THE SPACE OF BOUNDED SPHERICAL FUNCTIONS ON THE FREE TWO STEP NILPOTENT LIE GROUP

CHAL BENSON AND GAIL RATCLIFF

Abstract. Let N be a connected and simply connected 2-step nilpotent Lie group and K be a compact subgroup of $Aut(N)$. We say that (K, N) is a Gelfand pair when the set of integrable K-invariant functions on N forms an abelian algebra under convolution. In this paper, we construct a one-to-one correspondence between the set $\Delta(K, N)$ of bounded spherical functions for such a Gelfand pair and a set $\mathcal{A}(K, N)$ of K-orbits in the dual $\mathfrak{n}^*$ of the Lie algebra for N. The construction involves an application of the Orbit Method to spherical representations of $K \ltimes N$. We conjecture that the correspondence $\Delta(K, N) \leftrightarrow \mathcal{A}(K, N)$ is a homeomorphism. Our main result shows that this is the case for the Gelfand pair given by the action of the orthogonal group on the free 2-step nilpotent Lie group. In addition, we show how to embed the space $\Delta(K, N)$ for this example in a Euclidean space by taking eigenvalues for an explicit set of invariant differential operators. These results provide geometric models for the space of bounded spherical functions on the free 2-step group.

1. Introduction

This paper concerns the topological structure of spectra for Gelfand pairs that arise in analysis on nilpotent Lie groups. Suppose that N is a connected and simply connected nilpotent Lie group and that K is a compact Lie group acting smoothly on N via automorphisms. We say that (K, N) is a *Gelfand pair* when the algebra $L^1_K(N)$ of integrable K-invariant functions on N is commutative under convolution. It is shown in [BJR90] that when (K, N) is a Gelfand pair, N is necessarily 2-step (or abelian). The possibilities have been completely classified for the cases where N is a Heisenberg group [BR96], [Lea98]. Gelfand pairs of the sort (K, N) where N is a not a Heisenberg group are classified, subject to certain hypotheses, in [Vin01, Vin03] and [Yak05, Yak04]. Examples can also be found in [KR83], [Ric85], [Car87], [BJR90] and [Lau00]. Analysis in the non-Heisenberg setting has, however, not as yet been highly developed.

Consider the algebra $\mathbb{D}_K(N)$ of differential operators on N that are simultaneously invariant under left multiplication by N and under the action of K. It is known that $\mathbb{D}_K(N)$ is abelian whenever (K, N) is a Gelfand pair. In this case, a smooth function ϕ on N is said to be K-spherical if

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- ϕ is K-invariant,
- ϕ is an eigenfunction for all $D \in \mathbb{D}_K(N)$, and
- $\phi(e) = 1$, where $e \in N$ denotes the identity element.

We let $\Delta(K,N)$ denote the set of all *bounded* K-spherical functions for the Gelfand pair (K,N). One can identify $\Delta(K,N)$ with the Gelfand space (or spectrum) of the commutative Banach $\star$-algebra $L^1_K(N)$ via integration against spherical functions $\phi \in \Delta(K,N)$. The compact-open topology on $\Delta(K,N)$ (uniform convergence on compact sets) corresponds to the weak*-topology on the Gelfand space.

Below we introduce a correspondence between $\Delta(K,N)$ and a set $\mathcal{A}(K,N)$ of K-orbits in the dual $\mathfrak{n}^*$ of $\mathfrak{n}$ (Definition 1.3), which we call *K-spherical orbits*. The correspondence $\Delta(K,N) \leftrightarrow \mathcal{A}(K,N)$ is motivated by the *Orbit Method* in Representation Theory, which says that irreducible unitary representations of a Lie group should correspond to coadjoint orbits in the dual of its Lie algebra.

Let $G = K \ltimes N$ be the semidirect product of K with N. Now $L^1_K(N)$ coincides with $L^1(K\backslash G/K)$, the K-bi-invariant functions on G, via restriction of functions on G to N. So (K,N) is a Gelfand pair if and only if $L^1(K\backslash G/K)$ is abelian. Equivalently the space of K-fixed vectors for any irreducible unitary representation of G is at most one-dimensional [Gel50]. Theorem 1.1 below provides an orbital counterpart to this representation-theoretic criterion. Here we assume N is 2-step and identify $\mathfrak{n}^*$ with the annihilator of $\mathfrak{k}$ in $\mathfrak{g}^*$. The intersection $\mathcal{O} \cap \mathfrak{n}^*$ of any $Ad^*(G)$-orbit $\mathcal{O} \subset \mathfrak{g}^*$ with $\mathfrak{n}^*$ is K-saturated, i.e. a union of K-orbits.

Theorem 1.1. ([BJR99, Nis01]) *(K,N) is a Gelfand pair if and only if every coadjoint orbit in $\mathfrak{g}^*$ meets $\mathfrak{n}^*$ in at most one K-orbit.*

It is shown in [BJR99] that the orbit condition in Theorem 1.1 holds whenever (K,N) is a Gelfand pair. The converse is proved in [Nis01]. The result for Heisenberg groups was obtained first in [BJLR97].

There is an Orbit Method, due to Lipsman [Lip80, Lip82] and Pukanszky [Puk78], for semidirect products of compact with nilpotent groups. We discuss aspects of this below in Section 3, here specialized to $G = K \ltimes N$ where N is 2-step. The theory produces a well-defined coadjoint orbit $\mathcal{O}(\rho) \subset \mathfrak{g}^*$ for each irreducible unitary representation ρ of G. In this context, the orbit mapping

$$\widehat{G} \to \mathfrak{g}^*/Ad^*(G), \quad \rho \mapsto \mathcal{O}(\rho)$$

is, in general, finite-to-one, a fact which will require our subsequent attention.

Now suppose that (K,N) is a Gelfand pair and let $\widehat{G}_K$ denote the *K-spherical representations* of G:

$$\widehat{G}_K = \{\rho \in \widehat{G} \ : \ \rho \text{ has a 1-dimensional space of } K\text{-fixed vectors}\}.$$

The following proposition is proved in Section 5.2.

Proposition 1.2. *$\mathcal{O}(\rho) \cap \mathfrak{n}^* \neq \emptyset$ for each $\rho \in \widehat{G}_K$.*

Proposition 1.2 together with Theorem 1.1 show that for each $\rho \in \widehat{G}_K$ the intersection

$$\mathcal{K}(\rho) = \mathcal{O}(\rho) \cap \mathfrak{n}^*$$

is a K-orbit in $\mathfrak{n}^*$.

Definition 1.3. Let $\mathcal{A}(K,N)$ denote the set of K-orbits in $\mathfrak{n}^*$ given by

$$\mathcal{A}(K,N) = \{\mathcal{K}(\rho) \ : \ \rho \in \widehat{G}_K\}.$$

We call these the *K-spherical orbits* for the Gelfand pair (K,N).

In Section 5.4 we will prove the following.

Theorem 1.4. *The map $\mathcal{K} : \widehat{G}_K \to \mathcal{A}(K,N)$ is a bijection.*

The positive definite spherical functions for (K,N) correspond with $\widehat{G}_K$. Given a K-spherical representation, one obtains a spherical function by forming the diagonal matrix coefficient for a K-fixed vector of unit length. Such a spherical function is bounded by 1, its value at the identity element. Conversely it is known that every bounded spherical function for (K,N) is positive definite [BJR90]. Thus we can lift $\mathcal{K}$ to a mapping Ψ on the space $\Delta(K,N)$ of bounded K-spherical functions:

Definition 1.5. $\Psi : \Delta(K,N) \to \mathfrak{n}^*/K$ is defined as

$$\Psi(\phi) = \mathcal{K}(\rho^\phi)$$

where $\rho^\phi \in \widehat{G}_K$ is the K-spherical representation of G that yields ϕ.

The following assertion is now equivalent to Theorem 1.4.

Corollary 1.6. *The map $\Psi : \Delta(K,N) \to \mathcal{A}(K,N)$ is a bijection.*

We give $\mathcal{A}(K,N)$ the subspace topology from $\mathfrak{n}^*/K$. Note that $\mathfrak{n}^*/K$ is metrizable since K is compact. The compact-open topology on $\Delta(K,N)$ corresponds to the Fell topology on $\widehat{G}_K$. It is known that for nilpotent and exponential solvable groups, the Orbit Method provides a homeomorphism between the unitary dual and the space of coadjoint orbits [Bro73], [LL94]. Thus it is natural to conjecture that $\mathcal{K} : \widehat{G}_K \to \mathcal{A}(K,N)$ is a homeomorphism. Equivalently:

Conjecture 1.7. $\Psi : \Delta(K,N) \to \mathcal{A}(K,N)$ *is a homeomorphism*

There is a "degenerate" context in which Conjecture 1.7 is easily verified. This is the situation where $N \cong \mathbb{R}^n$ is abelian, discussed below in Section 6. See also [Wol06]. In this case $\mathcal{A}(K,N) = \mathfrak{n}^*/K$ is the set of all K-orbits in $\mathfrak{n}^*$, with $K \cdot \ell \in \mathfrak{n}^*/K$ corresponding, via Ψ, to the K-average of the unitary character $\chi_\ell(x) = e^{i\ell(x)}$. So Ψ can be viewed as the map obtained from the homeomorphism $\widehat{N} \cong \mathfrak{n}^*$, $\chi_\ell \leftrightarrow \ell$ by passing to K-orbits.

An alternate description of the map Ψ is preferable for purposes of calculation. As explained in Section 5.3, the bounded spherical functions $\phi \in \Delta(K,N)$ can be indexed by pairs of parameters (π, α). Here π and α are irreducible unitary representations of N and of the stabilizer K_π for $\pi \in \widehat{N}$. (The pair (π, α^*) are Mackey parameters for a K-spherical representation of G.) In Section 4 we define a *moment map* $\tau_{\mathcal{O}} : \mathcal{O} \to \mathfrak{k}_\pi^*$ for the action of K_π on the coadjoint orbit $(\mathcal{O} = \mathcal{O}^N(\pi)) \subset \mathfrak{n}^*$ associated to π. We show that the image of $\tau_{\mathcal{O}}$ includes the $Ad^*(K_\pi)$-orbit $\mathcal{O}^{K_\pi}(\alpha)$ associated to the representation $\alpha \in \widehat{K_\pi}$. Moreover one has

$$\Psi(\phi_{\pi,\alpha}) = K \cdot \ell_{\pi,\alpha}$$

where $\ell_{\pi,\alpha}$ denotes any point in $\mathcal{O}$ with $\tau_{\mathcal{O}}(\ell_{\pi,\alpha}) \in \mathcal{O}^{K_\pi}(\alpha)$. See Proposition 5.3 below.

In [BJR90] it is shown that the orthogonal group $O(d)$ acts on the F_d, the *free 2-step nilpotent Lie group* on d generators, to yield a Gelfand pair $(O(d), F_d)$. This example plays an important role in the theory of Gelfand pairs (K, N) since $O(d)$ is maximal compact in $Aut(F_d)$ and any 2-step group can be realized as a quotient of some F_d by a central subgroup. Some results concerning the spherical functions for $(O(d), F_d)$ can be found in [Str91] and [Fis06]. We discuss this example below, in a coordinate-free fashion, beginning in Section 8. Our main result is Theorem 8.1, which asserts that the correspondence $\Delta(O(d), F_d) \leftrightarrow \mathcal{A}(O(d), F_d)$ is indeed a homeomorphism.

There is another approach to constructing topological models for $\Delta(K, N)$. One can use the eigenvalues with respect to some set of operators $D \in \mathbb{D}_K(N)$ to map $\Delta(K, N)$ to a Euclidean space. This technique was used in [Wol92] to embed the spectrum for any Gelfand pair into an infinite dimensional Euclidean space by using all $D \in \mathbb{D}_K(N)$. For the Gelfand pair $(U(n), H_n)$, given by the action of the unitary group $U(n)$ on the Heisenberg group H_n, it suffices to use just two operators, the Heisenberg sub-Laplacian and the central derivative. This yields an embedding of $\Delta(U(n), H_n)$ in $\mathbb{R}^2$ whose image is called "the Heisenberg fan" [Bou81], [Far87], [Str91]. In [BJRW96], the Heisenberg fan construction is generalized to encompass Gelfand pairs of the form (K, H_n) where K is a closed subgroup of $U(n)$. The result is an embedding into a finite dimensional Euclidean space.

Our proof of Theorem 8.1, contained in Section 11, requires first establishing the analogous result for $(U(n), H_n)$. This is done in Section 7 by relating the space of spherical orbits for $(U(n), H_n)$ to the Heisenberg fan. For the case of $(O(d), F_d)$ we show that there is also a direct analog for the fan construction. That is, we describe a finite set of operators $D \in \mathbb{D}_{O(d)}(F_d)$ that can be used to embed $\Delta(O(d), F_d)$ in a finite dimensional Euclidean space. This construction is contained in Sections 9 and 10 below, culminating in Corollary 10.2. In Section 12 we describe our geometric models for $\Delta(O(d), F_d)$ explicitly in the case $d = 3$.

We conclude this overview of our results by listing the Gelfand pairs for which Conjecture 1.7 will be established. These are

- (K, N) with N abelian (Section 6),
- $(K, N) = (U(n), H_n)$ (Section 7),

- $(K, N) = (O(d), F_d)$ (Section 11).

We have also proved Conjecture 1.7 for the pair $(SO(d), F_d)$. In fact, as explained in Section 8, this can be derived as a corollary to the result for $(O(d), F_d)$.

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2. Preliminaries and notation

- Throughout this paper, N denotes a connected and simply connected 2-step nilpotent Lie group. K is a (possible disconnected) compact Lie group acting smoothly on N by automorphisms. We let $k \cdot x$ denote the result of applying $k \in K$ to $x \in N$.
- $G = K \ltimes N$ is the semidirect product, with group law
$$(k, x)(k', x') = (kk', x(k \cdot x')).$$
- A script letter indicates the Lie algebra for a corresponding group. We identify N with its Lie algebra $\mathfrak{n}$ via the exponential map. The derived action of $\mathfrak{k}$ on $\mathfrak{n}$ is written $A \cdot X$ for $A \in \mathfrak{k}$ and $X \in \mathfrak{n}$.
- $\widehat{H}$ denotes the unitary dual of a Lie group H. We identify representations modulo unitary equivalence and make no notational distinction between a representation and its equivalence class.
- The coadjoint actions of a Lie group H and its Lie algebra $\mathfrak{h}$ on $\mathfrak{h}^* = hom(\mathfrak{h}, \mathbb{R})$ are
$$Ad^*(h)\varphi = \varphi \circ Ad(h^{-1}),$$
$$ad^*(X)\varphi(Y) = \varphi \circ ad(-X)(Y) = -\varphi([X, Y])$$
for $h \in H$, $\varphi \in \mathfrak{h}^*$, and $X, Y \in \mathfrak{h}$. When H is nilpotent and is identified with its Lie algebra $\mathfrak{h}$, $Ad^*(X)$ for $X \in \mathfrak{h}$ denotes the coadjoint action of the group H.
- The symbol $\mathcal{O}$ indicates a coadjoint orbit. Given $\sigma \in \widehat{H}$, $\mathcal{O}(\sigma)$ is an associated coadjoint orbit in $\mathfrak{h}^*$. Sometimes we write $\mathcal{O}^H(\sigma)$ to clarify the group in question. We assume familiarity with Kirillov's Orbit Method for nilpotent Lie groups. (See [Kir62], [Kir04] or [CG90].) This establishes a one-to-one correspondence
$$\widehat{N} \leftrightarrow \mathfrak{n}^*/Ad^*(N), \quad \pi \leftrightarrow \mathcal{O}^N(\pi).$$
The Orbit Method for other groups that arise in this paper is discussed in Section 3.
- We will frequently extend linear functionals $\xi \in \mathfrak{h}^*$ from subalgebras $\mathfrak{h}$ of $\mathfrak{k}$ to all of $\mathfrak{k}$. For this purpose we fix at the outset a definite $Ad(K)$-invariant inner product $(\cdot, \cdot)_{\mathfrak{k}}$ on $\mathfrak{k}$. As an element of $\mathfrak{k}^*$, ξ is the unique extension which vanishes on the $(\cdot, \cdot)_{\mathfrak{k}}$-orthogonal complement of $\mathfrak{h}$. For concreteness one can

realize K as a Lie subgroup of a unitary group $U(n)$, via some faithful unitary representation, and use the negative definite inner product

$$(A, B)_{\mathfrak{k}} = tr(AB).$$

- Elements of $\mathfrak{g}^*$ are denoted $\varphi = (\xi, \ell)$, where $\xi \in \mathfrak{k}^*$ and $\ell \in \mathfrak{n}^*$. This means
$$\varphi(A, X) = \xi(A) + \ell(X)$$
for $A \in \mathfrak{k}$, $X \in \mathfrak{n}$. The set $\mathfrak{n}^*$ can be viewed as the subset $\{(0, \ell) \ : \ \ell \in \mathfrak{n}^*\}$ of $\mathfrak{g}^*$, the annihilator of $\mathfrak{k}$ in $\mathfrak{g}^*$, so that $\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{n}^*$.

3. The orbit method for $G = K \ltimes N$

A version of the Orbit Method, due to R. Lipsman [Lip80, Lip82] and L. Pukanszky [Puk78], associates a coadjoint orbit $\mathcal{O}(\rho)$ in $\mathfrak{g}^*$ to each irreducible unitary representation $\rho \in \widehat{G}$ of $G = K \ltimes N$. This construction is described in the current section. We do not, however, require the full strength of this theory, since N is here a 2-step group.

