THE SPHERICAL TRANSFORM OF A SCHWARTZ FUNCTION ON THE HEISENBERG GROUP

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Abstract. Suppose that $K \subset U(n)$ is a compact Lie group acting on the $(2n+1)$-dimensional Heisenberg group H_n. We say that (K, H_n) is a Gelfand pair if the convolution algebra $L^1_K(H_n)$ of integrable K-invariant functions on H_n is commutative. In this case, the Gelfand space $\Delta(K, H_n)$ is equipped with the Godement-Plancherel measure, and the spherical transform $^\wedge : L^2_K(H_n) \to L^2(\Delta(K, H_n))$ is an isometry. The main result in this paper provides a complete characterization of the set $\mathcal{S}_K(H_n)^\wedge = \{\widehat{f} \mid f \in \mathcal{S}_K(H_n)\}$ of spherical transforms of K-invariant Schwartz functions on H_n. We show that a function F on $\Delta(K, H_n)$ belongs to $\mathcal{S}_K(H_n)^\wedge$ if and only if the functions obtained from F via application of certain derivatives and difference operators satisfy decay conditions. We also consider spherical series expansions for K-invariant Schwartz functions on H_n modulo its center.

Contents

1. Introduction

Given a complex vector space V of dimension n with Hermitian inner product $\langle \cdot, \cdot \rangle$, one forms the Heisenberg group $H_n = V \times \mathbb{R}$ with group law

$$(z,t)(z',t') = \left(z + z', t + t' - \frac{1}{2} Im\langle z, z' \rangle \right).$$

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In order to present explicit formulae, one can use an orthonormal basis to identify V with $\mathbb{C}^n$ so that $\langle z, z' \rangle = z \cdot \overline{z'}$ for $z, z' \in \mathbb{C}^n$. The group $U(n)$ of unitary transformation of $(V, \langle \cdot, \cdot \rangle)$ acts by automorphisms on H_n via

$$k \cdot (z,t) := (kz, t) \quad \text{for } k \in U(n) \text{ and } (z,t) \in H_n$$

and on the space of polynomials $\mathbb{C}[V]$ via

$$k \cdot p(z) := p(k^{-1}z) \quad \text{for } k \in U(n),\, p \in \mathbb{C}[V] \text{ and } z \in V.$$

If K is a compact Lie subgroup of $U(n)$ then we say that (K, H_n) is a *Gelfand pair* when the algebra $L^1_K(H_n)$ of K-invariant L^1-functions on H_n is commutative under convolution. One has the following result:

Theorem 1.1 (cf. [Car87], [BJR90])**.** *(K, H_n) is a Gelfand pair if and only if the representation of K on $\mathbb{C}[V]$ is multiplicity free.*

Using Theorem 1.1, one sees that the group $U(n)$ and many of its proper subgroups K yield Gelfand pairs. Working from Theorem 1.1, one obtains a complete classification of all such subgroups [Kac80, BR96]. This classification shows that the theory of Gelfand pairs associated with Heisenberg groups is quite rich. There are, for example, twenty distinct families of Gelfand pairs (K, H_n) where K is connected and acts irreducibly on V. The current paper concerns analysis in this setting.

There is a well developed theory of spherical functions associated with Gelfand pairs of the form (K, H_n). We denote the space of bounded K-spherical functions, equipped with the compact-open topology, by $\Delta(K, H_n)$. There are two types of bounded K-spherical functions:

- Type 1: $\Delta_1(K, H_n) = \{\phi_{\alpha,\lambda} \mid \alpha \in \Lambda, \lambda \in \mathbb{R}^\times\}$. Here Λ is a countable index set that parameterizes the decomposition
$$\mathbb{C}[V] = \sum_{\alpha \in \Lambda} P_\alpha$$
of $\mathbb{C}[V]$ into K-irreducible subspaces. $\Delta_1(K, H_n)$ is dense and of full Godement-Plancherel measure in $\Delta(K, H_n)$.
- Type 2: $\Delta_2(K, H_n) = \{\eta_{Kw} \mid w \in V\}$, where Kw is a K-orbit in V.

The reader will find the relevant definitions below in Section 2 along with a summary of some results from earlier work. The K-spherical transform $\widehat{f} : \Delta(K, H_n) \to \mathbb{C}$ for a function $f \in L^1_K(H_n)$ is defined by

$$\widehat{f}(\psi) := \int_{H_n} f(z,t)\overline{\psi(z,t)}\, dzdt,$$

were "$dzdt$" denotes Haar measure for the group H_n. $\widehat{f}$ belongs to the space $C_\circ(\Delta(K, H_n))$ of continuous functions on $\Delta(K, H_n)$ that vanish at infinity.

