculation of $\widehat{L}_\beta(\phi_{\lambda,\alpha})$ and $\widehat{L}_\beta(\eta_{K\cdot w})$ respectively.)

**Remark 3.1.** It is known that the eigenvalues $\widehat{D}_\beta(\alpha)$ are non-negative real numbers [BR98]. So Equation 3.1 shows that $\widehat{L}_\beta(\phi_{\lambda,\alpha})$ is a non-negative real number. This motivates the factor of $(-1)^{|\beta|}$ in our definition of $L_\beta$. In [BJRW96] and elsewhere operators $L_p \in \mathbb{D}_K(H_V)$ are instead defined for given $p \in \mathbb{C}[V_\mathbb{R}]^K$ via $L_p = \mathrm{Sym}\big(p(Z,\overline{Z})\big)$, the *symmetrization* of $p(Z,\overline{Z})$. These symmetrized operators are canonical in the sense that they do not depend on the choice of orthonormal basis used to identify $V$ with $\mathbb{C}^n$. The simple relationship between the eigenvalues $\widehat{L}_\beta(\phi_{\lambda,\alpha})$ and $\widehat{D}_\beta(\alpha)$ given by Equation 3.1 explains our preference for *Wick ordered* operators $p(Z,\overline{Z})$ in this paper. Identities relating the eigenvalues for Wick, anti-Wick, and symmetrized operators can be found in [BR98].

3.3. **The Heisenberg fan $\mathcal{F}_\mathcal{L}(K,H_V)$.** [BJRW96, FR07]. Let $\mathcal{L} = \{L_1,\ldots,L_m\}$ be a finite set of generators for the algebra $\mathbb{D}_K(H_V)$. It is well known that the mapping

$$\big(\widehat{\mathcal{L}} := \widehat{L}_1 \times \cdots \times \widehat{L}_m\big) : \Delta(K,H_V) \to \mathbb{C}^m \tag{3.3}$$

is a homeomorphism onto its image, the *Heisenberg fan*

$$\mathcal{F}_\mathcal{L}(K,H_V) \subset \mathbb{C}^m.$$

If each $L_j$ is formally self-adjoint, then $\widehat{\mathcal{L}}$ embeds $\Delta(K,H_V)$ in $\mathbb{R}^m$. One such generating set is $\mathcal{L} = \big\{L_{\alpha_1},\ldots,L_{\alpha_r},T\big\}$ where $\{\alpha_1,\ldots,\alpha_r\}$ are the fundamental highest weights in $\Lambda$. By selecting other generating sets one produces equivalent fan models for the space $\Delta(K,H_V)$, potentially embedded in larger Euclidean spaces.

## 4. The orbital model $\mathcal{A}(K,H_V)$ for $\Delta(K,H_V)$

For $(z,t) \in \mathfrak{h}_V$, let $\ell_{(z,t)} \in \mathfrak{h}_V^*$ denote the linear functional

$$\ell_{(z,t)}(z',t') = \mathrm{Im}\langle z,z'\rangle + tt'.$$

We will use the $U(V)$-equivariant isomorphism

$$\mathfrak{h}_v \to \mathfrak{h}_V^*, \quad (z,t) \mapsto \ell_{(z,t)}$$

to identify $\mathfrak{h}_V^*$ with $\mathfrak{h}_V$ and $\mathfrak{h}_V^*/K$ with $\mathfrak{h}_V/K$. In particular, points in $\mathfrak{h}_V^*$ will be written as pairs $(z,t)$.

In [BR08] the mapping $\Psi$ defined by Equation 1.1 is given an alternate description which is less conceptual but more useful for purposes of computation. For Gelfand pairs $(K,H_V)$ we may adopt this alternate formulation as a definition.

**Definition 4.1.** [BR08] Let

$$\Psi : \Delta(K,H_V) \to \mathfrak{h}_V^*/K$$

be the mapping defined via

$$\Psi\big(\phi_{\lambda,\alpha}\big) = \mathcal{K}_{\lambda,\alpha} := \sqrt{2|\lambda|}\,\mathcal{K}_\alpha \times \{\lambda\} \qquad (\lambda \in \mathbb{R}^\times,\ \alpha \in \Lambda),$$
$$\Psi\big(\eta_{K\cdot w}\big) = (K\cdot w) \times \{0\},$$

where $\mathcal{K}_\alpha$ is as in Definition 2.3. (i.e. $\tau(\mathcal{K}_\alpha) = \mathcal{O}_\alpha$.)

In [BR08] we showed that $\Psi$ is injective. Let $\mathcal{A}(K, H_V)$ denote the image of $\Psi$ endowed with the subspace topology from the quotient topology on $\mathfrak{h}_V^*/K$. We conjecture that $\Psi$ is a homeomorphism $\Delta(K, H_V) \cong \mathcal{A}(K, H_V)$. Our main result, Theorem 1.2, shows that this is indeed the case provided that the multiplicity free action $K : V$ is well-behaved. The following result plays an essential role in our proof.

**Proposition 4.2.** *If $K : V$ is well-behaved then for each differential operator $L \in \mathbb{D}_K(H_V)$ there is a continuous $K$-invariant function $\varepsilon_L \in C(\mathfrak{h}_V^*)^K$ satisfying*

$$\varepsilon_L\big(\Psi(\varphi)\big) = \widehat{L}(\varphi)$$

*for all $\varphi \in \Delta(K, H_V)$.*

*Proof.* It suffices to prove this for the operators $L_\beta = (-1)^{|\beta|} p_\beta(Z, \overline{Z})$ $(\beta \in \Lambda)$ and $T = -i\partial/\partial t$, as these generate $\mathbb{D}_K(H_V)$. The eigenvalues $\widehat{L}(\varphi)$ for these operators are given by Equations 3.1, 3.2. Clearly

$$\varepsilon_T(z, t) = t$$

is continuous, $K$-invariant and satisfies

$$\begin{aligned} \varepsilon_T\big(\Psi(\phi_{\lambda,\alpha})\big) &= \varepsilon_T(\mathcal{K}_{\lambda,\alpha}) = \lambda = \widehat{T}(\phi_{\lambda,\alpha}), \\ \varepsilon_T\big(\Psi(\eta_{K\cdot w})\big) &= \varepsilon_T\big((K \cdot w) \times \{0\}\big) = 0 = \widehat{T}(\eta_{K\cdot w}). \end{aligned}$$