3.1. **Orbit Method for subgroups of K.** Let H be any *connected* Lie subgroup of K . Given an irreducible unitary representation $\nu \in \widehat{H}$ of H we use the highest weight theory for compact connected Lie groups to obtain a coadjoint orbit $\mathcal{O}(\nu) \subset \mathfrak{h}^*$. To begin, choose a maximal torus T in H and a system of positive roots. Let $i\xi : \mathfrak{t} \to i\mathbb{R}$ be the highest weight for ν. Extend $\xi \in \mathfrak{t}^*$ to an element of $\mathfrak{h}^*$ by using the $Ad(K)$-invariant inner product $(\cdot, \cdot)_{\mathfrak{k}}$, as discussed in Section 2. The coadjoint orbit $\mathcal{O}(\nu) \subset \mathfrak{h}^*$ is then defined as

$$\mathcal{O}(\nu) = \mathcal{O}^H(\nu) = Ad^*(H)\xi.$$

The map $\mathcal{O} : \widehat{H} \to \mathfrak{h}^*/Ad^*(H)$ is well-defined and injective.

Note that this approach does *not* incorporate the "ρ-shift" (half the sum of the positive roots) that appears elsewhere in the literature on the Orbit Method for compact groups. (See, for example, Chapter 5 in [Kir04].) The approach described here is better suited to our purposes.

Next suppose H is a *disconnected* Lie subgroup of K and $\alpha \in \widehat{H}$. Let $\nu \in \widehat{H^\circ}$ be an irreducible representation of the identity component H° occurring in the restriction α to H°. We let

$$\mathcal{O}(\alpha) = Ad^*(H)\mathcal{O}^{H^\circ}(\nu)$$

where $O^{H^\circ}(\nu) \subset \mathfrak{h}^*$ is the coadjoint orbit for $\nu \in \widehat{H^\circ}$, as defined above. Equivalently

$$\mathcal{O}(\alpha) = Ad^*(H)\xi$$

where $i\xi$ is *any* highest weight occurring in $\alpha|_{H^\circ}$.

Suppose that ν' is another irreducible representation of H° occurring in $\alpha|_{H^\circ}$. As H° is a normal subgroup of finite index in H, it follows that

$$\nu'(k) = (k_\circ \cdot \nu)(k) = \nu(k_\circ^{-1} k k_\circ)$$

for some $k_\circ \in H$. Hence if $\xi \in \mathcal{O}^{H^\circ}(\nu)$ then $Ad^*(k_\circ)\xi \in \mathcal{O}^{H^\circ}(\nu')$. We conclude that $Ad^*(H)\mathcal{O}^{H^\circ}(\nu) = Ad^*(H)\mathcal{O}^{H^\circ}(\nu')$. This shows that $\mathcal{O}(\alpha)$ is well defined, independent of the choice of $\nu \in \widehat{H^\circ}$ occurring in $\alpha|_{H^\circ}$.

When H is disconnected the orbit correspondence $\mathcal{O} : \widehat{H} \to \mathfrak{h}^*/Ad^*(H)$ is, in general, finite-to-one. That is, finitely many inequivalent representations of H can yield a common coadjoint orbit. For an example of this phenomenon one need only consider the situation when H is a *finite* subgroup of K.

3.2. **Aligned points in $\mathfrak{n}^*$.** Choose a positive definite inner product $(\cdot,\cdot)_\mathfrak{n}$ on $\mathfrak{n}$ that is invariant under the action of K. Let $\mathfrak{z}$ denote the center of $\mathfrak{n}$ and let $\mathcal{V} = \mathfrak{z}^\perp$, so that

$$\mathfrak{n} = \mathcal{V} \oplus \mathfrak{z}.$$

Let $\mathcal{O} \subset \mathfrak{n}^*$ be a coadjoint orbit and choose any point $\ell \in \mathcal{O}$, so that $\mathcal{O} = Ad^*(N)\ell$. Let $B_\mathcal{O}$ be the bilinear form

$$B_\mathcal{O}(X,Y) = \ell([X,Y])$$

on $\mathfrak{n}$ and let

$$\mathfrak{a}_\mathcal{O} = Rad(B_\mathcal{O}) \cap \mathcal{V} = \{X \in \mathcal{V} \ : \ \ell([X,\mathfrak{n}]) = 0\}.$$

As suggested by the notation, $B_\mathcal{O}$ and $\mathfrak{a}_\mathcal{O}$ do not depend on the choice of $\ell \in \mathcal{O}$, since N is 2-step nilpotent. Let

$$\mathfrak{w}_\mathcal{O} = \mathfrak{a}_\mathcal{O}^\perp \cap \mathcal{V}$$

so that

$$\mathfrak{n} = \mathfrak{a}_\mathcal{O} \oplus \mathfrak{w}_\mathcal{O} \oplus \mathfrak{z}. \tag{3.1}$$

Since N is 2-step, we see that the map

$$\mathfrak{w}_\mathcal{O} \to \mathcal{O}, \quad X \mapsto Ad^*(X)\ell = \ell - \ell[X,-]$$

is a homeomorphism. This identification of $\mathfrak{w}_\mathcal{O}$ with $\mathcal{O}$ does, however, depend on the choice of base point $\ell \in \mathcal{O}$. For our subsequent results, it is crucial that one can distinguish a *canonical* base point and use this to obtain a canonical identification $\mathfrak{w}_\mathcal{O} \cong \mathcal{O}$.

Definition 3.1. A point $\ell \in \mathcal{O}$ is said to be *aligned* if $\ell|_{\mathfrak{w}_\mathcal{O}} = 0$.

Lemma 3.2. *$\mathcal{O}$ contains exactly one aligned point.*

Proof. Let ℓ be any point in $\mathcal{O}$. Since $B_\mathcal{O}$ is non-degenerate on $\mathfrak{w}_\mathcal{O}$, we have

$$\ell|_{\mathfrak{w}_\mathcal{O}} = B_\mathcal{O}(X_\circ, -)$$

for some $X_\circ \in \mathfrak{w}_\mathcal{O}$. One checks easily that $\ell_\circ = Ad^*(X_\circ)\ell$ is the unique aligned point in $\mathcal{O}$. □

The compact group K acts on $\mathfrak{n}^*$ via the contragredient of its action on $\mathfrak{n}$:

$$(k \cdot \ell)(X) = \ell(k^{-1} \cdot X).$$

Since K acts by automorphisms on $\mathfrak{n}$, the action of K on $\mathfrak{n}^*$ takes coadjoint orbits to coadjoint orbits. Moreover

$$\mathfrak{a}_{k\cdot\mathcal{O}} = k \cdot \mathfrak{a}_{\mathcal{O}} \quad \text{and} \quad \mathfrak{w}_{k\cdot\mathcal{O}} = k \cdot \mathfrak{w}_{\mathcal{O}}$$

for elements $k \in K$, and coadjoint orbits $\mathcal{O} \subset \mathfrak{n}^*$. The following is now immediate.

Lemma 3.3. *If ℓ is aligned then so is $k \cdot \ell$.*

Now let $K_{\mathcal{O}} \subset K$ denote the stabilizer of the coadjoint orbit $\mathcal{O}$:

$$K_{\mathcal{O}} = \{k \in K \ : \ k \cdot \mathcal{O} = \mathcal{O}\}.$$

The action of $K_{\mathcal{O}}$ on $\mathfrak{n}$ preserves $\mathfrak{a}_{\mathcal{O}}$ and $\mathfrak{w}_{\mathcal{O}}$. Together Lemmas 3.2 and 3.3 imply:

Lemma 3.4. *Let $\ell_{\mathcal{O}}$ be the aligned point in $\mathcal{O}$. Then $K_{\mathcal{O}} = \{k \in K \ : \ k \cdot \ell_{\mathcal{O}} = \ell_{\mathcal{O}}\}$. That is, the stabilizer of a coadjoint orbit coincides with that of its aligned point.*

Our definition of aligned point depends, a priori, on the choice of K-invariant inner product $(\cdot,\cdot)_{\mathfrak{n}}$. Proposition 3.6 below will, however, relate Definition 3.1 to that found in [Lip80]. The latter does not involve a choice of inner product. In particular, we emphasize that the orbit method for G, described next, is independent of the chosen inner product.

3.3. **Coadjoint orbits and representations of G.** Our goal here is to obtain a coadjoint orbit $\mathcal{O}(\rho)$ in $\mathfrak{g}^*$ for each $\rho \in \widehat{G}$. First we recall how the *Mackey machine* describes $\widehat{G}$ in terms of representations of N and subgroups of K.

The group K acts on the unitary dual $\widehat{N}$ of N via

$$k \cdot \pi = \pi \circ k^{-1}$$

for $k \in K$, $\pi \in \widehat{N}$. Let K_π denote the stabilizer of π (up to unitary equivalence). Note that

$$K_\pi = K_{\mathcal{O}}$$

where $\mathcal{O} = \mathcal{O}^N(\pi) \subset \mathfrak{n}^*$ is the coadjoint orbit for π.

Lemma 2.3 in [BJR99] shows that there is a (non-projective) unitary representation

$$W_\pi : K_\pi \to U(\mathcal{H}_\pi)$$

of K_π in the representation space $\mathcal{H}_\pi$ for π that intertwines $k \cdot \pi$ with π:

$$(k \cdot \pi)(x) = W_\pi(k)^{-1}\pi(x)W_\pi(k)$$

for all $k \in K_\pi$, $x \in N$. Given any irreducible unitary representation α of K_π Mackey theory ensures that

$$\rho_{\pi,\alpha} = Ind_{K_\pi \ltimes N}^{K \ltimes N}\Big((k,x) \mapsto \alpha(k) \otimes \pi(x)W_\pi(k)\Big)$$

is an irreducible unitary representation of G. Moreover, up to unitary equivalence, all irreducible unitary representations of G have this form. That is:

$$\widehat{G} = \{\rho_{\pi,\alpha} \ : \ \pi \in \widehat{N},\ \alpha \in \widehat{K_\pi}\}.$$

We say that $\rho = \rho_{\pi,\alpha}$ has *Mackey parameters* (π, α). For our purposes it is important to note that the intertwining representation W_π can be *canonically* chosen, so that the parameters (π, α) completely determine $\rho_{\pi,\alpha}$. Corollary 3.2 in [Lip80] establishes this, via positive polarizations, in the general setting of Lie groups with co-compact nilradical. In the current context this observation amounts to the proof of Lemma 2.3 in [BJR99]. In outline one has the following.

Let ℓ be the aligned point in $\mathcal{O}$ and note that π factors through

$$N_{\mathcal{O}} = exp(\mathfrak{n}/Ker(\ell|_{\mathfrak{z}})).$$

When $\ell|_{\mathfrak{z}} \neq 0$ the group $N_{\mathcal{O}}$ is the product of a Heisenberg group H with the (possibly trivial) abelian group $\mathfrak{a}_{\mathcal{O}}$. Working from the inner product $(\cdot,\cdot)_{\mathfrak{n}}$ one constructs a unitary K_π-space V and and isomorphism φ from H to the standard Heisenberg group $H_V = V \times \mathbb{R}$. (See Section 5.1.) The element ℓ_φ in $\mathfrak{h}_V^*$ which corresponds to ℓ via φ satisfies

$$\ell_\varphi|_V = 0 \qquad \ell_\varphi(0,1) = 1.$$

So $\pi|_H$ can be realized, via φ, as the standard representation of H_V in the Fock space $\mathcal{F}_V$ on V. Thus also W_π is realized, via φ, as the restriction to K_π of the standard representation of $U(V)$ on $\mathcal{F}_V$. The equivalence class of W_π does not depend on the choice of inner product $(\cdot,\cdot)_{\mathfrak{n}}$ used to produce φ.

The coadjoint orbit $\mathcal{O}(\rho) \subset \mathfrak{g}^*$ for $\rho = \rho_{\pi,\alpha}$ is obtained from the Mackey parameters (π, α) as follows.

- Let $\mathcal{O}^N(\pi) \subset \mathfrak{n}^*$ be the coadjoint orbit corresponding to $\pi \in \widehat{N}$ and let ℓ_π denote the unique aligned point in $\mathcal{O}^N(\pi)$. (See Definition 3.1.)
- Let ξ be any point in the coadjoint orbit $\mathcal{O}^{K_\pi}(\alpha)$. (See Section 3.1.) Use the $Ad(K)$-invariant inner product $(\cdot,\cdot)_{\mathfrak{k}}$ on $\mathfrak{k}$ to lift ξ to a linear functional on all of $\mathfrak{k}$.
- Now set

$$\mathcal{O}(\rho) = Ad^*(G)(\xi, \ell_\pi). \tag{3.2}$$

To justify this definition, we will verify that $\mathcal{O}(\rho)$ does not depend on the various choices of data involved in its construction.

Lemma 3.5. *The coadjoint orbit $\mathcal{O}(\rho)$ depends only on ρ (up to unitary equivalence).*

Proof. Lemma 3.4 shows that $K_\pi = K_{\mathcal{O}^N(\pi)}$ coincides with the stabilizer of the aligned point $\ell_\pi \in \mathcal{O}^N(\pi)$:

$$K_\pi = \{k \in K \ : \ k \cdot \ell_\pi = \ell_\pi\}.$$

In addition observe that

$$Ad^*_G(k)(\xi,\ell) = (Ad^*_K(k)\xi, k\cdot\ell)$$

for $k \in K$ and $(\xi,\ell) \in \mathfrak{g}^*$.

- $\mathcal{O}(\rho)$ does not depend on the choice of $\xi \in \mathcal{O}^{K_\pi}(\alpha)$:

  Indeed if $\xi' = Ad^*(k_\circ)\xi$ for some $k_\circ \in K_\pi$ then

  $$(\xi',\ell_\pi) = (Ad^*_K(k_\circ)\xi, k_\circ\cdot\ell_\pi) = Ad^*_G(k_\circ)(\xi,\ell_\pi)$$

  since $k_\circ \cdot \ell_\pi = \ell_\pi$.
- $\mathcal{O}(\rho)$ does not depend on the choice of Mackey parameters (π,α) for ρ:

  Mackey theory dictates that $\rho_{\pi,\alpha} = \rho_{\pi',\alpha'}$ if and only if (π,α) and (π',α') *differ by the action of* K. This means

  $$\pi' = k_\circ\cdot\pi, \quad \alpha' = k_\circ\cdot\alpha$$

  for some $k_\circ \in K$ where

  $$K_{k_\circ\cdot\pi} = k_\circ K_\pi k_\circ^{-1}, \quad (k_\circ\cdot\alpha)(k) = \alpha(k_\circ^{-1}kk_\circ).$$

  We have $\mathcal{O}^N(\pi') = k_\circ\cdot\mathcal{O}^N(\pi)$ and hence

  $$\ell_{\pi'} = k_\circ\cdot\ell_\pi$$

  by Lemma 3.3. Moreover $\mathcal{O}^{K_{k_\circ\cdot\pi}}(\alpha') = Ad^*(k_\circ)\mathcal{O}^{K_\pi}(\alpha)$. Thus if $\xi \in \mathcal{O}^{K_\pi}(\alpha)$ then $\xi' = Ad^*_K(k_\circ)\xi$ is in $\mathcal{O}^{K_{k_\circ\cdot\pi}}(\alpha')$ and finally

  $$(\xi',\ell_{\pi'}) = (Ad^*_K(k_\circ)\xi, k_\circ\cdot\ell_\pi) = Ad^*_G(k_\circ)(\xi,\ell_\pi).$$

□

Note that the orbit correspondence

$$\widehat{G} \to \mathfrak{g}^*/Ad^*(G), \quad \rho \mapsto \mathcal{O}(\rho)$$

is, in general, finite-to-one. In fact $\mathcal{O}(\rho_{\pi,\alpha})$ can arise from more than one representation whenever the stabilizer K_π fails to be connected.

The following proposition relates Definition 3.1 to Lipsman's definition of *aligned point in* $\mathfrak{g}^*$. The point $(\xi,\ell_\pi) \in \mathfrak{g}^*$ in Equation 3.2 is, in particular, aligned in $\mathfrak{g}^*$. This reconciles our description of the orbit mapping $\rho \mapsto \mathcal{O}(\rho)$ with [Lip80, Lip82] and [Puk78].

Proposition 3.6. *Let* $\ell \in \mathfrak{n}^*$ *be aligned and* $\xi \in \mathfrak{k}^*_\ell \subset \mathfrak{k}^*$ *then* $\varphi = (\xi,\ell)$ *is an* aligned point in $\mathfrak{g}^*$ *in the sense of* [Lip80]. *That is,*

$$G_\ell = K_\ell N_\ell, \quad \textit{and} \quad G_\varphi = K_\varphi N_\varphi.$$

Proof. The adjoint action of G on $\mathfrak{g}$ can be written as

(3.3)
$$Ad_G(k,Y)(U,X) = \left(k\cdot U,\quad k\cdot X - (k\cdot U)\cdot Y + [Y, k\cdot X] - \frac{1}{2}[Y,(k\cdot U)\cdot Y]\right)$$

for $k \in K$, $U \in \mathfrak{k}$, $X, Y \in \mathfrak{n}$. Here $k\cdot U = Ad_K(k)U$ and we have identified N with $\mathfrak{n}$.