The spherical functions for the Gelfand pairs $(U(n), H_n)$ and $(\mathbb{T}^n, H_n)$ have been obtained from a variety of viewpoints in works including [BJR92, Far87, HR80, Kor80, Ste88, Str91]. The systematic study of other Gelfand pairs (K, H_n) and their spherical functions is less standard but no less natural. In any case, as explained below in Section 2.3, the K-spherical functions are joint eigenfunctions for $\partial/\partial t$ and the Heisenberg sub-Laplacian. As shown in [Str89, Str91], the joint spectral theory for these operators is central to harmonic analysis on the Heisenberg group and the spherical transform is its main tool. In [HR80] the spherical functions and spherical transform for $(\mathbb{T}^n, H_n)$ are used to establish a tangential convergence theorem for bounded harmonic functions on the hyperbolic space $SU(1, N+1)/U(n+1)$. The reader will find further applications of radial functions and the spherical transform in J. Faraut's book [Far87].

What can one say about the spherical transform of a K-invariant Schwartz function on H_n? More precisely, letting $\mathcal{S}_K(H_n)$ denote the space of K-invariant Schwartz functions on H_n, we seek to characterize the subspace

$$\mathcal{S}_K(H_n)^\wedge = \left\{ \widehat{f} \mid f \in \mathcal{S}_K(H_n) \right\}$$

of $C_\circ(\Delta(K, H_n))$. The main result in this paper is Theorem 6.1 below, which provides a complete solution to this problem. Before describing the contents of this theorem we wish to provide some background and motivation for the study of $\mathcal{S}_K(H_n)$ via the spherical transform.

Schwartz functions have played an important role in harmonic analysis with nilpotent groups since the work of Kirilov [Kir62]. Let N be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. The exponential map $\exp : \mathfrak{n} \to N$ is a polynomial diffeomorphism and one defines the (Fréchet) space $\mathcal{S}(N)$ of Schwartz functions on N via identification with the usual space $\mathcal{S}(\mathfrak{n})$ of Schwartz functions on the vector space $\mathfrak{n}$:

$$\mathcal{S}(N) := \{ f : N \to \mathbb{C} \mid f \circ \exp \in \mathcal{S}(\mathfrak{n}) \}.$$

$\mathcal{S}(N)$ is dense in $L^p(N)$ for each p and carries an algebra structure given by the convolution product. Moreover, it is known that the primitive ideal space for $\mathcal{S}(N)$ is isomorphic to that of both $L^1(N)$ and $C^*(N)$ [Lud88]. The Heisenberg groups H_n are the simplest groups for which $\mathcal{S}(N)$ is non-abelian.. Recall that the group Fourier transform for a function $f \in L^1(N)$ associates to $\pi \in \widehat{N}$, an irreducible unitary representation of N, the bounded operator

$$\pi(f) = \int_N f(x) \pi(x)^* dx$$

in the representation space of π. This generalizes the usual Euclidean Fourier transform for the case $N = \mathbb{R}^n$. The importance of Schwartz functions in Euclidean harmonic analysis arises from the fact that $\mathcal{S}(\mathbb{R}^n)$ is preserved by the Fourier transform. It is thus very natural to seek a characterization of $\mathcal{S}(N)$ via the group Fourier transform; a problem solved by R. Howe in [How77]. A related result for the case $N = H_n$ can be found in D. Geller's paper [Gel77].

One can sometimes obtain abelian subalgebras of $\mathcal{S}(N)$ by considering "radial" functions. This is of interest even when $N = \mathbb{R}^n$. Indeed, the algebra $\mathcal{S}_{O(n)}(\mathbb{R}^n)$ of radial Schwartz functions on $\mathbb{R}^n$ can be identified with $\mathcal{S}(\mathbb{R}^+)$ and the Fourier transform becomes a Hankel transform on $\mathcal{S}(\mathbb{R}^+)$. This is the spherical transform for the Gelfand pair obtained from the action of the orthogonal group $O(n)$ on $\mathbb{R}^n$ [Hel84].