For $\beta \in \Lambda$, we take $\varepsilon_{L_\beta} = \varepsilon_\beta$ where

$$\varepsilon_\beta(z, t) = \begin{cases} (2|t|)^{|\beta|} E_\beta\left(z/\sqrt{2|t|}\right) & \text{for } t \neq 0 \\ p_\beta(z) & \text{for } t = 0 \end{cases} \tag{4.1}$$

and $E_\beta \in \mathbb{C}[V_\mathbb{R}]^K$ is the eigenvalue polynomial given by Definition 2.1. As $\deg(E_\beta) = 2|\beta|$ and $\mathrm{top}(E_\beta) = p_\beta$ (Lemma 2.2), we have

$$\lim_{t \to 0} \varepsilon_\beta(z, t) = p_\beta(z) = \varepsilon_\beta(z, 0)$$

and this limit is uniform on compact sets in $z$. Hence $\varepsilon_\beta$ is a continuous function. The map $\varepsilon_\beta$ is moreover $K$-invariant since both $E_\beta$ and $p_\beta$ are $K$-invariant. Finally we compute

$$\begin{aligned} \varepsilon_\beta\big(\Psi(\phi_{\lambda,\alpha})\big) &= \varepsilon_\beta\big(\mathcal{K}_{\lambda,\alpha}\big) = (2|\lambda|)^{|\beta|} E_\beta(\mathcal{K}_\alpha) = (2|\lambda|)^{|\beta|} \widehat{D}_\beta(\alpha) = \widehat{L}_\beta\big(\phi_{\lambda,\alpha}\big), \\ \varepsilon_\beta\big(\Psi(\eta_{K\cdot w})\big) &= \varepsilon_\beta\big(K \cdot w \times \{0\}\big) = p_\beta(w) = \widehat{L}_\beta\big(\eta_{K\cdot w}\big), \end{aligned}$$

in view of Lemma 2.7(a) and Equations 3.1, 3.2. ∎

**Example 4.3.** Suppose now that the multiplicity free action $K : V$ is well-behaved and consider the operator $L_\circ \in \mathbb{D}_K(H_V)$ given by

$$L_\circ = -\sum_{j=1}^{n} Z_j \overline{Z}_j = -p_\circ(Z, \overline{Z}) \tag{4.2}$$

where

$$p_\circ(z) = |z|^2 = \sum_{|\beta|=1} p_\beta(z).$$

The corresponding operator $D_\circ \in \mathcal{PD}(V)^G$ is $D_\circ = -p_\circ(z, \partial)$ where

$$p_\circ(z, \partial) = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}$$

is the degree operator. Lemma 2.2 together with Lemma 2.7(b) show that the polynomial $E_\circ \in \mathbb{C}[V_\mathbb{R}]^K$ has $\deg(E_\circ) = 2$, $\mathrm{top}(E_\circ) = p_\circ$ and $E_\circ(0) = 0$. As $\mathcal{P}_1(V_\mathbb{R})^K = \{0\}$ it follows that $E_\circ = p_\circ$. Equation 4.1 now shows that the function $\varepsilon_\circ = \varepsilon_{L_\circ}$ is simply

$$\varepsilon_\circ(z, t) = p_\circ(z) = |z|^2. \tag{4.3}$$

We remark that $-\mathrm{Sym}(L_\circ)$ is the usual Heisenberg *sub-Laplacian.*

4.1. **Proof of Theorem 1.2.** Assume that the multiplicity free action $K : V$ is well-behaved. Let $(\varphi_n)_{n=1}^\infty$ be a sequence in $\Delta(K, H_V)$, $\varphi \in \Delta(K, H_V)$, and write $\mathcal{K}_n = \Psi(\varphi_n)$, $\mathcal{K} = \Psi(\varphi)$. We will show that

$$\varphi_n \longrightarrow \varphi \text{ in } \Delta(K, H_V) \iff \mathcal{K}_n \longrightarrow \mathcal{K} \text{ in } \mathfrak{h}_V^*/K.$$

The Heisenberg fan model shows that $\varphi_n \longrightarrow \varphi$ if and only if $\widehat{L}(\varphi_n) \longrightarrow \widehat{L}(\varphi)$ for every $L \in \mathbb{D}_K(H_V)$. So it suffices to prove that

$$\mathcal{K}_n \longrightarrow \mathcal{K} \iff \varepsilon_L(\mathcal{K}_n) \longrightarrow \varepsilon_L(\mathcal{K}) \text{ for every } L \in \mathbb{D}_K(H_V),$$

in view of Proposition 4.2. Continuity of the functions $\varepsilon_L$ ensures that if $\mathcal{K}_n \longrightarrow \mathcal{K}$ then $\varepsilon_L(\mathcal{K}_n) \longrightarrow \varepsilon_L(\mathcal{K})$. It remains to prove the converse.

Assume now that $\varepsilon_L(\mathcal{K}_n) \longrightarrow \varepsilon_L(\mathcal{K})$ for each $L \in \mathbb{D}_K(H_V)$ and choose points $(z_n, \lambda_n) \in \mathcal{K}_n$, $(z, \lambda) \in \mathcal{K}$. As $\varepsilon_T(z_n, \lambda_n) = \lambda_n$ and $\varepsilon_T(z, \lambda) = \lambda$ it follows that

$$\lambda_n \longrightarrow \lambda.$$

Applying $\varepsilon_\circ = \varepsilon_{L_\circ}$, with $L_\circ = -Z \cdot \overline{Z}$ as in Equation 4.2, one concludes that

$$|z_n|^2 \longrightarrow |z|^2,$$

in view of Equation 4.3. So $(z_n)_{n=1}^\infty$ is, in particular, bounded. By passing to a subsequence if necessary we can assume that $(z_n)_{n=1}^\infty$ converges and write

$$\lim z_n = z_\circ$$

say. Now $\mathcal{K}_n = K \cdot (z_n, \lambda_n)$ converges to $K \cdot (z_\circ, \lambda) = Kz_\circ \times \{\lambda\}$ whereas $\mathcal{K} = K \cdot (z, \lambda) = Kz \times \{\lambda\}$. To complete the proof we will show that $Kz_\circ = Kz$.

For each $\beta \in \Lambda$ we have $\varepsilon_\beta(z_{\mathfrak{n}}, \lambda_n) \longrightarrow \varepsilon_\beta(z, \lambda)$ since $\varepsilon_\beta(\mathcal{K}_n) \longrightarrow \varepsilon_\beta(\mathcal{K})$ by hypothesis. On the other hand as $(z_n, \lambda_n) \longrightarrow (z_\circ, \lambda)$ we have $\varepsilon_\beta(z_{\mathfrak{n}}, \lambda_n) \longrightarrow \varepsilon_\beta(z_\circ, \lambda)$ by continuity of $\varepsilon_\beta$ and hence

$$\varepsilon_\beta(z_\circ, \lambda) = \varepsilon_\beta(z, \lambda) \text{ for every } \beta \in \Lambda.$$

If $\lambda = 0$ Equation 4.1 now yields

$$p_\beta(z_\circ) = p_\beta(z) \text{ for every } \beta \in \Lambda.$$

It follows that $Kz_\circ = Kz$ since $\{p_\beta : \beta \in \Lambda\}$ is a basis for $\mathbb{C}[V_\mathbb{R}]^K$ and the invariants for a compact linear action separate orbits [OV90, Theorem 3.4.3]. When $\lambda \neq 0$ Equation 4.1 shows

$$E_\beta\big(z_\circ/\sqrt{2|\lambda|}\big) = E_\beta\big(z/\sqrt{2|\lambda|}\big) \text{ for every } \beta \in \Lambda.$$

As the set $\{E_\beta : \beta \in \Lambda\}$ is a basis for $\mathbb{C}[V_\mathbb{R}]^K$ it again follows that $Kz_\circ = Kz$. ☐