Let $(k,Y) \in G_\ell$. Applying (3.3) with $U = 0$ yields

$$\ell(k\cdot X) + \ell[Y, k\cdot X] = \ell(X) \quad \text{for all } X \in \mathfrak{n}. \tag{3.4}$$

Equivalently $k^{-1}\cdot(Ad_N^*(Y^{-1})\ell) = \ell$ and in particular, $k\cdot\ell \in Ad_N^*(N)\ell$. As ℓ is aligned this implies $k\cdot\ell = \ell$, in view of Lemmas 3.2 and 3.3. That is $k \in K_\ell$. Moreover (3.4) now becomes

$$\ell[Y, k\cdot X] = 0 \quad \text{for all } X \in \mathfrak{n},$$

which implies that $Y \in N_\ell$. So $G_\ell = K_\ell N_\ell$ as stated.

Next let $(k,Y) \in G_\varphi$. As $G_\varphi \subset G_\ell$ we have $k \in K_\ell$, $Y \in N_\ell$. Now (3.3) with $X = 0$ yields

$$\xi(k\cdot U) - \ell((k\cdot U)\cdot Y) = \xi(U) \quad \text{for all } U \in \mathfrak{k}. \tag{3.5}$$

This implies $\xi(k\cdot U) = \xi(U)$ when $U \in \mathfrak{k}_\ell$. But when $U \in \mathfrak{k}_\ell^\perp$ (orthogonal complement with respect to a definite $Ad(K)$-invariant inner product on $\mathfrak{k}$) we have $\xi(k\cdot U) = 0 = \xi(U)$, since $\xi \in \mathfrak{k}_\ell^* \subset \mathfrak{k}^*$. So

$$\xi(k\cdot U) = \xi(U) \quad \text{for all } U \in \mathfrak{k}. \tag{3.6}$$

From this it is easy to see that $k \in K_\varphi$. Moreover (3.5) and (3.6) together now give

$$\ell(A\cdot Y) = 0 \quad \text{for all } A \in \mathfrak{k}.$$

Using this and the fact that $Y \in N_\ell$ one can apply (3.3) to show

$$\varphi(Y\cdot(U,X)) = \varphi(U,X) \quad \text{for all } U \in \mathfrak{k},\ X \in N.$$

That is, $Y \in N_\varphi$. So $G_\varphi = K_\varphi N_\varphi$ as stated. □

Remark 3.7. The proof for Proposition 3.6 shows that one has $G_\ell = K_\ell N_\ell$ whenever $\ell = \varphi|_\mathfrak{n}$ is aligned. The condition that ξ belong to $\mathfrak{k}_\ell^*$ only enters the proof that $G_\varphi = K_\varphi N_\varphi$.

Lemma 3.8. *Let $\varphi = (\xi,\ell) \in \mathfrak{g}^*$ where $\ell \in \mathfrak{n}^*$ is aligned. Then*

$$Ad_G^*(N_\ell)\varphi = \varphi + (\mathfrak{k}_\ell + \mathfrak{n})^\perp.$$

Proof. Lemma 2 in [Puk78] shows that, in any case, $Ad_G^*(N_\ell)\varphi = \varphi + (\mathfrak{g}_\ell + \mathfrak{n})^\perp$. But alignment of ℓ gives $\mathfrak{g}_\ell + \mathfrak{n} = \mathfrak{k}_\ell + \mathfrak{n}$, in view of the preceding remark. □

4. The moment map for an $Ad^*(N)$-orbit

Definition 4.1. Let $\mathcal{O} \subset \mathfrak{n}^*$ be a coadjoint orbit for N, $K_{\mathcal{O}}$ the stabilizer of $\mathcal{O}$ in K and $\mathfrak{k}_{\mathcal{O}}$ its Lie algebra. The *moment map* $\tau_{\mathcal{O}} : \mathcal{O} \to \mathfrak{k}^*_{\mathcal{O}}$ is defined via[1]

$$\tau_{\mathcal{O}}(Ad^*(X)\ell_{\mathcal{O}})(A) = -\frac{1}{2}B_{\mathcal{O}}(X, A\cdot X) = -\frac{1}{2}\ell_{\mathcal{O}}[X, A\cdot X]$$

for $A \in \mathfrak{k}_{\mathcal{O}}$, $X \in \mathfrak{n}$. Here $\ell_{\mathcal{O}}$ is the unique aligned point in $\mathcal{O}$.

Lemma 4.2. *The map $\tau_{\mathcal{O}}$ is well defined.*

Proof. Suppose that $Ad^*(X_1)\ell_{\mathcal{O}} = Ad^*(X_2)\ell_{\mathcal{O}}$. It follows that $X_1 - X_2 \in Rad(B_{\mathcal{O}})$. Let $A \in \mathfrak{k}_{\mathcal{O}}$. We have $A\cdot\ell_{\mathcal{O}} = 0$ in view of Lemma 3.4 and an easy calculation yields $B_{\mathcal{O}}(X_1, A\cdot X_1) = B_{\mathcal{O}}(X_2, A\cdot X_2)$. □

Next note that for $k_\circ \in K$ and coadjoint orbits $\mathcal{O} \subset \mathfrak{n}^*$ one has

$$K_{k_\circ\cdot\mathcal{O}} = k_\circ K_{\mathcal{O}} k_\circ^{-1}, \quad \mathfrak{k}_{k_\circ\cdot\mathcal{O}} = Ad(k_\circ)(\mathfrak{k}_{\mathcal{O}}), \quad \text{and} \quad \mathfrak{k}^*_{k_\circ\cdot\mathcal{O}} = Ad^*(k_\circ)(\mathfrak{k}^*_{\mathcal{O}}).$$

The following equivariance property for moment maps is fundamental. The proof involves a routine calculation, which we leave to the reader.

Lemma 4.3. *The diagram*

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{k_\circ\cdot -} & k_\circ\cdot\mathcal{O} \\ \Big\downarrow \tau_{\mathcal{O}} & & \Big\downarrow \tau_{k_\circ\cdot\mathcal{O}} \\ \mathfrak{k}^*_{\mathcal{O}} & \xrightarrow{Ad^*(k_\circ)} & \mathfrak{k}^*_{k_\circ\cdot\mathcal{O}} \end{array}$$

commutes for any $k_\circ \in K$ and any coadjoint orbit $\mathcal{O} \subset \mathfrak{n}^*$. In particular, one has $\tau_{\mathcal{O}}(k\cdot\ell) = Ad^*(k)\tau_{\mathcal{O}}(\ell)$ for $\ell \in \mathcal{O}$, $k \in K_{\mathcal{O}}$.

The map $Ad^*(k_\circ) : \mathfrak{k}^*_{\mathcal{O}} \to \mathfrak{k}^*_{k_\circ\cdot\mathcal{O}}$ in the preceding diagram takes $Ad^*(K_{\mathcal{O}})$-orbits to $Ad^*(K_{k_\circ\cdot\mathcal{O}})$-orbits. For $\pi \in \widehat{N}$, $\alpha \in \widehat{K_\pi}$, $k_\circ \in K$ one has

$$K_\pi = K_{\mathcal{O}^N(\pi)}, \quad K_{k_\circ\cdot\pi} = K_{\mathcal{O}^N(k_\circ\cdot\pi)}, \quad k_\circ\cdot\alpha \in \widehat{K_{k_\circ\cdot\pi}}$$

and we conclude that

$$\mathcal{O}^{K_{k_\circ\cdot\pi}}(k_\circ\cdot\alpha) = Ad^*(k_\circ)\mathcal{O}^{K_\pi}(\alpha). \tag{4.1}$$

Proposition 4.4. *Consider a point $\varphi = (\xi, \ell)$ in $\mathfrak{g}^*$ where $\ell \in \mathfrak{n}^*$ is aligned and let $\mathcal{O} = Ad^*(N)\ell$. Then*

$$Ad^*(G)\varphi \cap \mathfrak{n}^* = \{k\cdot\ell' \ : \ k \in K,\ \ell' \in \mathcal{O} \text{ with } \tau_{\mathcal{O}}(\ell') = (-\xi)|_{\mathfrak{k}_{\mathcal{O}}}\},$$

the K-saturation of $\tau_{\mathcal{O}}^{-1}((-\xi)|_{\mathfrak{k}_{\mathcal{O}}})$. In particular, $Ad^*(G)\varphi \cap \mathfrak{n}^* \neq \emptyset$ if and only if $(-\xi)|_{\mathfrak{k}_{\mathcal{O}}}$ is in the image of $\tau_{\mathcal{O}}$.

[1] The minus sign in Definition 4.1 has been included to simplify the form of Equation 5.4 and Proposition 5.3 below.

Proof. First note that as ℓ is aligned we have $\mathfrak{k}_{\mathcal{O}} = \mathfrak{k}_\ell$, by Lemma 3.4. For $X \in \mathfrak{n}$ let $X \times \ell \in \mathfrak{k}^*$ be defined as $(X \times \ell)(A) = \ell(A \cdot X)$ and set

$$T_X\varphi = T_X(\xi, \ell) = \xi + X \times \ell + \frac{1}{2} X \times ad_N^*(X)\ell$$

From Equation 3.3 one obtains (see [BJR99])

$$Ad_G^*(X)\varphi = (T_X\varphi, Ad_N^*(X)\ell)$$

and hence

$$Ad^*(G)\varphi \cap \mathfrak{n}^* = \{k \cdot (Ad_N^*(X)\ell) \ : \ k \in K, X \in \mathfrak{n} \text{ with } T_X\varphi = 0\}$$

Observe that in this notation,

$$\tau_{\mathcal{O}}(Ad_N^*(X)\ell) = \frac{1}{2}(X \times ad_N^*(X)\ell)\Big|_{\mathfrak{k}_{\mathcal{O}}=\mathfrak{k}_\ell}.$$

Suppose that $k \in K$ and $\ell' = Ad_N^*(X_\circ)\ell$ where $X_\circ \in \mathfrak{n}$ satisfies $T_{X_\circ}\varphi = 0$, so that $k \cdot \ell' \in Ad^*(G)\varphi \cap \mathfrak{n}^*$. As $X_\circ \times \ell$ vanishes on $\mathfrak{k}_\ell$ the identity $T_{X_\circ}\varphi|_{\mathfrak{k}_\ell} = 0$ becomes $\tau_{\mathcal{O}}(\ell') = (-\xi)|_{\mathfrak{k}_\ell}$. So $Ad^*(G)\varphi \cap \mathfrak{n}^*$ is contained in the K-saturation of $\tau_{\mathcal{O}}^{-1}((-\xi)|_{\mathfrak{k}_{\mathcal{O}}})$.

Next assume that $(-\xi)|_{\mathfrak{k}_\ell} = \tau_{\mathcal{O}}(\ell')$ where $\ell' = Ad_N^*(X_\circ)\ell \in \mathcal{O}$ and set $\varphi' = Ad_G^*(X_\circ)\varphi$. Now φ' vanishes on $\mathfrak{k}_\ell$, since $X_\circ \times \ell|_{\mathfrak{k}_\ell} = 0$, and thus φ' and $(0, \ell')$ agree on $\mathfrak{k}_\ell + \mathfrak{n}$. Lemma 3.8 now implies that there is some $X_1 \in \mathfrak{n}_\ell$ with $Ad_G^*(X_1)\varphi' = (0, \ell')$. So $X_2 = X_1 + X_\circ + \frac{1}{2}[X_1, X_\circ] \in \mathfrak{n}$ has $Ad_G^*(X_2)\varphi = (0, \ell')$. That is, ℓ' belongs to $Ad^*(G)\varphi \cap \mathfrak{n}^*$. As $Ad^*(G)\varphi \cap \mathfrak{n}^*$ is K-saturated we conclude that the K-saturation of $\tau_{\mathcal{O}}^{-1}((-\xi)|_{\mathfrak{k}_{\mathcal{O}}})$ is contained in $Ad^*(G)\varphi \cap \mathfrak{n}^*$. □

5. The orbit method with Gelfand pairs (K, N)

Henceforth we assume that (K, N) is a Gelfand pair. Our goal here is to prove Proposition 1.2 and Theorem 1.4.

As in Section 3.3, given $\pi \in \widehat{N}$,

$$W_\pi : K_\pi \to U(\mathcal{H}_\pi)$$

denotes the canonical unitary representation of K_π intertwining $k \cdot \pi$ with π. The representation W_π is necessarily multiplicity free. In fact, (K, N) is a Gelfand pair if and only if W_π is a multiplicity free representation of K_π for all $\pi \in \widehat{N}$ [Car87, BJR90]. Let

$$\mathcal{H}_\pi = \bigoplus_{\alpha \in \Lambda_\pi} P_{\pi,\alpha} \tag{5.1}$$

denote the decomposition of $\mathcal{H}_\pi$ into $W_\pi(K_\pi)$-irreducible subspaces. This decomposition is canonical because W_π is multiplicity free. Here Λ_π is a countable index set that depends on $\pi \in \widehat{N}$. For concreteness we take

$$\Lambda_\pi = Spec(W_\pi) = \{\alpha \in \widehat{K_\pi} \ : \ \alpha \text{ occurs in } W_\pi\},$$

so that $W_\pi|_{P_{\pi,\alpha}} = \alpha \in \widehat{K_\pi}$.

Let $\rho = \rho_{\pi,\sigma} \in \widehat{G}$ have Mackey parameters $\pi \in \widehat{N}$, $\sigma \in \widehat{K_\pi}$. By Frobenius reciprocity

$$mult(1_K, \rho|_K) = mult(1_K, Ind_{K_\pi}^K \sigma \otimes W_\pi) = mult(1_{K_\pi}, \sigma \otimes W_\pi) = mult(\sigma^*, W_\pi).$$

Thus ρ is a K-spherical representation if and only if the representation σ^*, contragredient to σ, occurs in W_π. Hence

$$\widehat{G}_K = \{\rho_{\pi,\alpha^*} \ : \ \pi \in \widehat{N},\ \alpha \in \Lambda_\pi\}. \tag{5.2}$$

Lemma 5.1. *Let $\pi \in \widehat{N}$ and $\alpha \in \Lambda_\pi$, so that $\rho = \rho_{\pi,\alpha^*}$ belongs to $\widehat{G}_K$. Then*

$$\mathcal{O}(\rho) \cap \mathfrak{n}^* = K \cdot \tau_\pi^{-1}(\mathcal{O}^{K_\pi}(\alpha)),$$

where τ_π denotes the moment map $\tau_{\mathcal{O}^N(\pi)} : \mathcal{O}^N(\pi) \to \mathfrak{k}_\pi^*$. In particular, $\mathcal{O}(\rho) \cap \mathfrak{n}^* \neq \emptyset$ if and only if $\mathcal{O}^{K_\pi}(\alpha) \subset Image(\tau_\pi)$.

Proof. Choose a point $\xi \in \mathcal{O}^{K_\pi}(\alpha)$. Then $-\xi \in \mathcal{O}^{K_\pi}(\alpha^*)$ and ρ has coadjoint orbit $\mathcal{O}(\rho) = Ad^*(G)(-\xi, \ell_\pi)$. Proposition 4.4 shows $\mathcal{O}(\rho) \cap \mathfrak{n}^* = K \cdot \tau_\pi^{-1}(\xi|_{\mathfrak{k}_\pi})$ and the result now follows by K_π-equivariance of τ_π. □

Recall that Proposition 1.2 asserts that $\mathcal{O}(\rho) \cap \mathfrak{n}^* \neq \emptyset$ for all $\rho \in \widehat{G}_K$. Our proof, given below in Section 5.2, involves reduction to cases where N is a Heisenberg group.

5.1. **Gelfand pairs** (K, H_V)**.** Let V be a finite dimensional *complex* vector space and $\langle \cdot, \cdot \rangle$ be a positive definite Hermitian inner product on V. The associated Heisenberg group H_V has Lie algebra

$$\mathfrak{h}_V = V \oplus \mathbb{R} \ \text{ with Lie bracket } [(v,t),(v',t')] = (0, -Im\langle v, v'\rangle).$$

The unitary group $U(V)$ for $(V, \langle \cdot, \cdot \rangle)$ acts on H_V via automorphisms as

$$k \cdot (v,t) = (kv, t).$$

Let K be a closed Lie subgroup of $U(V)$. We know that (K, H_V) is a Gelfand pair if and only if the representation of K on the ring $\mathbb{C}[V]$ of (holomorphic) polynomials, given by

$$(k \cdot p)(v) = p(k^{-1}v), \tag{5.3}$$

is multiplicity free [BJR90]. Gelfand pairs of the sort (K, H_V) have been completely classified [Kac80, Bri85, BR96, Lea98].

Lemma 5.2. *Proposition 1.2 holds for Gelfand pairs (K, H_V).*

Proof. Let (K, H_V) be a Gelfand pair as above. In view of Lemma 5.1 it suffices to check that $\mathcal{O}^{K_\pi}(\alpha) \subset Image(\tau_\pi)$ for all $\pi \in \widehat{H_V}$, $\alpha \in \Lambda_\pi$. Letting $\mathcal{O} = \mathcal{O}^{H_V}(\pi)$ we will write

$$\text{“ } \Lambda_\pi \subset Image(\tau_{\mathcal{O}}) \text{ ”}$$

as shorthand for the statement

$$\mathcal{O}^{K_\pi}(\alpha) \subset Image(\tau_{\mathcal{O}}) \text{ for all } \alpha \in \Lambda_\pi.$$

The coadjoint orbits in $\mathfrak{h}_V^*$ are of two sorts. We will describe the moment map $\tau_{\mathcal{O}}$ for each type of orbit and verify that $\Lambda_\pi \subset Image(\tau_{\mathcal{O}})$ in each case.