In the case $N = H_n$, we are led to consider the abelian subalgebras $\mathcal{S}_K(H_n)$ arising from Gelfand pairs (K, H_n). The group Fourier transform on $\mathcal{S}_K(H_n)$ then reduces to the K-spherical transform as follows. The unitary dual of H_n can be written as

$$\widehat{H_n} = \{\pi_\lambda \mid \lambda \in \mathbb{R}^\times\} \cup \{\chi_w \mid w \in V\}$$

where each π_λ is an infinite dimensional representation and each χ_w is one dimensional. One can realize π_λ in a Fock space that contains $\mathbb{C}[V]$ as a dense subspace. For $f \in \mathcal{S}_K(H_n)$, $\pi_\lambda(f)$ is diagonalized by the decomposition $\mathbb{C}[V] = \sum_{\alpha\in\Lambda} P_\alpha$ and one has that

$$\pi_\lambda(f)|_{P_\alpha} = \widehat{f}(\phi_{\lambda,\alpha}) \quad \text{and} \quad \chi_w(f) = \widehat{f}(\eta_{Kw}).$$

Thus, characterizing $\mathcal{S}_K(H_n)^\wedge$ enables one to construct smooth functions on H_n which decay rapidly at infinity whose group Fourier transforms are prescribed in advance subject to some conditions. This provides an alternative to the application of the results from [How77] and to Dixmier's functional calculus [Dix60] in the Heisenberg group setting. J. Ludwig has recently applied the group Fourier transform and Dixmier's functional calculus to the study of hull minimal ideals in the algebra $\mathcal{S}(H_n)$ [Lud]. The authors believe that Theorem 6.1 (and Corollary 6.3) will provide a new technique to approach problems of this nature. We hope to pursue this idea elsewhere.

Theorem 6.1 provides conditions that are both necessary and sufficient for a function F on $\Delta(K, H_n)$ to belong to the space $\mathcal{S}_K(H_n)^\wedge$:

1. F is continuous on $\Delta(K, H_n)$.
2. $w \mapsto F(\eta_{Kw})$ is a Schwartz function on V.
3. $\lambda \mapsto F(\phi_{\alpha,\lambda})$ is smooth on $\mathbb{R}^\times$ and the functions $\partial_\lambda^m F(\phi_{\alpha,\lambda})$ satisfy certain decay conditions. In particular, $\partial_\lambda^m F(\phi_{\alpha,\lambda})$ is a rapidly decreasing sequence in α for each fixed $\lambda \in \mathbb{R}^\times$.
4. Certain "derivatives" of F also satisfy the three conditions above. These are defined on $\Delta_1(K, H_n)$ as specific combinations of ∂_λ and "difference operators" which play the role of differentiation in the discrete parameter $\alpha \in \Lambda$.

The precise formulation of these conditions can be found in Section 6. The "derivatives" of functions in $\mathcal{S}_K(H_n)^\wedge$ referred to above are operators corresponding to multiplication of functions in $\mathcal{S}_K(H_n)$ by certain polynomials. The difference operators in the discrete parameter $\alpha \in \Lambda$ are linear operators whose coefficients are "*generalized binomial coefficients*". These coefficients were introduced by Z. Yan in [Yan] and appear in many formulas concerning the type 1 spherical functions. A summary of their properties is given below in Section 4. A more complete discussion of generalized binomial coefficients will appear in [BR]. We remark that for the case $K = \mathbb{T}^n$, difference operators similar to the ones used here appear in the papers [Gel77], [dMM79] and [MS94].

One consequence of the estimates involved in our characterization of $\mathcal{S}_K(H_n)^\wedge$ is that $f \in \mathcal{S}_K(H_n)$ can be recovered from $F = \widehat{f}$ via inversion of the spherical transform. One has

$$f(z,t) = \left(\frac{1}{2\pi}\right)^{n+1} \int_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} \dim(P_\alpha) F\left(\phi_{\alpha,\lambda}\right) \phi_{\alpha,\lambda}(z,t)|\lambda|^n d\lambda. \tag{1.1}$$

The first two conditions in our characterization of $\mathcal{S}_K(H_n)^\wedge$ play a rather subtle role in the sufficiency proof. Indeed, the bounded K-spherical functions η_{Kw} of type 2 do not appear in Formula 1.1. In order to control the growth of $f(z,t)$ in the central direction, we introduce a λ-derivative into the right hand side of Formula 1.1. An application of integration by parts produces two boundary terms at $\lambda = 0$ and we need to show that these cancel. This is done in Section 7 where we prove that

$$\lim_{\lambda\to 0^\pm} \sum_{\alpha\in\Lambda} \dim(P_\alpha)|\lambda|^n F\left(\phi_{\alpha,\lambda}\right) \phi_{\alpha,\lambda}(z,0) = \left(\frac{1}{2\pi}\right)^n \int_V F\left(\eta_{Kw}\right) \eta_{Kw}(z) dw.$$

(See Proposition 7.1.) It is here that the continuity and behavior of F on $\Delta_2(K, H_n)$ comes into play.