4.2. **The models $\mathcal{A}(K, H_V)$ and $\mathcal{F}_\mathcal{L}(K, H_V)$.** We conclude this section by relating the orbital model $\mathcal{A}(K, H_V)$ for $\Delta(K, H_V)$ to that given by the Heisenberg fan $\mathcal{F}_\mathcal{L}(K, H_V)$. (See (3.3).) Suppose that $K : V$ is well-behaved, choose a set $\mathcal{L} = \{L_1, \ldots, L_m\}$ of generators for $\mathbb{D}_K(H_V)$, and let

$$\big(\mathcal{E}_\mathcal{L} := \varepsilon_{L_1} \times \cdots \times \varepsilon_{L_m}\big) : \mathfrak{h}_V^* \to \mathbb{C}^m.$$

Proposition 4.2 shows that the diagram

$$\begin{array}{ccccc} \Delta(K, H_V) & \xrightarrow{\Psi} & \mathcal{A}(K, H_V) & \hookrightarrow & \mathfrak{h}_V^*/K \\ & \searrow^{\widehat{\mathcal{L}}} & \downarrow \mathcal{E}_\mathcal{L} & & \\ & & \mathcal{F}_\mathcal{L}(K, H_V) & \hookrightarrow & \mathbb{C}^m \end{array}$$

commutes and we have seen that both $\Psi$ and $\widehat{\mathcal{L}}$ are homeomorphisms. This proves:

**Corollary 4.4.** *If $K : V$ is well-behaved then the map $\mathcal{E}_\mathcal{L} : \mathcal{A}(K, H_V) \to \mathcal{F}_\mathcal{L}(K, H_V)$ is a homeomorphism from the orbital model $\mathcal{A}(K, H_V)$ to the Heisenberg fan $\mathcal{F}_\mathcal{L}(K, H_V)$.*

## 5. Hermitian symmetric spaces

Let $G/K$ be a Hermitian symmetric space of non-compact type and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denote the Cartan decomposition for the Lie algebra of $G$. The real vector space $\mathfrak{p}$ inherits a complex Hermitian structure and $K$ acts unitarily on $\mathfrak{p}$ via $Ad$. The action $K : \mathfrak{p}$ is, in fact, multiplicity free [Joh80]. Here we will prove that $K : \mathfrak{p}$ is well-behaved (Theorem 1.3), so that $\Delta(K, H_\mathfrak{p})$ is homeomorphic to $\mathcal{A}(K, H_\mathfrak{p})$ via $\Psi$. We will, moreover, identify the spherical orbits in this setting (Proposition 4.2 below), rendering the model $\mathcal{A}(K, H_\mathfrak{p})$ relatively explicit. A concrete example is given at the end of this section. The reader may wish to study this in parallel with the general theory.

If $G/K$ is a reducible Hermitian symmetric space then $K : \mathfrak{p}$ splits as a product action. So we assume henceforth that $G/K$ is irreducible. The classification of irreducible Hermitian symmetric spaces [Hel78, Chapter X] shows that, up to geometric equivalence, $K : \mathfrak{p}$ is one of the following.

$$\left\{ \begin{array}{c|c|c} \big(U(n)\times U(m)\big) : \big(\mathbb{C}^n \otimes \mathbb{C}^m\big) & U(n) : S^2(\mathbb{C}^n) & U(n) : \Lambda^2(\mathbb{C}^n) \\ \hline \big(SO(n)\times \mathbb{T}\big) : \mathbb{C}^n & \big(Spin(10)\times\mathbb{T}\big) : \mathbb{C}^{16} & \big(E_6 \times \mathbb{T}\big) : \mathbb{C}^{27} \end{array} \right\}$$

5.1. **Structure theory.** We require some facts concerning the structure theory for irreducible Hermitian symmetric space of non-compact type. [Hel78, Chapter VIII] is a standard reference for this material.

$G$ is a connected non-compact simple Lie group and $K$ a maximal compact subgroup with center $Z(K) \cong \mathbb{T}$. The complex structure $J : \mathfrak{p} \to \mathfrak{p}$ (with $J^2 = -I$) on $\mathfrak{p}$ is given by $J = ad(Z_\circ)|_{\mathfrak{p}}$ for a distinguished element $Z_\circ \in Z(\mathfrak{k})$. The map $J$ extends to a complex-linear map on the complexification of $\mathfrak{p}$ which decomposes as

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$$

where $\mathfrak{p}_\pm$ are the $(\pm i)$-eigenspaces for $J : \mathfrak{p}_{\mathbb{C}} \to \mathfrak{p}_{\mathbb{C}}$. The complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ now decomposes as

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$$

where

$$[\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_\pm] \subset \mathfrak{p}_\pm, \quad [\mathfrak{p}_+, \mathfrak{p}_+] = 0 = [\mathfrak{p}_-, \mathfrak{p}_-], \quad [\mathfrak{p}_+, \mathfrak{p}_-] = \mathfrak{k}_{\mathbb{C}}.$$

The map $T_+ : \mathfrak{p} \to \mathfrak{p}_+$ given by

$$T_+(X) = \frac{1}{2}\big(X - iJ(X)\big) \tag{5.1}$$

is an isomorphism of complex vector spaces intertwining the adjoint actions of $K$ on $\mathfrak{p}$ and $\mathfrak{p}_+$. We proceed to work with the action $K : \mathfrak{p}_+$ in place of $K : \mathfrak{p}$.

The real subspace $\mathfrak{u} := \mathfrak{k} + i\mathfrak{p}$ is a compact real form in $\mathfrak{g}_{\mathbb{C}}$. We let $c_{\mathfrak{u}} : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ denote the conjugation mapping with respect to this real form.[4] Letting $B$ denote the Killing form,

$$\langle X, Y\rangle = B_{\mathfrak{u}}(X, Y) := -B(X, c_{\mathfrak{u}}(Y)) \tag{5.2}$$

is a positive definite Hermitian inner product on $\mathfrak{g}_{\mathbb{C}}$ and the adjoint action of $K$ on $\big(\mathfrak{p}_+, \langle\cdot,\cdot\rangle\big)$ is unitary.

As in Section 2 we choose a maximal torus $T$ in $K$, let $\mathfrak{t} := \mathrm{Lie}(T)$ and $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}}$, a Cartan subalgebra in $\mathfrak{k}_{\mathbb{C}}$. It is a crucial fact that $\mathfrak{h}$ is also a Cartan subalgebra for $\mathfrak{g}_{\mathbb{C}}$. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots for $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{h}$ and consider the (one-dimensional) root spaces

$$\mathfrak{g}^\delta := \big\{ X \in \mathfrak{g} \ : \ [H, X] = \delta(H)X \text{ for all } H \in \mathfrak{h} \big\} \quad (\delta \in \Delta).$$

[4] In [Hel78] and elsewhere this conjugation is written as $\tau$ but this conflicts with our notation for the moment map.

One has $c_{\mathfrak{u}}(\mathfrak{g}^\delta) = \mathfrak{g}^{-\delta}$. As $B(\mathfrak{g}^\delta, \mathfrak{g}^{\delta'}) = 0$ when $\delta + \delta' \neq 0$ it follows that the root spaces are orthogonal with respect to $\langle \cdot, \cdot \rangle$.