For $(v,t) \in \mathfrak{h}_V$ let $\ell_{(v,t)} \in \mathfrak{h}_V^*$ denote the functional

$$\ell_{(v,t)}(v',t') = Im\langle v, v'\rangle + tt'.$$

One has easily that

$$k \cdot \ell_{(v,t)} = \ell_{(kv,t)} \text{ for } k \in U(V).$$

Single Point Orbits: We have single point coadjoint orbits

$$\mathcal{O} = \{\ell_{(v_\circ,0)}\}$$

for $v_\circ \in V$. In this case $K_{\mathcal{O}} = \{k \in K \ : \ kv_\circ = v_\circ\}$ is the stabilizer of $v_\circ$ and $\tau_{\mathcal{O}} : \mathcal{O} \to \mathfrak{k}_{\mathcal{O}}^*$ is the zero map ($\ell_{(v_\circ,0)} \mapsto 0$). The representation $\pi \in \widehat{H_V}$ associated to $\mathcal{O}$ is the one dimensional representation

$$\pi(v,t) = e^{iIm\langle v_\circ, v\rangle}$$

and W_π is the trivial one dimensional representation $1_{K_{\mathcal{O}}}$ of $K_{\mathcal{O}}$. Thus $\Lambda_\pi = \{1_{K_{\mathcal{O}}}\}$. Since $\{0\} \subset \mathfrak{k}_{\mathcal{O}}^*$ is the coadjoint orbit that corresponds to $1_{K_{\mathcal{O}}}$, we see that $\Lambda_\pi \subset Image(\tau_{\mathcal{O}})$.

Planar Orbits: We have coadjoint orbits of the sort

$$\mathcal{O} = \{\ell_{(v,\lambda)} \ : \ v \in V\}$$

for fixed $\lambda \in \mathbb{R}^\times$. The stabilizer of $\mathcal{O}$ in K is $K_{\mathcal{O}} = K$. The aligned point in $\mathcal{O}$ is $\ell_{\mathcal{O}} = \ell_{(0,\lambda)}$ and one computes that

$$Ad^*(v)\ell_{\mathcal{O}} = \ell_{(\lambda v,\lambda)}.$$

Hence we have

$$\begin{aligned}
\tau_{\mathcal{O}}\left(\ell_{(v,\lambda)}\right)(A) &= \tau_{\mathcal{O}}\left(Ad^*\left(\frac{1}{\lambda}v\right)\ell_{\mathcal{O}}\right)(A) \\
&= -\frac{1}{2}\ell_{\mathcal{O}}\left(\left[\frac{1}{\lambda}v, \frac{1}{\lambda}Av\right]\right) \\
&= -\frac{1}{2\lambda^2}\ell_{(0,\lambda)}(0, -Im\langle v, Av\rangle) \\
&= \frac{1}{2\lambda}Im\langle v, Av\rangle
\end{aligned}$$

for $A \in \mathfrak{k}$. Thus letting $\eta : V \to \mathfrak{k}^*$ be the map

$$\eta(v)(A) = Im\langle v, Av\rangle,$$

we have that

$$\tau_{\mathcal{O}}(\ell_{(v,\lambda)}) = \frac{1}{2\lambda}\eta(v). \tag{5.4}$$

The map η is the (unnormalized) moment map for the action of K on V. Equation 5.4 shows that

$$\tau_{\mathcal{O}}(\mathcal{O}) = \left\{ \begin{array}{rl} \eta(V) & \text{for } \lambda > 0 \\ -\eta(V) & \text{for } \lambda < 0 \end{array} \right. .$$

The representation $\pi \in \widehat{H_V}$ that corresponds to $\mathcal{O}$ is infinite dimensional. When $\lambda > 0$ we can realize π in a Fock space that contains $\mathbb{C}[V]$ as a dense subspace. The intertwining representation W_π is given by Equation 5.3. Thus Λ_π is the spectrum of $\mathbb{C}[V]$. Proposition 4.1 in [BJLR97] asserts that $\Lambda_\pi \subset \eta(V)$. When $\lambda < 0$ we can realize π on the conjugate Fock space and W_π is contragredient to the representation given by Equation 5.3. In this case, Λ_π is the set of representations contragredient to those in the spectrum of $\mathbb{C}[V]$. These correspond to coadjoint orbits contained in $-\eta(V)$. Thus we see that $\Lambda_\pi \subset Image(\tau_{\mathcal{O}})$ holds in all cases. □

5.2. **Proof of Proposition 1.2.** We can now complete the proof of Proposition 1.2. Let $\rho \in \widehat{G}_K$ and $\mathcal{O}(\rho) = Ad^*(G)\varphi$, where $\varphi \in \mathfrak{g}^*$ and $\ell = \varphi|_{\mathfrak{n}}$ is aligned, as usual.

Let $\pi \in \widehat{N}$ be the representation corresponding to $Ad^*(N)\ell \subset \mathfrak{n}^*$. This representation factors through

$$N_\pi = N/Z_\pi$$

where $Z_\pi = \exp(\mathrm{Ker}(\ell|_{\mathfrak{z}}))$. The action of K_π preserves Z_π and hence descends to N_π. One has (see [BJR99]):

- (K_π, N_π) is a Gelfand pair.
- $\varphi' = \varphi|_{\mathfrak{k}_\pi + \mathfrak{n}_\pi}$ is a spherical point. That is, the coadjoint orbit $Ad^*(K_\pi N_\pi)\varphi'$ corresponds to a K_π-spherical representation of $K_\pi N_\pi$.

Now N_π is either a Heisenberg group, an abelian group or a product of a Heisenberg group with an abelian group. In the latter case, the action of K_π preserves the two factors. Lemma 5.2 now implies that

$$Ad^*(K_\pi N_\pi)\varphi' \cap \mathfrak{n}_\pi^* \neq \emptyset.$$

In particular, for some $X_\circ \in \mathfrak{n}$ we have

$$Ad_G^*(X_\circ)\varphi|_{k_\pi} = 0.$$

Applying Lemma 3.8, as in the proof for Proposition 4.4, it follows that $\mathcal{O}(\rho) \cap \mathfrak{n}^* \neq \emptyset$ as claimed. □

5.3. **The map $\mathbf{\Psi} : \mathbf{\Delta}(\mathbf{K}, \mathbf{N}) \to \mathcal{A}(\mathbf{K}, \mathbf{N})$.** Proposition 1.2 and Theorem 1.1 show that each K-spherical representation $\rho \in \widehat{G}_K$ yields a K-orbit

$$\mathcal{K}(\rho) = \mathcal{O}(\rho) \cap \mathfrak{n}^*$$

in $\mathfrak{n}^*$. As in Section 1 we let $\mathcal{A}(K, N) \subset \mathfrak{n}^*/K$ denote the set

$$\mathcal{A}(K, N) = \{\mathcal{K}(\rho) \ : \ \rho \in \widehat{G}_K\}$$

of K-spherical orbits in $\mathfrak{n}^*$ and lift $\mathcal{K}$ from $\widehat{G}_K$ to obtain a map Ψ on the space $\Delta(K, N)$ of bounded K-spherical functions. Proposition 5.3 below gives another point of view on this construction.

Equation 5.2 asserts that $\widehat{G}_K = \{\rho_{\pi,\alpha^*} \ : \ \pi \in \widehat{N}, \alpha \in \Lambda_\pi\}$. We let $\phi_{\pi,\alpha}$ denote the K-spherical function associated to $\rho_{\pi,\alpha^*} \in \widehat{G}_K$. This can be written as

$$\phi_{\pi,\alpha}(x) = \int_K \langle \pi(k \cdot x) v_{\pi,\alpha}, v_{\pi,\alpha} \rangle_\pi dk \tag{5.5}$$

where $\langle \cdot, \cdot \rangle_\pi$ is the Hilbert space structure on $\mathcal{H}_\pi = \bigoplus_{\alpha \in \Lambda_\pi} P_{\pi,\alpha}$ (see Equation 5.1) and $v_{\pi,\alpha}$ is any unit vector in $P_{\pi,\alpha}$ [BJR90]. The following result in an immediate consequence on Proposition 1.2 and Lemma 5.1.

Proposition 5.3. *For any $\pi \in \widehat{N}$, $\alpha \in \Lambda_\pi$ one has*

$$\mathcal{O}^{K_\pi}(\alpha) \subset Image\big(\tau_\pi : \mathcal{O}^N(\pi) \to \mathfrak{k}_\pi^*\big).$$

Moreover $\Psi : \Delta(K, N) \to \mathcal{A}(K, N)$ can be written as

$$\Psi(\phi_{\pi,\alpha}) = K \cdot \ell_{\pi,\alpha}$$

where $\ell_{\pi,\alpha}$ is any point in $\mathcal{O}^N(\pi)$ with $\tau_\pi(\ell_{\pi,\alpha}) \in \mathcal{O}^{K_\pi}(\alpha)$.

Proposition 5.3 allows one to compute $\Psi(\phi_{\pi,\alpha}) \in \mathfrak{n}^*/K$ without recourse to the semidirect product $G = K \ltimes N$. This is useful in connection with the examples treated below.

5.4. **Proof of Theorem 1.4.** Theorem 1.4 and Corollary 1.6 assert that the maps $\mathcal{K}$ and Ψ are bijective. Our proof requires the following lemma.

Lemma 5.4. *For each $\pi \in \widehat{N}$ the map*

$$\Lambda_\pi \to \mathfrak{k}_\pi^*/Ad^*(K_\pi), \quad \alpha \mapsto \mathcal{O}^{K_\pi}(\alpha)$$

is injective.

Proof. Let $\pi \in \widehat{N}$. As (K, N) is a Gelfand pair, so is (K°, N), by Proposition 2.5 in [BJR99]. It follows that $W_\pi|_{K_\pi^\circ}$ is a multiplicity free representation. Suppose that $\mathcal{O}^{K_\pi}(\alpha) = \mathcal{O}^{K_\pi}(\alpha')$ for some $\alpha, \alpha' \in \Lambda_\pi$. This means that some irreducible representation $\nu \in \widehat{K_\pi^\circ}$ of the identity component K_π° occurs in both $\alpha|_{K_\pi^\circ}$ and $\alpha'|_{K_\pi^\circ}$. We conclude that $\alpha = \alpha'$ since $W_\pi|_{K_\pi^\circ}$ is multiplicity free. □

We now turn to the proof of Theorem 1.4. Let $\pi, \pi' \in \widehat{N}$, $\alpha \in \Lambda_\pi$, $\alpha' \in \Lambda_{\pi'}$ so that

$$\rho = \rho_{\pi,\alpha^*}, \quad \rho' = \rho_{\pi',(\alpha')^*}$$

belong to $\widehat{G}_K$. By Proposition 5.3 there are points

$$\ell = \ell_{\pi,\alpha} \in \mathcal{O}^N(\pi), \quad \ell' = \ell_{\pi',\alpha'} \in \mathcal{O}^N(\pi')$$

with

$$\xi = \tau_\pi(\ell) \in \mathcal{O}^{K_\pi}(\alpha), \quad \xi' = \tau_{\pi'}(\ell') \in \mathcal{O}^{K_{\pi'}}(\alpha')$$

and one has

$$\mathcal{K}(\rho) = K \cdot \ell, \quad \mathcal{K}(\rho') = K \cdot \ell'.$$

Suppose that $\mathcal{K}(\rho) = \mathcal{K}(\rho')$. This means

$$\ell' = k_\circ \cdot \ell$$

for some $k_\circ \in K$. Thus also $k_\circ \cdot \mathcal{O}^N(\pi) = \mathcal{O}^N(\pi')$ and hence

$$\pi' = k_\circ \cdot \pi. \tag{5.6}$$

Moreover Lemma 4.3 yields

$$Ad^*(k_\circ)\xi = Ad^*(k_\circ)\tau_\pi(\ell) = \tau_{\pi'}(k_\circ \cdot \ell) = \tau_{\pi'}(\ell') = \xi'$$

which implies

$$\mathcal{O}^{K_{\pi'}}(\alpha') = Ad^*(k_\circ)\mathcal{O}^{K_\pi}(\alpha) = \mathcal{O}^{K_{\pi'}}(k_\circ \cdot \alpha),$$

using Equation 4.1. This gives

$$\alpha' = k_\circ \cdot \alpha \tag{5.7}$$

in view of Lemma 5.4. Equations 5.6 and 5.7 imply that ρ and ρ' are unitarily equivalent, as their Mackey parameters differ by the action of K. □

Remark 5.5. Recall that the orbit map $\mathcal{O} : \widehat{G} \to \mathfrak{g}^*/Ad^*(G)$ for a semidirect product $G = K \ltimes N$ can fail to be injective. Theorem 1.4 implies, however, that when (K, N) is a Gelfand pair, $\rho \mapsto \mathcal{O}(\rho)$ is one-to-one on $\widehat{G}_K$, the K-spherical representations.

5.5. **Eigenvalues for invariant differential operators.** A basic result concerning spherical functions and invariant differential operators will be needed in connection with the examples. Recall that $\mathbb{D}_K(N)$ denotes the set of differential operators on N that are invariant under both the action of K and left multiplication. The spherical functions are eigenfunctions for such operators. Given $D \in \mathbb{D}_K(N)$ and $\phi \in \Delta(K, N)$, we write $\widehat{D}(\phi)$ for the eigenvalue of D acting on ϕ, so that:

$$D\phi = \widehat{D}(\phi)\phi.$$

Since the spherical functions are normalized to have value 1 at the identity element $e \in N$, we have

$$\widehat{D}(\phi) = D\phi(e).$$

For $D \in \mathbb{D}_K(N)$ and $\pi \in \widehat{N}$, the operator $\pi(D)$ commutes with the action of K_π on $\mathcal{H}_\pi$ and hence preserves the subspaces $P_{\pi,\alpha}$ in Decomposition 5.1. Schur's Lemma shows, moreover, that $\pi(D)|_{P_{\pi,\alpha}}$ must be a scalar operator. From Equation 5.5 we see that

$$\widehat{D}(\phi_{\pi,\alpha}) = D\phi_{\pi,\alpha}(e) = \langle \pi(D)v_{\pi,\alpha}, v_{\pi,\alpha}\rangle_\pi$$

and conclude that:

Lemma 5.6. $\pi(D)|_{P_{\pi,\alpha}} = \widehat{D}(\phi_{\pi,\alpha})$.

6. The case of N abelian

Here we consider the map $\Psi : \Delta(K,N) \to \mathcal{A}(K,N)$ in the "degenerate" situation where the 2-step group N is in fact abelian. The entire group algebra $L^1(N)$ is now commutative and hence (K,N) is a Gelfand pair for *any* compact Lie group $K \subset Aut(N)$. One calls $G = K \ltimes N$ a *generalized Euclidean motion group.* A detailed study of the associated spherical functions can be found in [Wol06].

The unitary dual $\widehat{N}$ consists of characters

$$\widehat{N} = \{\chi_\ell \;:\; \ell \in \mathfrak{n}^*\}, \quad \chi_\ell(x) = e^{i\ell(x)}.$$

The space $\widehat{N}$ is homeomorphic to $\mathfrak{n}^*$ via $\chi_\ell \leftrightarrow \ell$. One has

$$\Lambda_{\chi_\ell} = \{1_{K_\ell}\}$$

because the intertwining representation W_{χ_ℓ} is trivial. We write $\phi_\ell = \phi_{\chi_\ell, 1_{K_\ell}}$ so that

$$\Delta(K,N) = \{\phi_\ell \;:\; \ell \in \mathfrak{n}^*\}.$$

Equation 5.5 here reduces to

$$\phi_\ell(x) = \int_K \chi_\ell(k\cdot x)\, dk = \int_K e^{i\ell(k\cdot x)}\, dk,$$

the K-average of χ_ℓ. Note that $\phi_\ell = \phi_{\ell'}$ if and only if $K\cdot\ell = K\cdot\ell'$. In fact $\Delta(K,N)$ is homeomorphic to $\widehat{N}/K$ via $\phi_\ell \leftrightarrow K\cdot\chi_\ell$.

Proposition 6.1. *Let N be abelian and K be a compact Lie group acting smoothly on N by automorphisms. In this context the map Ψ is simply*

$$\Psi : \Delta(K,N) \to \mathfrak{n}^*/K, \quad \Psi(\phi_\ell) = K\cdot\ell.$$

This is, moreover, a homeomorphism onto its image $\mathcal{A}(K,N) = \mathfrak{n}^*/K$

Proof. Fix $\ell \in \mathfrak{n}^*$. The Kirillov orbit for the representation χ_ℓ is

$$\mathcal{O} = \mathcal{O}^N(\chi_\ell) = \{\ell\},$$

a single point. Now $\ell \in \mathcal{O}$ is aligned because $\mathfrak{w}_{\mathcal{O}} = 0$ in Equation 3.1. The moment map $\tau_{\chi_\ell} : \mathcal{O} \to \mathfrak{k}_\ell^*$ sends ℓ to 0 since $\ell[\cdot,\cdot] = 0$ in Definition 4.1. Thus Proposition 5.3 yields

$$\Psi(\phi_\ell) = \Psi(\phi_{\chi_\ell, 1_{K_\ell}}) = K\cdot\ell$$

as claimed. Identifying $\Delta(K,N)$ with $\widehat{N}/K$ we see that Ψ is the mapping on K-orbits induced by

$$\widehat{N} \to \mathfrak{n}^*, \quad \chi_\ell \mapsto \ell.$$

As the latter is a homeomorphism, so is Ψ. □

7. The Gelfand pair $(U(V), H_V)$

The bounded spherical functions for $(U(V), H_V)$ have been computed independently by various authors. (See for example [HR80], [Kor80], [Far87], [Ste88], [Str91], [BJR92].) These spherical functions are of two distinct types, corresponding to the single point and planar coadjoint orbits discussed in Section 5.1.