We also consider the related problem of characterizing the the space $\mathcal{S}_K(V)$ of K-invariant Schwartz functions on V via the spherical transform. The functions ϕ°_α on V defined by $\phi^\circ_\alpha(z) = \phi_\alpha(z,0)$ form a complete orthogonal system in the space $L^2_K(V)$ of square integrable K-invariant functions on V (see Proposition 3.1). The ϕ°_α's are eigenfunctions for differential operators that arise from K-invariant and left-invariant differential operators on the Heisenberg group and (modulo dilations) for the Fourier transform. The coefficients in the L^2-expansion of $f \in L^2_K(V)$ as a series in $\{\phi^\circ_\alpha \mid \alpha \in \Lambda\}$ are (up to normalization) $\widehat{f}(\alpha) := \langle f, \phi^\circ_\alpha\rangle_2$. Theorem 5.1 asserts that F belongs to

$$\mathcal{S}_K(V)^\wedge = \{\widehat{f} \mid f \in \mathcal{S}_K(V)\}$$

if and only if $\{F(\alpha) \mid \alpha \in \Lambda\}$ is a rapidly decreasing sequence.

The remainder of this paper is structured as follows. Section 2 contains preliminary material and summarizes results from previous work concerning spherical functions. Section 3 concerns the spherical series expansions that arise by suppressing the central direction in the Heisenberg group. The material in this section appeared in [Yan] but we have provided new and complete proofs here. Section 4 concerns combinatorial properties of the generalized binomial coefficients. Our characterizations of $\mathcal{S}_K(V)^\wedge$ and $\mathcal{S}_K(H_n)^\wedge$ appear in Section 5 and Section 6 respectively. As a corollary to Theorem 6.1, we show that for $K = U(n)$ or $\mathbb{T}^n$, one can construct a function $f \in \mathcal{S}_K(H_n)$ whose spherical transform $\widehat{f}$ has a pre-determined compact support in

$\Delta(K, H_n)$. The sufficiency proof for Theorem 6.1 is completed in Section 7, which concerns the analysis of boundary terms.

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2. Notation and Preliminaries

We need to establish notation and recall some results concerning spherical functions associated with Gelfand pairs (K, H_n). We refer the reader to [BJR90], [BJR92] and [BJRW] for complete details on this material.

Throughout this paper, K will always denote a closed Lie subgroup of $U(n)$ for which (K, H_n) is a Gelfand pair.

2.1. **Decomposition of $\mathbb{C}[V]$.** We decompose $\mathbb{C}[V]$ into K-irreducible subspaces P_α,

$$\mathbb{C}[V] = \sum_{\alpha\in\Lambda} P_\alpha \tag{2.1}$$

where Λ is some countably infinite index set. Theorem 1.1 ensures that this decomposition is canonical. Since the representation of K on $\mathbb{C}[V]$ preserves the space $\mathcal{P}_m(V)$ of homogeneous polynomials of degree m, each P_α is a subspace of some $\mathcal{P}_m(V)$. We write $|\alpha|$ for the degree of homogeneity of the polynomials in P_α, so that $P_\alpha \subset \mathcal{P}_{|\alpha|}(V)$. We will write d_α for the dimension of P_α and denote by $0 \in \Lambda$ the index for the scalar polynomials $P_0 = \mathcal{P}_0(V) = \mathbb{C}$.

2.2. **Invariant polynomials.** Since the representation of K on $\mathbb{C}[V]$ is multiplicity free, there can be no non-constant K-invariant holomorphic polynomials. One does, however, have invariant polynomials on the underlying real vector space $V_\mathbb{R}$ for V. We denote the set of these by $\mathbb{C}[V_\mathbb{R}]^K$. One obtains a canonical basis for $\mathbb{C}[V_\mathbb{R}]^K$ as follows. Given $\alpha \in \Lambda$ and any orthonormal basis $\{v_1, \ldots, v_{d_\alpha}\}$ for P_α, we let

$$p_\alpha(z) := \frac{1}{d_\alpha}\sum_{j=1}^{d_\alpha} v_j(z)\overline{v_j(z)}. \tag{2.2}$$