Each root space lies in either $\mathfrak{k}_{\mathbb{C}}$ or $\mathfrak{p}_{\mathbb{C}}$. We let

$$C := \{\delta \in \Delta \ : \ \mathfrak{g}^\delta \subset \mathfrak{k}_{\mathbb{C}}\}, \qquad Q := \{\delta \in \Delta \ : \ \mathfrak{g}^\delta \subset \mathfrak{p}_{\mathbb{C}}\},$$

and call the roots in $C$ and $Q$ compact and non-compact respectively. One can choose an ordering for the roots, to produce sets $\Delta^\pm$ of positive/negative roots, in such a way that

$$\mathfrak{p}_\pm = \bigoplus_{\delta \in Q^\pm} \mathfrak{g}^\delta$$

where $Q^\pm := Q \cap \Delta^\pm$. Letting $C^\pm := C \cap \Delta^\pm$, $\mathfrak{n}_\pm := \bigoplus_{\delta \in C^\pm} \mathfrak{g}^\delta$ we now have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}} = \big(\mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-\big) \oplus \big(\mathfrak{p}_+ \oplus \mathfrak{p}_-\big)$$

and $\mathfrak{b} = \mathfrak{b}_+ := \mathfrak{h} \oplus \mathfrak{n}_+$ is a Borel subalgebra in $\mathfrak{k}_{\mathbb{C}}$.

A pair of roots $\delta, \delta' \in \Delta$ are said to be *strongly orthogonal* when neither $\delta + \delta'$ nor $\delta - \delta'$ is a root. Using our fixed choice of root ordering one inductively constructs a maximal ordered set of strongly orthogonal non-compact positive roots

$$\big(\Gamma := \{\delta_1, \ldots, \delta_r\}\big) \subset Q^+$$

as follows: Let $\delta_1$ be the lowest root in $Q^+$, and having chosen $\delta_1, \ldots, \delta_j$ let $\delta_{j+1}$ be the lowest root in $Q^+ \backslash \{\delta_1, \ldots, \delta_j\}$, strongly orthogonal to each of $\delta_1, \ldots, \delta_j$. We let $\Sigma \subset \mathfrak{p}_+$ be the subspace

$$\Sigma := \bigoplus_{\gamma \in \Gamma} \mathfrak{g}^\gamma = \mathfrak{g}^{\delta_1} \oplus \cdots \oplus \mathfrak{g}^{\delta_r}.$$

The value $r = |\Gamma| = \dim(\Sigma)$ is the *rank* of the symmetric space $G/K$. In fact $\mathfrak{a} := T_+^{-1}(\Sigma)$ is a maximal abelian subalgebra of $\mathfrak{p}$ and the rank is, by definition, the dimension of such a subalgebra.

**Remark 5.1.** $\Sigma$ is a cross section to the $K$-orbits in $\mathfrak{p}_+$. Equivalently, $\mathfrak{a}$ is a cross section to the $K$-orbits in $\mathfrak{p}$. As observed in [Kob07] this follows from the decomposition $G = KAK$. In fact $\mathfrak{a}$ is a *slice* for the action $K : \mathfrak{p}$ in the sense of [Kob05]. See also [Sas09].

5.2. **Decomposition of $\mathbb{C}[\mathfrak{p}_+]$.** Let $\Gamma = \{\delta_1, \ldots, \delta_r\}$ and $\mathfrak{b} = \mathfrak{b}_+$ be as above. The following result is due to Kenneth Johnson.

**Theorem 5.2.** [Joh80] *The action $K : \mathfrak{p}_+$ is multiplicity free and the fundamental highest weights occurring in $\mathbb{C}[\mathfrak{p}_+]$, relative to the Borel subalgebra $\mathfrak{b}$, are*

$$\{\alpha_j := -(\delta_1 + \cdots + \delta_j) \ : \ 1 \leq j \leq r\}$$

*Moreover, the representation in $\mathbb{C}[\mathfrak{p}_+]$ with highest weight $\alpha_j$ occurs in degree $|\alpha_j| = j$.*

Thus the rank of the multiplicity free action $K : \mathfrak{p}_+$ agrees with the rank of the symmetric space $G/K$ and

$\Lambda = \mathbb{Z}^+\text{-Span}(\alpha_1, \dots, \alpha_r) = \big\{-(k_1\delta_1 + \cdots + k_r\delta_r) \ : \ k_1 \geq k_2 \geq \cdots \geq k_r \geq 0, \ k_j \in \mathbb{Z}\big\}$

is the set of highest weights occurring in $\mathbb{C}[\mathfrak{p}_+]$.

5.3. **The moment map for $K : \mathfrak{p}_+$.** We now write

$$Q^+ = \big\{\delta_1, \dots, \delta_r, \delta_{r+1} \dots, \delta_n\big\},$$

where $\Gamma = \{\delta_1, \dots, \delta_r\}$ is as above, and choose root vectors $X_j \in \mathfrak{g}^{\delta_j}$ with $\langle X_j, X_j\rangle = 1$. As the root spaces are orthogonal, $\{X_1, \dots, X_n\}$ is an orthonormal basis for $\mathfrak{p}_+$ and $\{X_1, \dots, X_r\}$ is an orthonormal basis for the subspace $\Sigma$.

**Lemma 5.3.** *The restriction to $\Sigma$ of the moment map $\tau : \mathfrak{p}_+ \to \mathfrak{k}^*$ for the action $K : \mathfrak{p}_+$ is given by*

$$\tau\left(\sum_{j=1}^r c_j X_j\right) = i \sum_{j=1}^r |c_j|^2 \delta_j.$$

The right hand side of this formula belongs to $\mathfrak{h}^*$. We regard this as a $\mathbb{C}$-linear functional on all of $\mathfrak{k}_\mathbb{C}$ by extending as zero on $\mathfrak{n}_+ \oplus \mathfrak{n}_- = \oplus_{\delta\in C}\mathfrak{g}^\delta = \mathfrak{h}^\perp \cap \mathfrak{k}_\mathbb{C}$. The restriction of this linear functional from $\mathfrak{k}_\mathbb{C}$ to $\mathfrak{k}$ is real valued because each root $\delta \in \Delta$ takes pure imaginary values on $\mathfrak{t}$. This accounts for the factor of $i$ in the formula.

*Proof.* By definition, the moment map is given by

$$\tau(v)(A) = i\langle A \cdot v, v\rangle = -iB\big([A, v], c_\mathfrak{u}(v)\big)$$

for $A \in \mathfrak{k}$, $v \in \mathfrak{p}_+$. As usual we extent this to a $\mathbb{C}$-linear map on $\mathfrak{k}_\mathbb{C}$. This just amounts to applying the formula above with $A \in \mathfrak{k}_\mathbb{C} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_- = \mathfrak{h} \oplus \bigoplus_{\delta\in C} \mathfrak{g}^\delta$. So now

$$\frac{1}{i}\tau\left(\sum_{j=1}^r c_j X_j\right)(A) = \sum_{j,j'} c_j \overline{c_{j'}} \langle A \cdot X_j, X_{j'}\rangle.$$

But

$$\langle A \cdot X_j, X_{j'}\rangle = -B\big([A, X_j], c_\mathfrak{u}(X_{j'})\big) = -B\big([X_j, c_\mathfrak{u}(X_{j'})], A\big)$$

by a basic property of the Killing form. Here $c_\mathfrak{u}(X_{j'}) \in \mathfrak{g}^{-\delta_{j'}}$ so $[X_j, c_\mathfrak{u}(X_{j'})] \in [\mathfrak{g}^{\delta_j}, \mathfrak{g}^{-\delta_{j'}}]$. If $j \neq j'$ then $[\mathfrak{g}^{\delta_j}, \mathfrak{g}^{-\delta_{j'}}] = 0$ since $\delta_j - \delta_{j'}$ is not a root by strong orthogonality of $\Gamma$. Thus in fact

$$\frac{1}{i}\tau\left(\sum_{j=1}^r c_j X_j\right)(A) = \sum_{j=1}^r |c_j|^2 \langle A \cdot X_j, X_j\rangle.$$