Type 1 spherical functions: These are associated to the planar coadjoint orbits in $\mathfrak{h}_V$. For each $\lambda \in \mathbb{R}^\times$ and $m \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}$ we have the $U(V)$-spherical function

$$\phi_{\lambda,m}(v,t) = L_m^{(n-1)}\left(\frac{|\lambda||v|^2}{2}\right) e^{-|\lambda||v|^2/4} e^{i\lambda t}$$

where $L_m^{(n-1)}(x)$ denotes the Laguerre polynomial of order $n-1$ and degree m normalized to have value 1 at $x = 0$. This spherical function arises from the infinite dimensional representation $\pi = \pi_\lambda$ of H_V with central character $(0,t) \mapsto e^{i\lambda t}$. The associated coadjoint orbit is $\mathcal{O} = \mathcal{O}_\lambda = \{\ell_{(v,\lambda)} \ : \ v \in V\}$, with notation as in Section 5.1. For $\lambda > 0$ we realize W_π as the standard representation of $U(V)$ on $\mathbb{C}[V]$ (see Equation 5.3). For $\lambda < 0$, we have the conjugate of this representation. The space $\mathbb{C}[V]$ decomposes under the action of $U(V)$ as

$$\mathbb{C}[V] = \sum_{m=0}^{\infty} \mathcal{P}_m(V)$$

where $\mathcal{P}_m(V)$ denotes the space of homogeneous polynomials of degree m. In terms of the notation used in the preceding section, we have $\phi_{\lambda,m} = \phi_{\pi_\lambda,\alpha_m}$ where α_m is the representation of $U(V)$ on $\mathcal{P}_m(V)$.

One can use an orthonormal basis to identify V with $\mathbb{C}^n$ and $U(V)$ with the group $U(n)$ of $n \times n$ unitary matrices. The standard maximal torus in $U(n)$ has Lie algebra

$$\mathfrak{t} = \left\{ A_\theta = \begin{bmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_n \end{bmatrix} \ : \ \theta_1, \dots, \theta_n \in \mathbb{R} \right\}.$$

The polynomial $(z_1, \dots, z_n) \mapsto z_1^m$ on $V = \mathbb{C}^n$ is a highest weight vector in $\mathcal{P}_m(V)$ with highest weight $A_\theta \mapsto -im\theta_1$. Using Equation 5.4 we compute that for

$v = \left(\sqrt{2|\lambda| m}, 0, \ldots, 0\right) \in V$ one has

$$\tau_{\mathcal{O}}\left(\ell_{(v,\lambda)}\right)(A_\theta) = \frac{1}{2\lambda}\eta(v)(A_\theta) = \frac{1}{2\lambda} Im\langle v, A_\theta v\rangle = \frac{\left(\sqrt{2|\lambda| m}\right)^2}{2\lambda}(-\theta_1)$$
$$= \begin{cases} -m\theta_1 & \text{for } \lambda > 0 \\ m\theta_1 & \text{for } \lambda < 0 \end{cases}.$$

Using Proposition 5.3, we conclude that the $U(V)$-spherical orbit $\Psi(\phi_{\lambda,m})$ is

(7.1) $$K_{\lambda,m} = U(V)\cdot \ell_{(v,\lambda)} = \left\{\ell_{(v,\lambda)} \;:\; |v| = \sqrt{2|\lambda| m}\right\}.$$

Type 2 spherical functions: For each real number $r \geq 0$ we have a $U(V)$-spherical function

$$\psi_r(v,t) = \int_{U(V)} e^{iRe\langle w_r, kv\rangle} dk = \int_{U(V)} e^{iIm\langle w_r, kv\rangle} dk$$

where $w_r \in V$ is any vector with $|w_r| = r$. More explicitly we have

$$\psi_r(v,t) = \frac{2^{n-1}(n-1)!}{(r|v|)^{n-1}} J_{n-1}(r|v|)$$

for $r > 0$ and $\psi_0(v,t) \equiv 1$. Here J_{n-1} is the Bessel function (of the first kind) with order $n-1$. The function ψ_r is the $U(V)$-average of the unitary character $\pi(v,t) = \chi_{w_r}(v) = e^{iIm\langle w_r, v\rangle}$. In terms of the notation from Section 5.3, we have $\psi_r = \phi_{\pi,1}$ where 1 is the trivial one-dimensional representation of $K_\pi = K_{w_r}$. As π is associated to the single point coadjoint orbit $\mathcal{O} = \{\ell_{(w_r,0)}\}$, we see that the $U(V)$-spherical orbit $\Psi(\psi_r)$ is

(7.2) $$K_r = U(V)\cdot \ell_{(w_r,0)} = \{\ell_{(v,0)} \;:\; |v| = r\}.$$

In summary, we have shown that

- $\mathcal{A}(U(V), H_V) = \{K_{\lambda,m} \;:\; \lambda \in \mathbb{R}^\times, m \in \mathbb{Z}^+\} \cup \{K_r \;:\; r \geq 0\}$ where $K_{\lambda,m}$ and K_r are as in Equations 7.1 and 7.2, and
- the map $\Psi : \Delta(U(V), H_V) \to \mathcal{A}(U(V), H_V)$ is given by $\Psi(\phi_{\lambda,m}) = K_{\lambda,m}$ and $\Psi(\psi_r) = K_r$.

We can now establish Conjecture 1.7 for the Gelfand pair $(U(V), H_V)$.

Proposition 7.1. *The map $\Psi : \Delta(U(V), H_V) \to \mathcal{A}(U(V), H_V)$ is a homeomorphism.*

Proof. From our description of the spherical orbits $K_{\lambda,m}$ and K_r we see that the map $F : \mathcal{A}(U(V), H_V) \to \mathbb{R}^+ \times \mathbb{R}$ defined by

$$F(K_{\lambda,m}) = \left(\sqrt{2|\lambda| m}, \lambda\right), \quad F(K_r) = (r, 0)$$

is a homeomorphism onto its image. On the other hand, the "Heisenberg fan" model for $\Delta(U(V), H_V)$ ([Far87],[Str91],[BJRW96]) asserts that the map $E : \Delta(U(V), H_V) \to \mathbb{R}^+ \times \mathbb{R}$ given by

$$E(\phi_{\lambda,m}) = (|\lambda|(2m+n), \lambda), \quad E(\psi_r) = (r^2, 0)$$

is also a homeomorphism onto its image. The result now follows since $F \circ \Psi$ and E differ by the homeomorphism

$$\mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}, \quad (r, \lambda) \mapsto (r^2 + n|\lambda|, \lambda).$$

□

We recall that the map E in the Heisenberg fan construction is

$$E(\phi) = \left(|\widehat{\mathcal{L}}(\phi)|, \widehat{T}(\phi)\right)$$

where $T = \frac{\partial}{\partial t}$ and $\mathcal{L}$ is the Heisenberg sub-Laplacian. A key point is that $\pi_\lambda(\mathcal{L})$ is the quantum harmonic oscillator which acts on $\mathcal{P}_m(V) \subset \mathbb{C}[V]$ via the scalar $-|\lambda|(2m+n)$. From Lemma 5.6 we see that $\widehat{\mathcal{L}}(\phi_{\lambda,m}) = -|\lambda|(2m+n)$.

8. Spherical functions on the free 2-step group

Let $V \cong \mathbb{R}^d$ be a d-dimensional real vector space. The *free 2-step group* F_V has Lie algebra

$$\mathfrak{f}_V = V \oplus \mathfrak{z} = V \oplus \Lambda^2(V) \text{ with Lie bracket } [(u, A), (v, B)] = (0, u \wedge v).$$

This construction is degenerate when $d = 1$ and yields a Heisenberg group when $d = 2$. Thus we take $d \geq 3$ below. Choose any positive definite inner product $(\cdot, \cdot)$ on V and identify $\Lambda^2(V)$ with $so(V) = \{A \in gl(V) : A^t = -A\}$ so that $u \wedge v$ corresponds to the map

$$w \mapsto (u, w)v - (v, w)u.$$

Here A^t denotes the transpose of $A \in gl(V)$ with respect to $(\cdot, \cdot)$. The group $O(V)$ acts on $N = F_V$ by automorphisms via

$$k \cdot (v, A) = (kv, kAk^t),$$

yielding a maximal compact subgroup in $Aut(F_V)$.

It is shown in [BJR90] that $(O(V), F_V)$, and in fact $(SO(V), F_V)$, is a Gelfand pair, but that (K, F_V) fails to be a Gelfand pair for proper closed subgroups K of $SO(V)$. Our goal is the following result, which will be proved in Section 11.

Theorem 8.1. *The map $\Psi : \Delta(O(V), F_V) \to \mathcal{A}(O(V), F_V)$ is a homeomorphism.*

Likewise Conjecture 1.7 holds for $(SO(V), F_V)$:

Corollary 8.2. *The map $\Psi : \Delta(SO(V), F_V) \to \mathcal{A}(SO(V), F_V)$ is a homeomorphism.*

We will not present the proof details for Corollary 8.2 here. The spaces $\Delta(O(V), F_V)$ and $\Delta(SO(V), F_V)$ are, in any case, closely related. Detailed parameterizations for both spaces were obtained by Fischer in [Fis06]. Corollary 8.2 can be derived from Theorem 8.1 by reasoning with these parameters. We prefer to work primarily with $O(V)$ as this simplifies some aspects of our presentation.

The inner product on V extends to a positive definite $O(V)$-invariant inner product on all of $\mathfrak{f}_V$ via

$$\Big((u,A),(v,B)\Big) = (u,v) + \frac{1}{2}tr(A^tB) = (u,v) - \frac{1}{2}tr(AB). \tag{8.1}$$

For $u, v \in V$ and $B \in so(V)$ one has

$$\Big(B,[u,v]\Big) = (Bu,v). \tag{8.2}$$

From this one sees that

$$\Big((b,B), Ad(a,A)(u,U)\Big) = \Big((b+Ba,B),(u,U)\Big),$$

and thus we can also write

$$Ad^*(a,A)(b,B) = (b - Ba, B),$$

where here we are using the inner product (8.1) to identify $\mathfrak{f}_V^*$ with $\mathfrak{f}_V$. The coadjoint orbit $\mathcal{O} = Ad^*(F_V)(b,B)$ through $(b,B) \in \mathfrak{f}_V^*$ is thus

$$\mathcal{O} = \{(b+Bu,B) \;:\; u \in V\} = (b,B) + Image(B).$$

By $Image(B)$ we mean the image as a map from V to V. Using Equation 8.2 one sees that

$$\mathfrak{a}_{\mathcal{O}} = Ker(B) \quad\text{and}\quad \mathfrak{w}_{\mathcal{O}} = \mathfrak{a}_{\mathcal{O}}^{\perp} \cap V = Image(B),$$

with notation as in Section 3. The point (b,B) is aligned if and only if $Bb = 0$. In this case the stabilizer $K_{\mathcal{O}}$ of $\mathcal{O}$ in $O(V)$ is, by Lemma 3.4,

$$K_{\mathcal{O}} = \{k \in O(V) \;:\; kb = b,\; kBk^t = B\}.$$

We continue to suppose that $(b,B) \in \mathfrak{f}_V^* \cong \mathfrak{f}_V$ is aligned and that $\mathcal{O} = Ad^*(F_V)(b,B)$. The eigenvalues for $B \in so(V)$ are of the form $\pm i\lambda$ $(\lambda > 0)$ and perhaps 0. The symmetric operator B^2 has eigenvalues $-\lambda^2$. Let V_λ denote the $(-\lambda^2)$-eigenspace for B^2, so that

$$V = \sum_{\lambda \geq 0} V_\lambda, \quad \mathfrak{a}_{\mathcal{O}} = V_0, \quad \mathfrak{w}_{\mathcal{O}} = \sum_{\lambda > 0} V_\lambda. \tag{8.3}$$

These are orthogonal direct sums. Letting

$$m(\lambda) = \begin{cases} \dim(V_0) & \text{for } \lambda = 0 \\ \dim(V_\lambda)/2 & \text{for } \lambda > 0 \end{cases} \tag{8.4}$$

we see that

$$K_{\mathcal{O}} = O(b^\perp \cap V_0) \times \prod_{\lambda>0} U(V_\lambda) \cong \begin{cases} O(m(0)) \times \prod_{\lambda>0} U(m(\lambda)) & \text{for } b = 0 \\ O(m(0)-1) \times \prod_{\lambda>0} U(m(\lambda)) & \text{for } b \neq 0 \end{cases},$$

where $U(V_\lambda)$ denotes the unitary group for V_λ equipped with a suitable complex Hermitian structure.

The space $\mathbb{C}[\mathfrak{w}_{\mathcal{O}}]$ decomposes under $K_{\mathcal{O}}|_{\mathfrak{w}_{\mathcal{O}}} = \prod_{\lambda>0} U(V_\lambda)$ as

$$\mathbb{C}[\mathfrak{w}_{\mathcal{O}}] = \bigotimes_{\lambda>0} \mathbb{C}[V_\lambda] = \bigoplus_{\alpha} \left(\bigotimes_{\lambda>0} \mathcal{P}_{\alpha(\lambda)}(V_\lambda) \right),$$

where $\alpha = (\alpha(\lambda) \ : \ \lambda > 0)$ is a set of non-negative integers. We obtain $(K_{\mathcal{O}}|_{\mathfrak{w}_{\mathcal{O}}})$-spherical functions

$$\phi_{1,\alpha}(w,t) = e^{it} \prod_{\lambda>0} L_{\alpha(\lambda)}^{(m(\lambda)-1)} \left(\frac{|w(\lambda)|^2}{2} \right) e^{-|w(\lambda)|^2/4}$$

on $H_{\mathfrak{w}_{\mathcal{O}}}$ where $w = \sum_{\lambda>0} w(\lambda) \in \sum_{\lambda>0} V_\lambda = \mathfrak{w}_{\mathcal{O}}$. Each of these spherical functions is associated to the coadjoint orbit through $\ell_1 \in \mathfrak{h}_{\mathfrak{w}_{\mathcal{O}}}^*$. Pulling $\phi_{1,\alpha}$ up to F_V yields the following:

Proposition 8.3. (See [Str91], [Fis06].) *The bounded $O(V)$-spherical functions on F_V can be described as follows: Given $\pi \in \widehat{F_V}$, there is an aligned point (b,B) in the coadjoint orbit associated with π. The space V decomposes as $V = \sum_{\lambda\geq 0} V_\lambda$ with respect to B. The representation space of π decomposes, with respect to K_π, as $\bigoplus_\alpha \left(\bigotimes_{\lambda>0} \mathcal{P}_{\alpha(\lambda)}(V_\lambda) \right)$, where $\alpha = (\alpha(\lambda) \ : \ \lambda > 0)$ is a set of non-negative integers. The spherical function $\phi_{\pi,\alpha}$ is the $O(V)$-average of*

$$(a,A) \mapsto e^{i(b,a(0))} e^{i(B,A)} \prod_{\lambda>0} L_{\alpha(\lambda)}^{(m(\lambda)-1)} \left(\frac{\lambda |a(\lambda)|^2}{2} \right) e^{-\lambda|a(\lambda)|^2/4} \tag{8.5}$$

where $a = a(0) + \sum_{\lambda>0} a(\lambda) \in V_0 + \sum_{\lambda>0} V_\lambda = V$.

We remark that Proposition 8.3 includes cases where $B = 0$. In such cases, $\mathcal{O} = \{(b, O)\}$ is a single point, $V_0 = V$ has dimension $m(0) = d$, the representation space of π has dimension 1, and the product in Proposition 8.3 is empty. We adopt the convention that the α-parameter in $\{\phi_{\pi,\alpha} \ : \ \pi, \ \alpha\}$ is empty when π is one dimensional. We obtain a single $O(V)$-spherical function on F_V, namely the $O(V)$-average of $(a,A) \mapsto e^{i(b,a)}$. This is, more explicitly,

$$(a,A) \mapsto \frac{2^{(d-2)/2}\Gamma(d/2)}{(r|a|)^{(d-2)/2}} J_{\frac{d-2}{2}}(r|a|) \tag{8.6}$$

when $r = |b|$ is non-zero and $(a,A) \mapsto 1$ when $b = 0$.

The derivation of Proposition 8.3 is easily adapted to encompass $SO(V)$-spherical functions. One obtains an $SO(V)$-spherical function for each $\alpha = (\alpha(\lambda) \ : \ \lambda > 0)$ as above, namely the $SO(V)$-average of (8.5). We denote this function by $\phi^\circ_{\pi,\alpha}$. Note that although

$$\Delta(O(V),F_V)=\{\phi_{\pi,\alpha}\ :\ \pi,\ \alpha\}\,,\qquad \Delta(SO(V),F_V)=\left\{\phi^\circ_{\pi,\alpha}\ :\ \pi,\ \alpha\right\},$$

one has $\phi_{\pi,\alpha}=\phi_{\pi',\alpha'}$ (resp. $\phi^\circ_{\pi,\alpha}=\phi^\circ_{\pi',\alpha'}$) whenever (π',α') differs from (π,α) by the action of $O(V)$ (resp. $SO(V)$). Parameterizations for $\Delta(O(V),F_V)$ and $\Delta(SO(V),F_V)$ are given in [Fis06]. The formulation of Proposition 8.3 will, however, suffice for our proof of Theorem 8.1.

9. Some invariant differential operators on F_V

One verifies that the following polynomials on $\mathfrak{f}_V=V\oplus\Lambda^2(V)=V\oplus so(V)$ are invariant under the action of $O(V)$.

- For $j=1,\ldots,\lfloor d/2\rfloor$ we define $c_j(a,A)=c_j(A)$ where
$$\det(I-xA)=1+\sum_{j=1}^{\lfloor d/2\rfloor}c_j(A)x^{2j}.$$
Here recall that $d=\dim(V)$. The polynomial c_j is homogeneous of degree $2j$ on $\mathfrak{z}=\Lambda^2(V)=so(V)$. Note that the characteristic polynomial for A can be written as $\det(xI-A)=x^n+\sum_j c_j(A)x^{n-2j}$.
- For $\ell\geq 0$ we have polynomials p_ℓ defined by
$$p_\ell(a,A)=\Big(a,A^{2\ell}a\Big).$$
Note that $p_0(a,A)=|a|^2$, independent of A.

From these polynomials, we obtain differential operators

$$c_j(Z),\quad p_\ell(U,Z)\quad\in\mathbb{D}_{O(V)}(F_V)$$

as follows.

Let $\mathcal{B}_V=\{U_1,\ldots,U_d\}$ be any orthonormal basis for V and set $Z_{ij}=U_i\wedge U_j$ so that $\mathcal{B}_\mathfrak{z}=\{Z_{ij}\ :\ 1\leq i<j\leq d\}$ is also an orthonormal basis for $\mathfrak{z}=\Lambda^2(V)$. We express $c_j:\mathfrak{f}_V\to\mathbb{R}$ and $p_\ell:f_V\to\mathbb{R}$ as polynomial functions in coordinates (u_i,z_{ij}) with respect to the basis $\mathcal{B}_V\cup\mathcal{B}_\mathfrak{z}$ for $\mathfrak{f}_V$. The resulting expressions do not depend on the choice of basis. Indeed, let $\mathcal{B}'_V=\{U'_1,\ldots,U'_d\}$ be another such basis and $\mathcal{B}'_\mathfrak{z}=\{Z'_{ij}\}$ where $Z'_{ij}=U'_i\wedge U'_j$. The coordinates (u'_i,z'_{ij}) with respect to $\mathcal{B}'_V\cup\mathcal{B}'_\mathfrak{z}$ are related to (u_i,z_{ij}) via $(u',z')=(ku,kuk^t)$ for some $k\in O(d)$. Since the polynomials c_j, p_ℓ are $O(V)$-invariant, we see that the expressions for c_j and p_ℓ in the two coordinate systems correspond under the change of variables $u_i\mapsto u'_i$, $z_{ij}\mapsto z'_{ij}$.

Since U_j and Z_{ij} are elements of $\mathfrak{f}_V$, we can view these as left-invariant vector fields on F_V. The operators $c_j(Z)$ and $p_\ell(U,Z)$ are obtained by replacing the variables u_j and z_{ij} by U_i and Z_{ij} in the expressions for c_j and p_ℓ with respect to the basis $\mathcal{B}_V\cup\mathcal{B}_\mathfrak{z}$. The preceding paragraph shows these to be well defined. Since the operators U_i are non-central, there is, however, an issue regarding the ordering of variables u_i within monomials in the expression for p_ℓ. We specify an ordering as follows. Let $a \in V$ have coordinates (a_i) with respect to $\mathcal{B}_V$ and let $A \in \mathfrak{z}$. Using the basis $\mathcal{B}_V$, A can be regarded as a $d \times d$ skew-symmetric matrix $(A_{ij} = (U_i, A(U_j)))$. Let

$$A^{2\ell} = \left(q_{ij}^{2\ell}(A)\right)_{ij}.$$

That is, $q_{ij}^{2\ell}(A)$ is the (i,j)'th entry of the $d \times d$ symmetric matrix $A^{2\ell}$. The polynomial $q_{ij}^{2\ell} : \mathfrak{z} \to \mathbb{R}$ is homogeneous of degree 2ℓ and we have

$$p_\ell(a, A) = \sum_{i,j} a_i q_{ij}^{2\ell}(A) a_j.$$

We define the operator $p_\ell(U, Z)$ unambiguously as

$$p_\ell(U, Z) = \sum_{i,j} U_i q_{ij}^{2\ell}(Z) U_j = \sum_{i,j} U_i U_j q_{ij}^{2\ell}(Z), \tag{9.1}$$

where "$q_{ij}^{2\ell}(Z)$" denotes the central operator obtained by replacing z_{ij} by Z_{ij} in the expression for $q_{ij}^{2\ell} : \mathfrak{z} \to \mathbb{R}$ in the basis $\mathcal{B}_\mathfrak{z}$.

The following result describes the eigenvalues that arise when $c_j(Z)$ and $p_\ell(U, Z)$ are applied to bounded $O(V)$-spherical functions on F_V.

Lemma 9.1. *Let (b, B) be an aligned point in $\mathfrak{f}_V^*$, $\pi \in \widehat{F_V}$ be the representation that corresponds to the coadjoint orbit through (b, B), $V = \sum_{\lambda \geq 0} V_\lambda$ be the eigenspace decomposition of V from Equation 8.3, and $m(\lambda)$ be as in Equation 8.4. Let $\alpha = (\alpha(\lambda) : \lambda > 0)$ be a set of non-negative integers and $\phi_{\pi,\alpha} \in \Delta(O(V), F_V)$ be the spherical function from Proposition 8.3. We have the following expressions for the eigenvalues of invariant differential operators:*

(a) $c_j(Z)^\wedge(\phi_{\pi,\alpha}) = (-1)^j c_j(B)$.

(b) $p_0(U, Z)^\wedge(\phi_{\pi,\alpha}) = -\sum_{\lambda > 0} \lambda(2\alpha(\lambda) + m(\lambda)) - |b|^2$.

(c) $p_\ell(U, Z)^\wedge(\phi_{\pi,\alpha}) = -\sum_{\lambda > 0} \lambda^{2\ell+1}(2\alpha(\lambda) + m(\lambda))$ *for* $\ell > 0$.

Proof. The representation π has central character $\pi(0, A) = e^{i(B,A)}$. So for $Z \in \mathfrak{z}$ we have the scalar operator

$$\pi(Z) = \left.\frac{d}{dt}\right|_{t=0} e^{i(B,\, tZ)} = i(B, Z).$$

Thus $\pi(Z_{ij}) = iB_{ij}$ and if f is any polynomial on $\mathfrak{z}$ then

$$\pi(f(Z)) = f(iB). \tag{9.2}$$

Using this fact together with Lemma 5.6 gives

$$c_j(Z)^\wedge(\phi_{\pi,\alpha}) = c_j(iB) = i^{2j} c_j(B) = (-1)^j c_j(B),$$

independent of α. This proves (a).

We choose an orthonormal basis $\mathcal{B}_V = \{U_1, \ldots, U_d\}$ for V that is compatible with the eigenspace decomposition $V = \sum_{\lambda \geq 0} V_\lambda$. That is, each U_i belongs to some V_λ.

This is possible since the eigenspaces for B^2 are mutually orthogonal. The operator $p_0(U,Z)$ is

$$p_0(U,Z) = U_1^2 + \cdots + U_d^2,$$

the sub-Laplacian for F_V. We write this as

$$p_0(U,Z) = \sum_{\lambda \geq 0} \mathcal{L}_\lambda \quad \text{where} \quad \mathcal{L}_\lambda = \sum_{\{i\,:\,U_i \in V_\lambda\}} U_i^2.$$

As explained in Section 8, π can be realized in a Hilbert space completion of $\mathbb{C}[\mathfrak{w}_\mathcal{O}] = \bigotimes_{\lambda>0} \mathbb{C}[V_\lambda]$ and $\phi_{\pi,\alpha}$ is associated with the subspace $P_\alpha = \bigotimes_{\lambda>0} \mathcal{P}_{\alpha(\lambda)}(V_\lambda)$. (When $B = 0$, we just have $\mathbb{C}[\mathfrak{w}_\mathcal{O}] = \mathbb{C}$.) For $\lambda > 0$, $\pi(\mathcal{L}_\lambda)$ acts on $\mathcal{P}_{\alpha(\lambda)}(V_\lambda)$ via the scalar

$$-\lambda(2\alpha(\lambda) + m(\lambda))$$

and annihilates $\mathcal{P}_{\alpha(\lambda')}(V_{\lambda'})$ for $\lambda' \neq \lambda$. Thus $\pi(\mathcal{L}_\lambda)$ acts on P_α as the scalar $-\lambda(2\alpha(\lambda) + m(\lambda))$. For $a \in V_0$, $\pi(a)$ acts on all of $\mathbb{C}[\mathfrak{w}_\mathcal{O}]$ via the scalar $e^{i(b,a)}$. As (b,B) is aligned, $b \in V_0 = \mathfrak{a}_\mathcal{O}$ and we see that $\pi(\mathcal{L}_0)$ acts by $-|b|^2$. We conclude that $\pi(p_0(U,Z)) = \sum_{\lambda \geq 0} \pi(\mathcal{L}_\lambda)$ acts on P_α by the scalar

$$-\sum_{\lambda>0} \lambda(2\alpha(\lambda) + m(\lambda)) - |b|^2.$$

In view of Lemma 5.6, this proves (b).

Next recall that for $\ell \geq 1$, p_ℓ is defined by $p_\ell(U,Z) = \sum_{i,j} U_i U_j q_{ij}^{2\ell}(Z)$, as in Equation 9.1. From Equation 9.2 we have

$$\pi(q_{ij}^{2\ell}(Z)) = q_{ij}^{2\ell}(iB) = (-1)^\ell q_{ij}^{2\ell}(B).$$

But $B^2|_{V_\lambda} = -\lambda^2$ and hence $q_{ij}^{2\ell}(B) = (-\lambda^2)^\ell$ for $i = j$ with $U_i \in V_\lambda$ and $q_{ij}^{2\ell}(B) = 0$ for $i \neq j$. Thus we have

$$\pi(p_\ell(U,Z)) = \sum_{\lambda \geq 0} \lambda^{2\ell} \pi(\mathcal{L}_\lambda) = \sum_{\lambda>0} \lambda^{2\ell} \pi(\mathcal{L}_\lambda).$$

Since $\pi(\mathcal{L}_\lambda)$ acts on P_α as $-\lambda(2\alpha(\lambda) + m(\lambda))$. we conclude that $\pi(p_\ell(U,Z))$ acts on P_α as

$$-\sum_{\lambda>0} \lambda^{2\ell+1}(2\alpha(\lambda) + m(\lambda)).$$

Again using Lemma 5.6, this proves (c). □

10. Convergence in the space $\Delta(O(V), F_V)$

Theorem 10.1. *Let $\phi \in \Delta(O(V), F_V)$ and $(\phi_n)_{n=1}^\infty$ be a sequence in $\Delta(O(V), F_V)$. Then $(\phi_n)_{n=1}^\infty$ converges to ϕ in the space $\Delta(O(V), F_V)$ if and only if*

$$\lim_{n\to\infty} c_j(Z)^\wedge(\phi_n) = c_j(Z)^\wedge(\phi) \quad \textit{and} \quad \lim_{n\to\infty} p_\ell(U,Z)^\wedge(\phi_n) = p_\ell(U,Z)^\wedge(\phi)$$

for $j = 1, \ldots, \lfloor d/2 \rfloor$ and $\ell = 0, \ldots, \lfloor d/2 \rfloor$.

Proof. Convergence in $\Delta(O(V), F_V)$ is uniform convergence on compact sets. If (ϕ_n) converges to ϕ in $\Delta(O(V), F_V)$ then it follows that

$$(D\phi_n)(0,0) \to (D\phi)(0,0)$$

so that

$$\widehat{D}(\phi_n) \to \widehat{D}(\phi)$$

for all $D \in \mathbb{D}_{O(V)}(F_V)$. It remains to prove the converse.

Let $\phi_n = \phi_{\pi_n,\alpha_n}$ where $\pi_n \in \widehat{F_V}$ is given by the aligned point $(b_n, B_n) \in \mathfrak{f}_V^* \cong \mathfrak{f}_V$. Similarly, let $\phi = \phi_{\pi,\alpha}$ where π is given by the aligned point (b, B). We have

$$c_j(Z)^\wedge(\phi_n) \to c_j(Z)^\wedge(\phi),$$

so in view of Lemma 9.1(a),

$$(-1)^j c_j(B_n) \to (-1)^j c_j(B).$$

Since the values $c_j(B_n)$, $c_j(B)$ yield the coefficients in the characteristic polynomials for B_n and B, we conclude that the characteristic polynomial for B_n converges to that for B uniformly on compact sets. It follows that the eigenvalues for B_n, together with their multiplicities, converge to those for B. More precisely, this means the following. Each B_n has pure imaginary eigenvalues $\pm i\mu$ and perhaps 0. If we list these eigenvalues with multiplicity in increasing order in $i\mathbb{R}$ then we obtain $d = \dim(V)$ sequences. Each of these converges to an eigenvalue for B and every eigenvalue for B, together with its multiplicity, is obtained in this way.

Suppose that the non-zero eigenvalues for B_n are $\pm i\mu_j(n)$ for $j = 1, \ldots, I(n)$ where

$$0 < \mu_1(n) < \mu_2(n) < \cdots < \mu_{I(n)}(n).$$

Let $\mathcal{V}_j(n)$ be the $(-\mu_j(n)^2)$-eigenspace for B_n^2 and let $\mathcal{V}_0(n) = \ker(B_n)$. The eigenspace decomposition with respect to B_n, as in Equation 8.3, reads

$$V = \sum_{j=0}^{I(n)} \mathcal{V}_j(n).$$

Note that $\mathcal{V}_0(n) = \{0\}$ when 0 is not an eigenvalue for B. We can partition the sequence $(\phi_n)_{n=1}^\infty$ into finitely many subsequences in which the values $I(n)$ and $\dim(\mathcal{V}_j(n))$ are constant in n. It suffices to show that each of these subsequences converges to ϕ. Thus we suppose henceforth that

$$I = I(n), \quad m_j = \frac{1}{2}\dim(\mathcal{V}_j(n)) \ \ (j = 1, \ldots, I), \quad m_0 = \dim(\mathcal{V}_0(n)), \tag{10.1}$$

independent of n. Let

$$\mathcal{S}^+ = \{\lambda > 0 \ : \ -\lambda^2 \text{ is an eigenvalue for } B^2\}, \quad \text{and} \quad \mathcal{S} = \mathcal{S}^+ \cup \{0\}. \tag{10.2}$$

The eigenspace decomposition (8.3) with respect to B is

$$V = \sum_{\lambda \in \mathcal{S}} V_\lambda.$$

Recall that $m(\lambda) = \frac{1}{2}\dim(V_\lambda)$ for $\lambda \neq 0$ and $m(0) = \dim(V_0)$. We have now the following facts.

- $\lim_{n\to\infty} \mu_j(n) \in \mathcal{S}$ for $j = 1, \ldots, I$.
- If $\lambda \in \mathcal{S}^+$ then $\lambda = \lim_{n\to\infty} \mu_j(n)$ for some $j \in \{1, \ldots, I\}$. We write $S_\lambda = \{j \ : \ \mu_j(n) \to \lambda\}$.
- For each $\lambda \in \mathcal{S}^+$, $m(\lambda) = \sum_{j \in S_\lambda} m_j$.
- $m(0) = m_0 + 2\sum_{j\in S_0} m_j$

Note that the data (π_n, α_n) and (π, α), which determine the spherical functions ϕ_n and ϕ, are only unique modulo the action of $K = O(V)$. By conjugating each B_n by a suitably chosen element $k_n \in O(V)$, we can assume that the subspace $\mathcal{V}_j(n)$ does not depend on n and is contained in V_0 for $j = 0$ and in V_λ where $\lambda = \lim_n \mu_j(n)$ for $j \geq 1$. In this regard, recall that, by Lemma 3.3, the action of $O(V)$ takes aligned points to aligned points. We let

$$\mathcal{V}_j = \mathcal{V}_j(n) \tag{10.3}$$

for $j = 0, \ldots, I$, independent of n, and now have:

- $V = \sum_{j=0}^{I} \mathcal{V}_j$ is the common eigenspace decomposition for V with respect to the B_n's. That is, $\mathcal{V}_0 = \ker(B_n)$ and for $j = 1, \ldots, I$, $\mathcal{V}_j$ is the $(-\mu_j(n)^2)$-eigenspace for B_n^2. We have $m_0 = \dim(\mathcal{V}_0)$ and $m_j = \dim(\mathcal{V}_j)/2$ for $j = 1, \ldots, I$.
- For each $\lambda \in \mathcal{S}^+$, $V_\lambda = \sum_{j\in S_\lambda} \mathcal{V}_j$.
- $V_0 = \mathcal{V}_0 + \sum_{j\in S_0} \mathcal{V}_j$.

Recall that the parameter α for the spherical function $\phi = \phi_{\pi,\alpha}$ is a set of non-negative integers $\{\alpha(\lambda) : \lambda \in \mathcal{S}^+\}$. For ease of notation, we write

$$\alpha_j(n) = \alpha(\mu_j(n))$$

for the parameters associated with ϕ_n.