(a) If $A \in \mathfrak{h}$ then $A \cdot X_j = \delta_j(A) X_j$ and we obtain

$$\frac{1}{i}\tau\left(\sum_{j=1}^r c_j X_j\right)(A) = \sum_{j=1}^r |c_j|^2 \delta_j(A)\langle X_j, X_j\rangle = \sum_{j=1}^r |c_j|^2 \delta_j(A)$$

(b) If $A \in \mathfrak{n}_+ \oplus \mathfrak{n}_-$ then $\langle A \cdot X_j, X_j \rangle = 0$ and hence $(1/i)\tau\left(\sum_{j=1}^r c_j X_j\right)(A) = 0$. Indeed $\langle A \cdot X_j, X_j \rangle = -B\big([X_j, c_{\mathfrak{u}}(X_j)], A\big)$ and $[X_j, c_{\mathfrak{u}}(X_j)] \in [\mathfrak{g}^{\delta_j}, \mathfrak{g}^{-\delta_j}] \subset \mathfrak{h}$. As $\mathfrak{h}$ is orthogonal to $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ with respect to the Killing form we obtain $\langle A \cdot X_j, X_j \rangle = 0$ as stated. □

5.4. **Spherical points for the action $K : \mathfrak{p}_+$.**

**Proposition 5.4.** *Let $\alpha \in \Lambda$ be given by $\alpha = -(k_1\delta_1 + \cdots + k_r\delta_r)$ where $k_1, \ldots, k_r \in \mathbb{Z}$ satisfy $k_1 \geq k_2 \geq \cdots \geq k_r \geq 0$. Then*

$$v_\alpha := \sqrt{k_1} X_1 + \cdots + \sqrt{k_r} X_r$$

*is a spherical point for $\alpha$.*

*Proof.* The point $v_\alpha$ lies in the subspace $\Sigma$. In view of Lemma 5.3 we have

$$\tau(v_\alpha) = i \sum_{j=1}^r k_j \delta_j = -i\alpha = \alpha_{\mathfrak{k}}. \quad \square$$

The following is equivalent to Theorem 1.3, stated in the Introduction to this paper.

**Theorem 5.5.** *The multiplicity free action $K : \mathfrak{p}_+$ is well-behaved.*

*Proof.* Let $(z_1, \ldots, z_n)$ be the coordinate functions on $\mathfrak{p}_+$ with respect to the orthonormal basis $\{X_1, \ldots, X_n\}$. Now $\mathbb{C}[\mathfrak{p}_+] = \mathbb{C}[z_1, \ldots, z_n]$ and each monomial

$$m_{\mathbf{I}}(z) = z^{\mathbf{I}} := z_1^{i_1} \cdots z_n^{i_n}$$

is a weight vector for $\mathfrak{h}$ with weight $-(i_1\delta_1 + \cdots + i_n\delta_n)$. A fundamental highest weight vector $h_j := h_{\alpha_j}$ $(1 \leq j \leq r)$ is thus a linear combination of monomials $m_{\mathbf{I}}$ each with weight $\alpha_j = -(\delta_1 + \cdots + \delta_j)$. That is,

$$h_j(z) = \sum c_{\mathbf{I}} m_{\mathbf{I}}(z)$$

for some scalars $c_{\mathbf{I}}$ where the sum is over all $\mathbf{I} = (i_1, \ldots, i_n)$ with $i_1 + \cdots + i_n = j$ and

$$i_1\delta_1 + \cdots + i_n\delta_n = \delta_1 + \cdots + \delta_j.$$

One such monomial is

$$m_j(z) := m_{(1^j, 0^{n-j})}(z) = z_1 \cdots z_j.$$

Note that the restriction $m_{\mathbf{I}}|_\Sigma$ of a monomial to $\Sigma = \{(z_1, \ldots, z_r, 0, \ldots, 0)\}$ vanishes unless $i_p = 0$ for each $p > r$. As $\{\delta_1, \ldots, \delta_r\}$ are linearly independent (they are strongly orthogonal) we conclude that $m_j$ is the only monomial with weight $\alpha_j$ whose restriction to $\Sigma$ is non-zero. Thus we have

$$h_j(z) = c_j m_j(z) + s_j(z)$$

for some $c_j \in \mathbb{C}$ where $s_j|_\Sigma = 0$. Moreover we must have $h_j|_\Sigma \neq 0$ since the open $B$-orbit meets $\Sigma$. Indeed, $v_\circ := X_1 + \cdots + X_r$ is a point in $\Sigma$ with $\mathfrak{b} \cdot v_\circ = \mathfrak{p}_+$. So $c_j \neq 0$ here and replacing $h_j$ by $(1/c_j)h_j$ we can assume

$$h_j(z) = m_j(z) + s_j(z) = z_1 \cdots z_j + s_j(z) \quad \text{where} \quad s_j|_\Sigma = 0.$$

Now fix a weight $\alpha \in \Lambda$, and write

$$\alpha = a_1\alpha_1 + \cdots a_r\alpha_r = -\big(k_1\delta_1 + \cdots + k_r\delta_r\big)$$

where $a_1, \ldots, a_r \in \mathbb{Z}^+$ and

$$k_j := a_j + \cdots + a_r \quad (1 \leq j \leq r).$$

Proposition 5.4 shows that $v_\alpha := \sqrt{k_1}X_1 + \cdots + \sqrt{k_r}X_r$ is a spherical point for $\alpha$. An associated highest weight vector $h_\alpha \in \mathbb{C}[\mathfrak{p}_+]$ is given by

$$h_\alpha := h_1^{a_1} \cdots h_r^{a_r} = m_1^{a_1} \cdots m_r^{a_r} + s_\alpha$$

where

$$s_\alpha = \prod_{j=1}^{r} \left( \sum_{\ell_j=1}^{a_j} \binom{a_j}{\ell_j} m_j^{a_j-\ell_j} s_j^{\ell_j} \right) = \sum_{\ell_1=1}^{a_1} \sum_{\ell_2=1}^{a_2} \cdots \sum_{\ell_r=1}^{a_r} \binom{a_1}{\ell_1} \cdots \binom{a_r}{\ell_r} m_1^{a_1-\ell_1} \cdots m_r^{a_r-\ell_r} s_1^{\ell_1} \cdots s_r^{\ell_r}. \tag{5.3}$$

We will show that the spherical point $v_\alpha$ ($\alpha \in \Lambda$) is well-adapted to the highest weight vector $h_\alpha$ in the sense of Definition 2.4.

Equation 5.3 shows that $s_\alpha|_\Sigma = 0$. As $v_\alpha$ lies in $\Sigma$ this gives

$$h_\alpha(v_\alpha) = m_1(v_\alpha)^{a_1} \cdots m_r(v_\alpha)^{a_r} = k_1^{k_1/2} \cdots k_r^{k_r/2}.$$

In particular, $h_\alpha(v_\alpha) \neq 0$, verifying condition (i) in Definition 2.4.