Using Lemma 9.1 and the hypotheses that $p_\ell(U,Z)^\wedge(\phi_n) \to p_\ell(U,Z)^\wedge(\phi)$ for $\ell = 0, \ldots, \lfloor d/2 \rfloor$ we obtain, as $n \to \infty$,

$$\sum_{j=1}^{I} \mu_j(n)(2\alpha_j(n) + m_j) + |b_n|^2 \to \sum_{\lambda\in\mathcal{S}^+} \lambda(2\alpha(\lambda) + m(\lambda)) + |b|^2 \tag{10.4}$$

and

$$\sum_{j=1}^{I} \mu_j(n)^{2\ell+1}(2\alpha_j(n) + m_j) \to \sum_{\lambda\in\mathcal{S}^+} \lambda^{2\ell+1}(2\alpha(\lambda) + m(\lambda)) \tag{10.5}$$

for $\ell = 1, \ldots, \lfloor d/2 \rfloor$. Since all terms in (10.4) are non-negative, it follows that

$$\{\mu_j(n)\alpha_j(n) : n = 1 \ldots \infty\} \text{ is bounded for } j = 1, \ldots, I. \tag{10.6}$$

Hence for $\ell \geq 1$ we have $\lim_{n\to\infty} \mu_j(n)^{2\ell+1}\alpha_j(n) = 0$ whenever $\lim_{n\to\infty} \mu_j(n) = 0$. Thus we can write

$$\begin{aligned}\lim_{n\to\infty} \sum_{j=1}^{I} &\mu_j(n)^{2\ell+1}(2\alpha_j(n) + m_j) \\ &= \sum_{\lambda\in\mathcal{S}^+} \lim_{n\to\infty} \left[\sum_{j\in S_\lambda} \mu_j(n)^{2\ell+1}(2\alpha_j(n) + m_j)\right] \\ &= \sum_{\lambda\in\mathcal{S}^+} \left\{\lim_{n\to\infty} \left[\sum_{j\in S_\lambda} 2\mu_j(n)^{2\ell+1}\alpha_j(n)\right] + \lambda^{2\ell+1}m(\lambda)\right\},\end{aligned}$$

using the identity $m(\lambda) = \sum_{j\in S_\lambda} m_j$. Comparing the above with (10.5) we see that

$$\sum_{\lambda\in\mathcal{S}^+} \lambda^{2\ell+1}\alpha(\lambda) = \lim_{n\to\infty} \sum_{\lambda\in\mathcal{S}^+} \sum_{j\in S_\lambda} \mu_j(n)^{2\ell+1}\alpha_j(n).$$

If $\lim_{n\to\infty} \mu_j(n) \neq 0$, then $\{\alpha_j(n) : n = 1 \ldots \infty\}$ is bounded by (10.6). Since $\alpha_j(n)$ is an integer, we can suppose, by partitioning $(\phi_n)_{n=1}^\infty$ into a finite number of subsequences, that

$$\alpha_j(n) = \alpha_j$$

is constant in n for all j with $\lim_{n\to\infty} \mu_j(n) \neq 0$. We now have

$$\sum_{\lambda\in\mathcal{S}^+} \lambda^{2\ell+1}\alpha(\lambda) = \lim_{n\to\infty} \sum_{\lambda\in\mathcal{S}^+} \sum_{j\in S_\lambda} \mu_j(n)^{2\ell+1}\alpha_j = \sum_{\lambda\in\mathcal{S}^+} \lambda^{2\ell+1} \left(\sum_{j\in S_\lambda} \alpha_j\right).$$

As this holds for all $\ell = 1, \ldots, \lfloor d/2 \rfloor$ and $|\mathcal{S}^+| \leq \lfloor d/2 \rfloor$ we conclude that

$$\sum_{j\in S_\lambda} \alpha_j = \alpha(\lambda) \quad \text{for all } \lambda \in \mathcal{S}^+. \tag{10.7}$$

Recall that $\phi_n(a, A)$ is the $O(V)$-average of

$$e^{i(b_n,a)}e^{i(B_n,A)} \prod_{j=1}^{I} L_{\alpha_j(n)}^{(m_j-1)}\left(\frac{\mu_j(n)|a(j)|^2}{2}\right) e^{-\mu_j(n)|a(j)|^2/4} \tag{10.8}$$

where $a = \sum_{j=0}^{I} a(j)$ with $a(j) \in \mathcal{V}_j$. For $\lambda \in \mathcal{S}^+$ and $j \in S_\lambda$ we have $\alpha_j(n) = \alpha_j$ in this expression. The factors

$$\prod_{j\in S_\lambda} L_{\alpha_j}^{(m_j-1)}\left(\frac{\mu_j(n)|a(j)|^2}{2}\right) e^{-\mu_j(n)|a(j)|^2/4}$$

converge as $n \to \infty$ to

$$\prod_{j \in S_\lambda} L_{\alpha_j}^{(m_j-1)}\left(\frac{\lambda|a(j)|^2}{2}\right) e^{-\lambda|a(j)|^2/4}.$$

Averaging over $U(V_\lambda)$ gives

$$L_{\alpha(\lambda)}^{(m(\lambda)-1)}\left(\frac{\lambda|a(\lambda)|^2}{2}\right) e^{-\lambda|a(\lambda)|^2/4},$$

where $a = \sum_{\lambda \in \mathcal{S}} a(\lambda)$ with $a(\lambda) \in V_\lambda$. Here we have used $m(\lambda) = \sum_{j \in S_\lambda} m_j$. and Equation 10.7.

It remains to consider the factors

$$e^{i(b_n,a)} \prod_{j \in S_0} L_{\alpha_j(n)}^{(m_j-1)}\left(\frac{\mu_j(n)|a(j)|^2}{2}\right) e^{-\mu_j(n)|a(j)|^2/4} \tag{10.9}$$

from Formula 10.8. We will show that the $O(V_0)$-average of (10.9) converges to

$$\psi_b(a_0) = \int_{O(V_0)} e^{i(kb,a_0)} dk.$$

Equation (10.4) says that

$$\lim_{n\to\infty}\left(\sum_{j=1}^{I} \mu_j(n)(2\alpha_j(n) + m_j) + |b_n|^2\right) = \sum_{\lambda \in \mathcal{S}^+} \lambda(2\alpha(\lambda) + m(\lambda)) + |b|^2.$$

For $\lambda \in \mathcal{S}^+$ we have

$$\lim_{n\to\infty} \sum_{j \in S_\lambda} \mu_j(n)(2\alpha_j(n) + m_j) = \lambda(2\alpha(\lambda) + m(\lambda)),$$

again using $m(\lambda) = \sum_{j \in S_\lambda} m_j$ and $\sum_{j \in S_\lambda} \alpha_j = \alpha(\lambda)$. Hence we see that

$$\lim_{n\to\infty}\left(\sum_{j \in S_0} 2\mu_j(n)\alpha_j(n) + |b_n|^2\right) = |b|^2. \tag{10.10}$$

For $j \in S_0$, it may not be true that the sequence $\alpha_j(n)$ is bounded. Since (10.10) converges and all terms are non-negative, we see that $\{|b_n|^2 : n = 1 \ldots \infty\}$ and $\{\mu_j(n)\alpha_j(n) : n = 1 \ldots \infty\}$ must be bounded. Pass to *any* subsequence of (10.9). We need only show that this subsequence itself has some subsequence whose $O(V)$-average converges to $\psi_b(a_0)$. For this, we use a sub-subsequence for which $|b_n|^2$ converges and $\mu_j(n)\alpha_j(n)$ converges for each $j \in S_0$. Thus we now suppose that

$$\lim_{n\to\infty} 2\mu_j(n)\alpha_j(n) = h_j$$

say, for each $j \in S_0$ and that

$$\lim_{n\to\infty} |b_n|^2 = h_0.$$

Choose any vectors $c_j \in \mathcal{V}_j$ with $|c_j|^2 = h_j$. For $j \in S_0$ we have

$$\lim_{n\to\infty} L_{\alpha_j(n)}^{(m_j-1)}\left(\frac{\mu_j(n)|a(j)|^2}{2}\right)e^{-\mu_j(n)|a(j)|^2/4} = \int_{U(\mathcal{V}_j)} e^{i(kc_j,a(j))}dk.$$

This follows from the description of $\Delta(U(V_j), H_{V_j})$ presented in Section 7. We now see that (10.9) converges to

$$e^{i(c_0,a)}\prod_{j\in S_0}\int_{U(\mathcal{V}_j)} e^{i(kc_j,a(j))}dk = \int_{[\prod_{j\in S_0} U(\mathcal{V}_j)]} e^{i(kc,a)}dk,$$

where $c = c_0 + \sum_{j\in S_0} c_j$. Note that $c \in V_0$ since $V_0 = \mathcal{V}_0 + \sum_{j\in S_0}\mathcal{V}_j$. Averaging over $O(V_0)$ gives $\psi_c(a_0)$. But (10.10) yields

$$|c|^2 = |c_0|^2 + \sum_{j\in S_0}|c_j|^2 = h_0 + \sum_{j\in S_0} h_j = |b|^2$$

and hence $\psi_c(a_0) = \psi_b(a_0)$ as desired.

We have now shown that the $(O(V_0)\times\prod_{\lambda\in\mathcal{S}^+} U(V_\lambda))$-average of (10.8) converges to

$$\psi_b(a_0)e^{i(B,A)}\prod_{\lambda\in\mathcal{S}^+} L_{\alpha(\lambda)}^{(m(\lambda)-1)}\left(\frac{\lambda|a(\lambda)|^2}{2}\right)e^{-\lambda|a(\lambda)|^2/4}.$$

This is also the $O(V_0)$-average of

$$e^{i(b,a)}e^{i(B,A)}\prod_{\lambda\in\mathcal{S}^+} L_{\alpha(\lambda)}^{(m(\lambda)-1)}\left(\frac{\lambda|a(\lambda)|^2}{2}\right)e^{-\lambda|a(\lambda)|^2/4},$$

which is a function whose $O(V)$-average is ϕ. Thus ϕ_n converges to ϕ in $\Delta(O(V), F_V)$ as claimed. □

Lemma 9.1 shows that the eigenvalues $c_j(Z)^\wedge(\phi)$ and $p_\ell(U,Z)^\wedge(\phi)$ are real numbers and that $p_\ell(U,Z)^\wedge(\phi)$ is non-positive for all $\phi \in \Delta(O(V), F_V)$. Thus we obtain the following corollary to Theorem 10.1.

Corollary 10.2. *The map*

$$E : \Delta(O(V), F_V) \to (\mathbb{R}^+)^{\lfloor d/2\rfloor+1}\times(\mathbb{R})^{\lfloor d/2\rfloor}$$

defined by

$$E(\phi) = \Big(|p_0(U,Z)^\wedge(\phi)|, \ldots, |p_{\lfloor d/2\rfloor}(U,Z)^\wedge(\phi)|,\ c_1(Z)^\wedge(\phi), \ldots, c_{\lfloor d/2\rfloor}(Z)^\wedge(\phi)\Big)$$

is a homeomorphism onto its image.

This provides an analogue for $(O(V), F_V)$ of the Heisenberg fan model for $(U(V), H_V)$ and its generalization to Gelfand pairs (K, H_V) [BJRW96].

11. Proof of Theorem 8.1

As in the proof of Theorem 10.1, we let $\{\phi_n = \phi_{\pi_n,\alpha_n} : n = 1\ldots\infty\}$ and $\phi = \phi_{\pi,\alpha}$ be bounded $O(V)$-spherical functions on F_V. Let $\mathcal{O}_n = Ad^*(F_V)(b_n, B_n)$ and $\mathcal{O} = Ad^*(F_V)(b, B)$ be the coadjoint orbits associated to π_n and π, where the points (b_n, B_n) and (b, B) are aligned in $\mathfrak{f}_V^* \cong \mathfrak{f}_V$. We have

$$\mathcal{O}_n = \{(b_n + v, B_n) \ : \ v \in \mathfrak{w}_{\mathcal{O}_n}\}, \qquad \mathcal{O} = \{(b + v, B) \ : \ v \in \mathfrak{w}_{\mathcal{O}}\}$$

where $\mathfrak{w}_{\mathcal{O}_n} = Image(B_n)$, $\mathfrak{w}_{\mathcal{O}} = Image(B)$. Proposition 5.3 ensures that

$$\Psi(\phi_n) = O(V) \cdot (b_n + u_n, B_n), \qquad \Psi(\phi) = O(V) \cdot (b + u, B),$$

for some points $u_n \in \mathfrak{w}_{\mathcal{O}_n}$, $u \in \mathfrak{w}_{\mathcal{O}}$ which satisfy

$$\tau_{\mathcal{O}_n}(b_n + u_n, B_n) \in \mathcal{O}^{O(V)_{\pi_n}}(\alpha_n), \qquad \tau_{\mathcal{O}}(b + u, B) \in \mathcal{O}^{O(V)_\pi}(\alpha).$$

We will show that $(\phi_n)_{n=1}^\infty$ converges to ϕ in $\Delta(O(V), F_V)$ if and only if $(O(V) \cdot (u_n + b_n, B_n))_{n=1}^\infty$ converges to $O(V) \cdot (b + u, B)$ in $\mathcal{A}(O(V), F_V)$.

First suppose that $(\phi_n)_{n=1}^\infty$ converges to ϕ. Theorem 10.1 shows that $c_j(Z)^\wedge(\phi_n) \to c_j(Z)^\wedge(\phi)$ for $j = 1, \ldots, \lfloor d/2 \rfloor$ and $p_\ell(U, Z)^\wedge(\phi_n) \to p_\ell(U, Z)^\wedge(\phi)$ for $\ell = 0, \ldots, \lfloor d/2 \rfloor$. We will continue to employ the notation for eigenvalues and eigenspaces developed in the proof of Theorem 10.1. In particular, the proof shows that we can assume V has a common eigenspace decomposition "$V = \sum_{j=0}^I \mathcal{V}_j$" with respect to all of the B_n's and that this is related to the eigenspace decomposition "$V = \sum_{\lambda \in \mathcal{S}} V_\lambda$" with respect to B as explained in connection with Equations 10.1, 10.2 and 10.3.

The coadjoint orbits $\mathcal{O}_n$ and $\mathcal{O}$ correspond to coadjoint orbits in Heisenberg groups $H_{\mathfrak{w}_{\mathcal{O}_n}}$ and $H_{\mathfrak{w}_{\mathcal{O}}}$, as discussed prior to Proposition 8.3. Equation 7.1 now shows that

$$u_n = \sum_{j=1}^I \widetilde{u}_j(n) \quad \text{where} \quad \widetilde{u}_j(n) \in \mathcal{V}_j, \ |\widetilde{u}_j(n)|^2 = 2\mu_j(n)\alpha_j(n), \tag{11.1}$$

$$\text{and} \ \ u = \sum_{\lambda \in \mathcal{S}^+} u_\lambda \quad \text{where} \quad u_\lambda \in V_\lambda, \ |u_\lambda|^2 = 2\lambda\alpha(\lambda). \tag{11.2}$$

By using the action of $\prod_{j=1}^I U(\mathcal{V}_j) \subset O(V)$, we can suppose that

$$\widetilde{u}_j(n) = \sqrt{2\mu_j(n)\alpha_j(n)}\, \widetilde{e}_j$$

where $\widetilde{e}_j \in \mathcal{V}_j$ is any fixed unit vector, independent of n.

As in the proof of Theorem 10.1, we can suppose that for $j \in S_\lambda$ with $\lambda > 0$ we have $\alpha_j(n) = \alpha_j$, independent of n. Thus for $\lambda \in \mathcal{S}^+$, we can define

$$v_\lambda := \sum_{j \in S_\lambda} \left(\sqrt{2\lambda\alpha_j}\right) \widetilde{e}_j = \lim_{n\to\infty} \sum_{j \in S_\lambda} \widetilde{u}_j(n).$$

We have $v_\lambda \in V_\lambda$ since $V_\lambda = \sum_{j\in S_\lambda} \mathcal{V}_j$, and

$$|v_\lambda|^2 = 2\lambda \left[\sum_{j\in S_\lambda} \alpha_j\right] = 2\lambda\alpha(\lambda),$$

in view of Equation 10.7. Thus, using the fact that $|v_\lambda|^2 = |u_\lambda|^2$, we see that

$$\begin{aligned}\lim_{n\to\infty} \sum_{\lambda\in\mathcal{S}^+} \sum_{j\in S_\lambda} \widetilde{u}_j(n) &= \sum_{\lambda\in\mathcal{S}^+} v_\lambda \\ &\in \left(\prod_{\lambda\in\mathcal{S}^+} U(V_\lambda)\right)\left(\sum_{\lambda\in\mathcal{S}^+} u_\lambda\right) = \left(\prod_{\lambda\in\mathcal{S}^+} U(V_\lambda)\right) u.\end{aligned}$$

Letting $u_0(n) = b_n + \sum_{j\in S_0} \widetilde{u}_j(n)$, we have $u_0(n) \in V_0$ since $V_0 = \mathcal{V}_0 + \sum_{j\in S_0} \mathcal{V}_j$. Moreover

$$\begin{aligned}|u_0(n)|^2 &= |b_n|^2 + \sum_{j\in S_0} |\widetilde{u}_j(n)|^2 \\ &= |b_n|^2 + \sum_{j\in S_0} 2\mu_j(n)\alpha_j(n) \xrightarrow[n\to\infty]{} |b|^2\end{aligned}$$

by (10.10). Thus the $\big(O(V_0) \times \prod_{\lambda\in\mathcal{S}^+} U(V_\lambda)\big)$-orbit through $b_n + u_n = u_0(n) + \sum_{\lambda\in\mathcal{S}^+}\sum_{j\in S_\lambda} \widetilde{u}_j(n)$ converges to the $\big(O(V_0) \times \prod_{\lambda\in\mathcal{S}^+} U(V_\lambda)\big)$-orbit through $b + u$. Hence also $(O(V)\cdot(u_n + b_n, B_n))_{n=1}^\infty$ converges to $O(V)\cdot(b+u, B)$.

Conversely, suppose that $O(V)\cdot(u_n+b_n, B_n) \to O(V)\cdot(b+u, B)$ in $\mathcal{A}(O(V), F_V)$. Since c_j and p_ℓ are $O(V)$-invariant polynomials, it follows that

$$c_j(B_n) \xrightarrow[n\to\infty]{} c_j(B) \quad \text{for } j = 1, \dots, \lfloor d/2\rfloor \tag{11.3}$$

$$\text{and} \quad p_\ell(b_n + u_n, B_n) \xrightarrow[n\to\infty]{} p_\ell(b+u, B) \quad \text{for all } l \geq 0. \tag{11.4}$$

From (11.3) and Lemma 9.1(a) we have that

$$c_j(Z)^\wedge(\phi_n) \xrightarrow[n\to\infty]{} c_j(Z)^\wedge(\phi) \tag{11.5}$$

for $j = 1, \dots, \lfloor d/2\rfloor$. Also, as in the proof of Theorem 10.1, it follows from (11.3) that the eigenvalues for B_n converge to those for B. Thus we can assume that we have compatible eigenspace decompositions as in the first part of this proof. Since $\tau_{\mathcal{O}_n}(b_n + u_n, B_n) = \alpha_n$ and $\tau_{\mathcal{O}}(b+u, B) = \alpha$, Equations 11.1 and 11.2 hold. Thus we have

$$\begin{aligned}p_0(b_n + u_n, B_n) &= |b_n|^2 + |u_n|^2 = \sum_{j=1}^{I} 2\mu_j(n)\alpha_j(n) + |b_n|^2 \\ \text{and} \quad p_0(b+u, B) &= \sum_{\lambda\in\mathcal{S}^+} 2\lambda\alpha(\lambda) + |b|^2.\end{aligned}$$

Since $p_0(b_n+u_n,B_n)\to p_0(b+u,B)$ and $m(\lambda)=\sum_{j\in S_\lambda} m_j$ for $\lambda\in\mathcal{S}^+$, we conclude that

$$\left[\sum_{j=1}^{I}\mu_j(n)(2\alpha_j(n)+m_j)+|b_n|^2\right]\xrightarrow[n\to\infty]{}\left[\sum_{\lambda\in\mathcal{S}^+}\lambda(2\alpha(\lambda)+m(\lambda))+|b|^2\right].$$

But this gives

$$p_0(U,Z)^\wedge(\phi_n)\xrightarrow[n\to\infty]{}p_0(U,Z)^\wedge(\phi), \tag{11.6}$$

via Lemma 9.1(b). For $\ell\geq 1$ we have

$$p_\ell(b_n+u_n,B_n)=\Big(b_n+u_n,B_n^{2\ell}(b_n+u_n)\Big)=\Big(u_n,B_n^{2\ell}u_n\Big)$$

since $b_n\in\ker(B_n)=\mathcal{V}_0$. As $u_n=\sum_{j=1}^{I}\left(\sqrt{2\mu_j(n)\alpha_j(n)}\right)\widetilde{e}_j$ and $B_n^2|_{\mathcal{V}_j}=-\mu_j(n)^2$ we conclude that

$$p_\ell(b_n+u_n,B_n)=(-1)^\ell\sum_{j=1}^{I}2\mu_j(n)^{2\ell+1}\alpha_j(n).$$

Similarly

$$p_\ell(b+u,B)=(-1)^\ell\sum_{\lambda\in\mathcal{S}^+}2\lambda^{2\ell+1}\alpha(\lambda).$$

Using $p_\ell(b_n+u_n,B_n)\to p_\ell(b+u,B)$ and Lemma 9.1(c), we conclude that

$$p_\ell(U,Z)^\wedge(\phi_n)\xrightarrow[n\to\infty]{}p_\ell(U,Z)^\wedge(\phi) \tag{11.7}$$

for $\ell\geq 1$, just as for the case $\ell=0$ above.

Having established (11.5), (11.6) and (11.7), it now follows from Theorem 10.1 that $\phi_n\to\phi$ in $\Delta(O(V),F_V)$. This completes the proof of Theorem 8.1. □

12. Spherical functions on F_3

In this section we examine the models for $\Delta(O(V),F_V)$ provided by Corollary 10.2 and Theorem 8.1 in the simplest case: $d=\dim(V)=3$. We will write

$$K=O(3),\quad \mathfrak{n}=\mathbb{R}^3\times\Lambda^2(\mathbb{R}^3)=\mathbb{R}^3\times so(3),\quad N=\exp(\mathfrak{n})$$

and for $\lambda\in\mathbb{R}$ let

$$B_\lambda=\left[\begin{array}{c|cc}0&0&0\\ \hline 0&0&\lambda\\ 0&-\lambda&0\end{array}\right].$$

One can check that each K-orbit in $\mathfrak{n}^*\cong\mathfrak{n}$ through an aligned point contains a unique aligned point with one of two possible forms:

$$\big((r,0,0),\ B_\lambda\big)\text{ with } r\geq 0,\ \lambda>0\qquad\text{or}\qquad\big((r,0,0),\ 0\big)\text{ with } r\geq 0.$$

The space $\Delta(O(V),F_V)$ can be parameterized by the set

$$\mathcal{P}=\{(r,\lambda,m)\ :\ r\geq 0,\ \lambda>0,\ m\in\mathbb{Z}^+\}\cup\{(r,0)\ :\ r\geq 0\}.$$

The spherical function $\phi_{r,\lambda,m}$ for parameter $(r,\lambda,m)\in\mathcal{P}$ is the K-average of

$$(a,A)\mapsto e^{ira_1}e^{i\lambda A_{2,3}}L_m^{(0)}\left(\frac{\lambda(a_2^2+a_3^2)}{2}\right)e^{-\lambda(a_2^2+a_3^2)/4}.$$

This follows from Proposition 8.3, since $(B_\lambda,A)=-tr(B_\lambda A)/2=\lambda A_{2,3}$. The spherical functions $\phi_{r,0}$ associated to parameters $(r,0)\in\mathcal{P}$ are $\phi_{0,0}=1$ and

$$\phi_{r,0}(a,A)=\frac{2^{1/2}\Gamma(3/2)}{(r|a|)^{1/2}}J_{\frac{1}{2}}(r|a|)=\frac{\sin(r|a|)}{r|a|}$$

for $r>0$. Here we have used (8.6) together with the classical identities

$$\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2},\qquad J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sin(x).$$

We now consider the map

$$E:\Delta(K,N)\to(\mathbb{R}^+)^2\times\mathbb{R},\quad E(\phi)=(|p_0(U,Z)^\wedge(\phi)|,|p_1(U,Z)^\wedge(\phi)|,c_1(Z)^\wedge(\phi))$$

given by Corollary 10.2. Using Lemma 9.1 we compute

$$\begin{aligned}
p_0(U,Z)^\wedge(\phi_{r,\lambda,m})&=-\lambda(2m+1)-r^2\\
p_0(U,Z)^\wedge(\phi_{r,0})&=-r^2\\
p_1(U,Z)^\wedge(\phi_{r,\lambda,m})&=-\lambda^3(2m+1)\\
p_1(U,Z)^\wedge(\phi_{r,0})&=0\\
c_1(Z)^\wedge(\phi_{r,\lambda,m})&=-c_1(B_\lambda)=-\lambda^2\\
c_1(Z)^\wedge(\phi_{r,0})&=-c_1(0)=0.
\end{aligned}$$

Thus we have

$$E(\phi_{r,\lambda,m})=(\lambda(2m+1)+r^2,\ \lambda^3(2m+1),\ -\lambda^2),\quad E(\phi_{r,0})=(r^2,\ 0,\ 0).$$

For $m\in\mathbb{Z}^+$ let $\mathcal{S}_m\subset(\mathbb{R}^+)^3$ be defined as

$$\mathcal{S}_m=\{(\lambda(2m+1)+r^2,\ \lambda^3(2m+1),\ \lambda^2)\ :\ r\geq0,\ \lambda\geq0\}$$

We see that the image $E(\Delta(K,N))$ of $\Delta(K,N)$ in $(\mathbb{R}^+)^2\times\mathbb{R}$ is homeomorphic to

$$\mathcal{E}=\bigcup_{m=0}^{\infty}\mathcal{S}_m\subset(\mathbb{R}^+)^3 \tag{12.1}$$

Finally we consider the space $\mathcal{A}(K,N)$, which is homeomorphic to $\Delta(K,N)$ by Theorem 8.1. From Equation 11.2 we see that $\ell=((r,\sqrt{2\lambda m},0),B_\lambda)$ is a spherical point in $\mathcal{O}=Ad^*(N)((r,0,0),B_\lambda)$ with $\tau_{\mathcal{O}}(\ell)=m$. Thus we have

$$\Psi(\phi_{r,\lambda,m})=K\cdot((r,(2\lambda m)^{1/2},0),B_\lambda),\quad\Psi(\phi_{r,0})=K\cdot((r,0,0),0).$$

So $\mathcal{A}(K,N) = \mathcal{X}/K$ where $\mathcal{X}$ is the closed subset of $\mathfrak{n}^* = \mathfrak{n}$ given by

$$\mathcal{X} = (\mathbb{R}^3 \times \{0\}) \cup \left\{ (b,B) \ : \ \frac{||b_1||^2}{2||B||} \in \mathbb{Z} \right\}$$

and $b = b_0 + b_1$ denotes the Fitting decomposition for $b \in \mathbb{R}^3$ with respect to $B \in so(3)$. The inverse mapping for Ψ is given on $\mathcal{X}/K$ by

$$K \cdot (b,B) \mapsto \begin{cases} \phi_{||b_0||,\,||B||,\,||b_1||^2/2||B||} & \text{for } B \neq 0 \\ \phi_{||b||,\,0} & \text{for } B = 0 \end{cases}$$

and the model $\mathcal{X}/K$ is homeomorphic to $\mathcal{E}$ via

$$\mathcal{X}/K \to \mathcal{E}, \quad K \cdot (b,B) \mapsto \big(||b||^2 + ||B||, \ ||b_1||^2||B|| + ||B||^3, \ ||B||^2 \big).$$

From either model one sees, for example, that a sequence of spherical functions $(\phi_{r_n,\lambda_n,m_n})_{n=1}^{\infty}$ converges in $\Delta(K,N)$ to $\phi_{r,0}$ when (r_n), (λ_n) and $(\lambda_n m_n)$ are convergent with $\lim \lambda_n = 0$ and $\lim(r_n^2 + 2\lambda_n m_m) = r^2$.

References

[BJLR97] C. Benson, J. Jenkins, R. Lipsman, and G. Ratcliff. A geometric criterion for Gelfand pairs associated with the Heisenberg group. *Pacific J. Math.*, 178:1–36, 1997.

[BJR90] C. Benson, J. Jenkins, and G. Ratcliff. On Gelfand pairs associated with solvable Lie groups. *Trans. Amer. Math. Soc.*, 321:85–116, 1990.

[BJR92] C. Benson, J. Jenkins, and G. Ratcliff. Bounded K-spherical functions on Heisenberg groups. *J. Funct. Anal.*, 105:409–443, 1992.

[BJR99] C. Benson, J. Jenkins, and G. Ratcliff. The orbit method and Gelfand pairs associated with nilpotent groups. *J. Geometric Analysis*, 9:569–582, 1999.

[BJRW96] C. Benson, J. Jenkins, G. Ratcliff, and T. Worku. Spectra for Gelfand pairs associated with the Heisenberg group. *Colloq. Math.*, 71:305–328, 1996.

[Bou81] P. Bougerol. Théorème central limite local sur certains groupes de Lie. *Ann. Scient. Éc. Norm. Sup.*, 14:403–432, 1981.

[BR96] C. Benson and G. Ratcliff. A classification for multiplicity free actions. *J. Algebra*, 181:152–186, 1996.

[Bri85] M. Brion. Représentations exceptionnelles des groupes semi-simples. *Ann. Sci. École Norm. Sup. (4)*, 18(2):345–387, 1985.

[Bro73] I. D. Brown. Dual topology of a nilpotent lie group. *Ann. Sci. École Norm. Sup.*, 6:407–411, 1973.

[Car87] G. Carcano. A commutativity condition for algebras of invariant functions. *Boll. Un. Mat. Italiano*, 7:1091–1105, 1987.

[CG90] L. Corwin and F. Greenleaf. *Representations of nilpotent Lie groups and their applications, Part 1: Basic theory and examples*, volume 18 of *Studies in advanced math.* Cambridge Univ. Press, Cambridge, 1990.

[Far87] J. Faraut. Analyse harmonique et fonctions speciales. In *Deux Courses d'Analyse Harmonique*. Birkhäuser, Boston, 1987.

[Fis06] V. Fischer. The bounded spherical functions on the free two-step nilpotent Lie group. *J. Lie Theory*, 16:351–370, 2006.

[Gel50] I. M. Gelfand. Spherical functions on symmetric spaces. *Dokl. Akad. Nauk USSR*, 70:5–8, 1950. Amer. Math. Soc. Transl. 37 (1964), 39-44.

[HR80] A. Hulanicki and F. Ricci. A tauberian theorem and tangential convergence of bounded harmonic functions on balls in $\mathbb{C}^n$. *Inventiones Math.*, 62:325–331, 1980.

[Kac80] V. Kac. Some remarks on nilpotent orbits. *J. Algebra*, 64:190–213, 1980.

[Kir62] A. A. Kirillov. Unitary representations of nilpotent Lie groups. *Uspeki Mat. Nauk*, 17:57–110, 1962. Russian Math. Surveys 17 (1962), 57-110.

[Kir04] A. A. Kirillov. *Lectures on the Orbit Method*, volume 64 of *Graduate Studies in Math.* Amer. Math. Soc., Providence, Rhode Island, 2004.

[Kor80] A. Koranyi. Some applications of Gelfand pairs in classical analysis. *C.I.M.E.*, 1980.

[KR83] A. Kaplan and F. Ricci. *Harmonic analysis on groups of Heisenberg type*, pages 416–435. Lecture Notes in Math. 992. Springer Verlag, Berlin, 1983.

[Lau00] J. Lauret. Gelfand pairs attached to representations of compact Lie groups. *Transformation Groups*, 5:307–324, 2000.

[Lea98] A. Leahy. A classification of multiplicity free representations. *J. Lie Theory*, 8:367–391, 1998.

[Lip80] R. Lipsman. Orbit theory and harmonic analysis on Lie groups with co-compact nilradical. *J. Math. Pure Appl.*, 59:337–374, 1980.

[Lip82] R. Lipsman. Orbit theory and representations of Lie groups with co-compact radical. *J. Math. Pure Appl.*, 61:17–39, 1982.

[LL94] H. Leptin and J. Ludwig. *Unitary Representation Theory of Exponential Lie Groups.* Walter de Gruyter and Co., Berlin, 1994.

[Nis01] N. Nishihara. A geometric criterion for Gelfand pairs associated with nilpotent Lie groups. *J. Funct. Anal.*, 183:148–163, 2001.

[Puk78] L. Pukanszky. Unitary representations on Lie groups with co-compact radical and applications. *Trans. Amer. Math. Soc.*, 236:1–49, 1978.

[Ric85] F. Ricci. Commutative algebras of invariant functions on groups of Heisenberg type. *J. London Math. Soc.*, 32:265–271, 1985.

[Ste88] K. Stempac. An agebra associated with the generalized sublaplacian. *Studia Math.*, 88:245–256, 1988.

[Str91] R. Strichartz. L^p harmonic analysis and Radon transforms on the Heisenberg group. *J. Funct. Anal.*, 96:350–406, 1991.

[Vin01] E. B. Vinberg. Commutative homogeneous spaces and co-isotropic symplectic actions. *Russian Math. Surveys*, 56:1–60, 2001.

[Vin03] E. B. Vinberg. Commutative homogeneous spaces of Heisenberg type. *Trans. Moscow Math. Soc.*, 64:45–78, 2003.

[Wol92] J. Wolf. The uncertainty principle for Gelfand pairs. *Nova J. Algebra Geom.*, 1:383–396, 1992.

[Wol06] J. Wolf. Spherical functions on Euclidean space. *J. Funct. Anal.*, 239(1):127–136, 2006.

[Yak04] O. S. Yakimova. Saturated commutative homogeneous spaces of Heisenberg type. *Acta Applicandae Math.*, 81:339–345, 2004.

[Yak05] O. S. Yakimova. *Gelfand pairs.* Bonner Mathematische Schriften [Bonn Mathematical Publications], 374. Universität Bonn Mathematisches Institut, Bonn, 2005. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2004.

Dept of Mathematics, East Carolina University, Greenville, NC 27858
E-mail address: bensonf@ecu.edu, ratcliffg@ecu.edu