To establish condition (ii) in Definition 2.4 it suffices to show that

$$\big(\partial_i h_\alpha\big)(v_\alpha) = \langle X_i, v_\alpha \rangle h_\alpha(v_\alpha)$$

for $1 \leq i \leq n$ where $\partial_i := \partial_{X_i} = \partial/\partial z_i$. As

$$\langle X_i, v_\alpha \rangle = \begin{cases} \sqrt{k_i} & \text{if } i \leq r \\ 0 & \text{if } i > r \end{cases}$$

we must check that

(ii-a) $\big(\partial_i h_\alpha\big)(v_\alpha) = \sqrt{k_i}\, h_\alpha(v_\alpha)$ for $1 \leq i \leq r$, and
(ii-b) $\big(\partial_i h_\alpha\big)(v_\alpha) = 0$ for $i > r$.

We will now assume that $\big(\partial_i s_\alpha\big)|_\Sigma = 0$, to be proved in Lemma 5.6 below.

First suppose that $1 \leq i \leq r$. We have

$$\big(\partial_i h_\alpha\big)(z) = \partial_i\big(z_1^{k_1} \cdots z_r^{k_r} + s_\alpha(z)\big).$$

As $v_\alpha \in \Sigma$ and $\big(\partial_i s_\alpha\big)|_\Sigma = 0$ one obtains

$$\big(\partial_i h_\alpha\big)(v_\alpha) = k_i k_1^{k_1/2} \cdots k_i^{(k_i-1)/2} \cdots k_r^{k_r/2} = \sqrt{k_i}\,\big(k_1^{k_1/2} \cdots k_r^{k_r/2}\big) = \sqrt{k_i}\, h_\alpha(v_\alpha)$$

as required for (ii-a). Next let $i > r$. Now

$$\big(\partial_i h_\alpha\big)(z) = \partial_i\big(z_1^{k_1} \cdots z_r^{k_r} + s_\alpha(z)\big) = \big(\partial_i s_\alpha\big)(z).$$

As $v_\alpha \in \Sigma$ and $\big(\partial_i s_\alpha\big)|_\Sigma = 0$ this gives $\big(\partial_i h_\alpha\big)(v_\alpha) = 0$ as required for (ii-b). □

**Lemma 5.6.** $\big(\partial_i s_\alpha\big)|_\Sigma = 0$.

*Proof.* Applying the product rule to terms in the summation formula for $s_\alpha$, given in Equation 5.3, and using the fact that $s_1|_\Sigma = \cdots = s_r|_\Sigma = 0$ we see that it suffices to prove that

$$\big(\partial_i s_j\big)|_\Sigma = 0 \text{ for all } 1 \le j \le r \text{ and all } 1 \le i \le n. \tag{5.4}$$

Let $j$ be fixed ($1 \le j \le r$). The polynomial $s_j$ is a linear combination of monomials $m_{\mathbf{L}}(z) = z_1^{\ell_1} \cdots z_n^{\ell_n}$ where

(a) $\ell_1 + \cdots + \ell_n = j$,
(b) $\ell_1\delta_1 + \cdots + \ell_n\delta_n = \delta_1 + \cdots + \delta_j$, and
(c) $\ell_p > 0$ for at least one index $p > r$.

In view of condition (c) we have that $\big(\partial_i m_{\mathbf{L}}\big)|_\Sigma = 0$ for all such monomials $m_{\mathbf{L}}$ when $i \le r$. Thus $\big(\partial_i s_j\big)|_\Sigma = 0$ for $i \le r$.

Next suppose that $i > r$. For $m_{\mathbf{L}}$ as above the restriction of $\partial_i m_{\mathbf{L}}$ to $\Sigma = \{(z_1, \ldots, z_r, 0 \ldots, 0)\}$ can be non-zero only if $m_{\mathbf{L}}$ has the form

$$m_{\mathbf{L}}(z) = z_1^{\ell_1} \cdots z_r^{\ell_r} z_i.$$

We claim that no such monomials appear in the expression for $s_j$ and hence that $\big(\partial_i s_j\big)|_\Sigma = 0$. Indeed condition (b) above gives $\ell_1\delta_1 + \cdots + \ell_r\delta_r + \delta_i = \delta_1 + \cdots + \delta_j$ and hence

$$\delta_i = (1-\ell_1)\delta_1 + \cdots + (1-\ell_j)\delta_j - \ell_{j+1}\delta_{j+1} - \cdots - \ell_r\delta_r.$$

In particular $\delta_i$ is an integral linear combination of $\Gamma = \{\delta_1, \ldots, \delta_r\}$. But this contradicts the *Restricted Roots Theorem.* [Hel94, Proposition V.4.8]. In more detail, this goes as follows. The restriction of the Killing form to $i\mathfrak{t}$ is a positive definite inner product. For each $\gamma \in \Gamma$ let $\widetilde{H}_\gamma \in i\mathfrak{t}$ be the unique vector with $B(\cdot, \widetilde{H}_\gamma) = \gamma|_{i\mathfrak{t}}$ and set

$$\big(\mathfrak{t}^- := \mathbb{R}\text{-Span}\big(i\widetilde{H}_\gamma \;:\; \gamma \in \Gamma\big)\big) \subset \mathfrak{t}.$$

The roots $\gamma \in \Gamma$ remain linearly independent upon restriction to $\mathfrak{t}^-$ and the Restricted Roots Theorem asserts, in part, that the restriction $\delta|_{\mathfrak{t}^-}$ of any positive non-compact root $\delta \in Q^+ \backslash \Gamma$ to $\mathfrak{h}^-$ must belong to

$$\left\{\frac{1}{2}\delta_1|_{\mathfrak{t}^-}, \ldots, \frac{1}{2}\delta_r|_{\mathfrak{t}^-}\right\} \cup \left\{\frac{1}{2}(\delta_p + \delta_q)|_{\mathfrak{t}^-} \;:\; 1 \le p < q \le r\right\}.$$

Thus $\delta_i$ cannot be expressed as an integral linear combination of $\Gamma$. $\square$

5.5. **An Example.** We will illustrate our results using the Hermitian symmetric space $G/K = SU(r,r)/S(U(r) \times U(r))$. We write matrices $X \in M_{2r}(\mathbb{C})$ in block form as $X = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ $\big(A, B, C, D \in M_r(\mathbb{C})\big)$.

The Lie algebra $\mathfrak{g} = su(r,r)$ for $G = SU(r,r)$,

$$\mathfrak{g} = \left\{ \left[\begin{array}{c|c} A & B \\ \hline B^* & D \end{array}\right] \ : \ A^* = -A,\ D^* = -D,\ tr(A) + tr(D) = 0 \right\} \quad (A^* := \overline{A^t}),$$

carries the Cartan involution $\theta(X) = -X^*$, yielding $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where

$$\mathfrak{k} = s(u(r) \times u(r)) = \left\{ \left[\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right] \ : \ A^* = -A,\ D^* = -D,\ tr(A) + tr(D) = 0 \right\},$$
$$\mathfrak{p} = \left\{ \left[\begin{array}{c|c} 0 & B \\ \hline B^* & 0 \end{array}\right] \ : \ B \in M_r(\mathbb{C}) \right\}.$$

The matrix $Z_\circ := \frac{1}{2}\left[\begin{array}{c|c} iI & 0 \\ \hline 0 & -iI \end{array}\right]$ spans $Z(\mathfrak{k})$ and $J := ad(Z_\circ)|_{\mathfrak{p}}$ gives the complex structure on $\mathfrak{p}$, namely $J\left(\left[\begin{array}{c|c} 0 & B \\ \hline B^* & 0 \end{array}\right]\right) = \left[\begin{array}{c|c} 0 & iB \\ \hline -iB^* & 0 \end{array}\right]$. The $(\pm i)$-eigenspaces $\mathfrak{p}_\pm$ for the extension of $J$ to $\mathfrak{p}_{\mathbb{C}} = \left\{ \left[\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right] \ : \ B, C \in M_r(\mathbb{C}) \right\}$ are

$$\mathfrak{p}_+ = \left\{ \left[\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right] \ : \ B \in M_r(\mathbb{C}) \right\}, \quad \mathfrak{p}_- = \left\{ \left[\begin{array}{c|c} 0 & 0 \\ \hline C & 0 \end{array}\right] \ : \ C \in M_r(\mathbb{C}) \right\},$$

and the map $T_+$, given in Equation 5.1, is here simply $T_+\left(\left[\begin{array}{c|c} 0 & B \\ \hline B^* & 0 \end{array}\right]\right) = \left[\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right]$.

We have $\mathfrak{g}_{\mathbb{C}} = sl(2r, \mathbb{C})$, $\mathfrak{k}_{\mathbb{C}} = s(gl(r,\mathbb{C}) \times gl(r,\mathbb{C}))$ and the compact real form $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ is now $\mathfrak{u} = su(2r)$. The conjugation mapping $c_{\mathfrak{u}} : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{u}$ is $c_{\mathfrak{u}}(Z) = -Z^*$ and Equation 5.2 for the Hermitian inner product on $\mathfrak{p}_+$ becomes

$$\left\langle \left[\begin{array}{c|c} 0 & B_1 \\ \hline 0 & 0 \end{array}\right], \left[\begin{array}{c|c} 0 & B_2 \\ \hline 0 & 0 \end{array}\right] \right\rangle = -tr\left( ad\left[\begin{array}{c|c} 0 & B_1 \\ \hline 0 & 0 \end{array}\right] ad\left[\begin{array}{c|c} 0 & 0 \\ \hline -B_2^* & 0 \end{array}\right]\right) = 4r\, tr(B_1 B_2^*).$$

Hence the mapping

$$\left[\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right] \mapsto \frac{1}{2\sqrt{r}} B$$

is an isometry from $(\mathfrak{p}_+, \langle \cdot, \cdot \rangle)$ to $M_r(\mathbb{C})$ with its standard Hermitian inner product. Under this identification the adjoint action of $K = S(U(r) \times U(r))$ on $\mathfrak{p}_+$ becomes

$$(u_1, u_2) \cdot Z = u_1 Z u_2^{-1} = u_1 Z u_2^*.$$

Twisting by the automorphism $u_2 \mapsto \big((u_2^t)^{-1} = \overline{u_2}\big)$ on the second factor gives the standard action $(u_1, u_2) \cdot Z = u_1 Z u_2^t$. Thus $K : \mathfrak{p}_+$ is geometrically equivalent to $(U(r) \times U(r)) : (\mathbb{C}^r \otimes \mathbb{C}^r)$, one of the classical multiplicity free actions.

Let $T \subset K$ and $(\mathfrak{h} := \mathfrak{t}_{\mathbb{C}}) \subset \mathfrak{g}_{\mathbb{C}}$ be the usual maximal torus and Cartan subalgebra, consisting of diagonal matrices. Letting $\varepsilon_i \in \mathfrak{h}^*$ denote the functional $\varepsilon_i\big(diag(z_1, \ldots, z_{2r})\big) := z_i$ we have

$$\text{roots: } \Delta = \{\varepsilon_i - \varepsilon_j \;:\; 1 \le i, j \le 2r,\; i \ne j\}, \quad \text{root spaces: } \mathfrak{g}^{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{ij}.$$

For $1 \le j \le r$ let $\varepsilon_j' := \varepsilon_{r+j}$. Each functional on the real vector space $i\mathfrak{t}$ can be uniquely written as

$$(a_r\varepsilon_r + \cdots + a_1\varepsilon_1) + (b_1\varepsilon_1' + \cdots + b_r\varepsilon_r')$$

for some scalars $a_j, b_j \in \mathbb{R}$ with $a_r + \cdots + a_1 + b_1 + \cdots + b_r = 0$. We use lexicographic ordering on the coordinates $(a_r, \ldots, a_1; b_1, \ldots, b_r)$ to impose an ordering on $(i\mathfrak{t})^*$. This convention yields positive compact roots

$$C^+ = \{\varepsilon_i - \varepsilon_j \;:\; 1 \le j < i \le r\} \cup \{\varepsilon_i' - \varepsilon_j' \;:\; 1 \le i < j \le r\}$$

so that $\mathfrak{b} = \mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$ becomes

$$\mathfrak{b} = \left\{ \left[\begin{array}{c|c} L & 0 \\ \hline 0 & U \end{array}\right] \in \mathfrak{k}_{\mathbb{C}} \;:\; L \text{ lower triangular, } U \text{ upper triangular} \right\}.$$

Moreover the positive non-compact roots are

$$Q^+ = \{\varepsilon_i - \varepsilon_j' \;:\; 1 \le i, j \le 2r\},$$

and $\sum_{\delta \in Q^+} \mathfrak{g}^\delta = \mathfrak{p}_+$ as required.

The inductive construction of a maximal ordered set of strongly orthogonal non-compact positive roots, working from our root ordering, yields $\Gamma = \{\delta_1, \ldots, \delta_r\}$ where

$$\delta_j := \varepsilon_j - \varepsilon_j'.$$

Identifying $\mathfrak{p}_+ \cong M_r(\mathbb{C})$ the slice $\Sigma := \oplus_{j=1}^r \mathfrak{g}^{\delta_j}$ becomes the set of diagonal matrices. Our symmetric space $G/K$ and action $K : \mathfrak{p}_+$ have rank $r$.

Theorem 5.2 gives fundamental $\mathfrak{b}$-highest weights

$$\alpha_j = (\varepsilon_1' + \cdots + \varepsilon_j') - (\varepsilon_1 + \cdots + \varepsilon_j) \qquad (1 \le j \le r)$$

and the set of all highest weights occurring in $\mathbb{C}[\mathfrak{p}_+]$ is

$$\Lambda = \{(k_1\varepsilon_1' + \cdots + k_r\varepsilon_r') - (k_1\varepsilon_1 + \cdots + k_r\varepsilon_r) \;:\; k_1 \ge \cdots \ge k_r \ge 0\}.$$

Identifying $\mathfrak{p}_+ \cong M_r(\mathbb{C})$ the Borel subalgebra $\mathfrak{b}$ acts via

$$(L, U) \cdot Z = LZ - ZU \qquad (L \text{ lower triangular, } U \text{ upper triangular})$$

and one can verify that the leading minor determinant

$$h_j(Z) = \begin{vmatrix} z_{11} & \cdots & z_{1j} \\ \vdots & & \vdots \\ z_{j1} & \cdots & z_{jj} \end{vmatrix}$$

is a $\mathfrak{b}$-highest weight vector in $\mathbb{C}[M_r(\mathbb{C})]$ with weight $\alpha_j$.

For $\alpha = (k_1\varepsilon_1' + \cdots + k_r\varepsilon_r') - (k_1\varepsilon_1 + \cdots + k_r\varepsilon_r) \in \Lambda$ the spherical point $v_\alpha \in \Sigma$, furnished by Proposition 5.4, is

$$v_\alpha = diag\big(\sqrt{k_1}, \ldots, \sqrt{k_r}\big)$$

and we have the $K$-spherical orbit

$$\mathcal{K}_\alpha = \Big\{u_1\, diag\big(\sqrt{k_1}, \ldots, \sqrt{k_r}\big)\, u_2 \ : \ u_1, u_2 \in U(r)\Big\}$$

in $M_r(\mathbb{C})$. On the other hand the open Borel orbit in $M_r(\mathbb{C})$ is the set of all non-singular matrices which admit an $LU$-factorization. These are precisely the matrices $Z \in M_r(\mathbb{C})$ whose leading minor determinants $h_j(Z)$ are all non-zero. The point $v_\alpha$ lies in this open Borel orbit if and only if $k_r \neq 0$ and $\mathcal{K}_\alpha$ is entirely contained in the complement of the open Borel orbit whenever $k_r = 0$. See Remark 2.6.

## References


[ADBR07] Francesca Astengo, Bianca Di Blasio, and Fulvio Ricci. Gelfand transforms of polyradial Schwartz functions on the Heisenberg group. *J. Funct. Anal.*, 251(2):772–791, 2007.

[ADBR09] Francesca Astengo, Bianca Di Blasio, and Fulvio Ricci. Gelfand pairs on the Heisenberg group and Schwartz functions. *J. Funct. Anal.*, 256(5):1565–1587, 2009.

[BJLR97] Chal Benson, Joe Jenkins, Ronald L. Lipsman, and Gail Ratcliff. A geometric criterion for Gelfand pairs associated with the Heisenberg group. *Pacific J. Math.*, 178(1):1–36, 1997.

[BJR90] Chal Benson, Joe Jenkins, and Gail Ratcliff. On Gel′fand pairs associated with solvable Lie groups. *Trans. Amer. Math. Soc.*, 321(1):85–116, 1990.

[BJR92] Chal Benson, Joe Jenkins, and Gail Ratcliff. Bounded $K$-spherical functions on Heisenberg groups. *J. Funct. Anal.*, 105(2):409–443, 1992.

[BJRW96] Chal Benson, Joe Jenkins, Gail Ratcliff, and Tefera Worku. Spectra for Gelfand pairs associated with the Heisenberg group. *Colloq. Math.*, 71(2):305–328, 1996.

[Bou81] Philippe Bougerol. Théorème central limite local sur certains groupes de Lie. *Ann. Sci. École Norm. Sup. (4)*, 14(4):403–432 (1982), 1981.

[BR96] Chal Benson and Gail Ratcliff. A classification of multiplicity free actions. *J. Algebra*, 181(1):152–186, 1996.

[BR98] Chal Benson and Gail Ratcliff. Combinatorics and spherical functions on the Heisenberg group. *Represent. Theory*, 2:79–105 (electronic), 1998.

[BR04] Chal Benson and Gail Ratcliff. On multiplicity free actions. In *Representations of real and p-adic groups*, volume 2 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 221–304. Singapore Univ. Press, Singapore, 2004.

[BR08] Chal Benson and Gail Ratcliff. The space of bounded spherical functions on the free 2-step nilpotent Lie group. *Transform. Groups*, 13(2):243–281, 2008.

[Car87] Giovanna Carcano. A commutativity condition for algebras of invariant functions. *Boll. Un. Mat. Ital. B (7)*, 1(4):1091–1105, 1987.

[DP96] Andrzej Daszkiewicz and Tomasz Przebinda. On the moment map of a multiplicity free action. *Colloq. Math.*, 71(1):107–110, 1996.

[FH87] Jacques Faraut and Khélifa Harzallah. *Deux cours d'analyse harmonique*, volume 69 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1987. Papers from the Tunis summer school held in Tunis, August 27–September 15, 1984.

[FR07] Fabio Ferrari Ruffino. The topology of the spectrum for Gelfand pairs on Lie groups. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, 10(3):569–579, 2007.

[Hel78] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 80 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.

[Hel94] Sigurdur Helgason. *Geometric analysis on symmetric spaces*, volume 39 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1994.

[HU91] Roger Howe and Tōru Umeda. The Capelli identity, the double commutant theorem, and multiplicity-free actions. *Math. Ann.*, 290(3):565–619, 1991.

[Joh80] Kenneth D. Johnson. On a ring of invariant polynomials on a Hermitian symmetric space. *J. Algebra*, 67(1):72–81, 1980.

[Kac80] V. G. Kac. Some remarks on nilpotent orbits. *J. Algebra*, 64(1):190–213, 1980.

[Kno98] Friedrich Knop. Some remarks on multiplicity free spaces. In *Representation theories and algebraic geometry (Montreal, PQ, 1997)*, volume 514 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 301–317. Kluwer Acad. Publ., Dordrecht, 1998.

[Kob05] Toshiyuki Kobayashi. Multiplicity-free representations and visible actions on complex manifolds. *Publ. Res. Inst. Math. Sci.*, 41(3):497–549, 2005.

[Kob07] Toshiyuki Kobayashi. Visible actions on symmetric spaces. *Transform. Groups*, 12(4):671–694, 2007.

[Lea98] Andrew S. Leahy. A classification of multiplicity free representations. *J. Lie Theory*, 8(2):367–391, 1998.

[Lip80] Ronald L. Lipsman. Orbit theory and harmonic analysis on Lie groups with co-compact nilradical. *J. Math. Pures Appl. (9)*, 59(3):337–374, 1980.

[Lip82] Ronald L. Lipsman. Orbit theory and representations of Lie groups with co-compact radical. *J. Math. Pures Appl. (9)*, 61(1):17–39, 1982.

[OV90] A. L. Onishchik and È. B. Vinberg. *Lie groups and algebraic groups*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990. Translated from the Russian and with a preface by D. A. Leites.

[Puk78] L. Pukanszky. Unitary representations of Lie groups with cocompact radical and applications. *Trans. Amer. Math. Soc.*, 236:1–49, 1978.

[Sas09] Atsumu Sasaki. Visible actions on irreducible multiplicity-free spaces. *Int. Math. Res. Not. IMRN*, (18):3445–3466, 2009.

[Str91] Robert S. Strichartz. $L^p$ harmonic analysis and Radon transforms on the Heisenberg group. *J. Funct. Anal.*, 96(2):350–406, 1991.

[Vin86] È. B. Vinberg. Complexity of actions of reductive groups. *Funktsional. Anal. i Prilozhen.*, 20(1):1–13, 96, 1986.

[Wil92] N. J. Wildberger. The moment map of a Lie group representation. *Trans. Amer. Math. Soc.*, 330(1):257–268, 1992.

Dept of Mathematics, East Carolina University, Greenville, NC 27858
*E-mail address*: bensonf@ecu.edu, ratcliffg@ecu.edu