The polynomial p_α on $V_\mathbb{R}$ is $\mathbb{R}^+$-valued, K-invariant, and homogeneous of degree $2|\alpha|$. The definition of p_α does not depend on the choice of basis for P_α and $\{p_\alpha \mid \alpha \in \Lambda\}$ is a vector space basis for $\mathbb{C}[V_\mathbb{R}]^K$. One computes that

$$\sum_{|\alpha|=m} d_\alpha p_\alpha(z) = \frac{1}{m!}\gamma(z)^m \tag{2.3}$$

where $\gamma(z)$ is defined by

$$\gamma(z) := |z|^2/2.$$

A result in [HU91] ensures that the algebra $\mathbb{C}[V_{\mathbb{R}}]^K$ is freely generated by a canonical finite subset $\{\gamma_1, \gamma_2, \ldots, \gamma_d\} \subset \{p_\alpha \mid \alpha \in \Lambda\}$,

$$\mathbb{C}[V_{\mathbb{R}}]^K = \mathbb{C}[\gamma_1, \gamma_2, \ldots, \gamma_d]. \tag{2.4}$$

We call the generators $\{\gamma_1, \ldots, \gamma_d\}$ the *fundamental invariants* for the action of K on V. When K acts irreducibly on V, the polynomial γ is one of the fundamental invariants and we can suppose that $\gamma_1 = \gamma$.

2.3. **Invariant differential operators.** One basis for the Lie algebra of left invariant vector fields on H_n is written as $\{Z_1, Z_2, \ldots, Z_n, \overline{Z}_1, \overline{Z}_2, \ldots, \overline{Z}_n, T\}$ where

$$Z_j = 2\frac{\partial}{\partial \overline{z}_j} + i\frac{z_j}{2}\frac{\partial}{\partial t}, \quad \overline{Z}_j = 2\frac{\partial}{\partial z_j} - i\frac{\overline{z}_j}{2}\frac{\partial}{\partial t}, \tag{2.5}$$

and

$$T := \frac{\partial}{\partial t}.$$

With these conventions one has $[Z_j, \overline{Z}_j] = -2iT$. The first order operators $Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n, T$ generate the algebra $\mathbb{D}(H_n)$ of left-invariant differential operators on H_n. We denote the subalgebra of K-invariant differential operators by

$$\mathbb{D}_K(H_n) := \{D \in \mathbb{D}(H_n) \mid D(f \circ k) = D(f) \circ k \text{ for } k \in K,\ f \in C^\infty(H_n)\}.$$

Since (K, H_n) is a Gelfand pair, $\mathbb{D}_K(H_n)$ is an abelian algebra. A result due to Thomas [Tho82] shows that the converse is also true, at least when K is connected.

One differential operator will play a key role in this paper. This is the *Heisenberg sub-Laplacian* defined by

$$\mathcal{L} := \frac{1}{2}\sum_{j=1}^{n}\left(Z_j\overline{Z}_j + \overline{Z}_j Z_j\right). \tag{2.6}$$

$\mathcal{L}$ is $U(n)$-invariant and hence belongs to $\mathbb{D}_K(H_n)$ for all Gelfand pairs (K, H_n). Since the formal adjoints of Z_j and $\overline{Z}_j$ as operators on $L^2(H_n)$ are $Z_j^* = -\overline{Z}_j$ and $\overline{Z}_j^* = -Z_j$, we see that $\mathcal{L}$ is essentially self-adjoint on $L^2(H_n)$.

The algebra $\mathbb{D}_K(H_n)$ is generated by $\{L_{\gamma_1}, \ldots, L_{\gamma_d}, T\}$, where L_{γ_j} is the operator obtained from $\gamma_j(Z, \overline{Z})$ by symmetrization.

2.4. **Spherical functions.** A smooth function $\psi : H_n \to \mathbb{C}$ is called *K-spherical* if

1. ψ is K-invariant,
2. ψ is an eigenfunction for every $D \in \mathbb{D}_K(H_n)$, and
3. $\psi(0,0) = 1$.

We write $\widehat{D}(\psi)$ for the eigenvalue of $D \in \mathbb{D}_K(H_n)$ on a K-spherical function ψ, that is $D(\psi) = \widehat{D}(\psi)\psi$.

We denote the set of positive definite K-spherical functions on H_n by $\Delta(K, H_n)$. In [BJR90], it is shown that every bounded K-spherical function is positive definite, so $\Delta(K, H_n)$ is also the set of bounded K-spherical functions. We remark that this result contrasts with the situation for symmetric spaces. (See eg. [GV88].) As shown in [BJR92], the bounded K-spherical functions can be derived from the representation theory for H_n together with the action of K on V.

The infinite dimensional irreducible unitary representations of H_n can be realized in *Fock space*. This is the space $\mathcal{F}$ consisting of entire functions $f : V \to \mathbb{C}$ which are square integrable with respect to $e^{-|z|^2/2}dz$ with Hilbert space structure

$$\langle f, g\rangle_{\mathcal{F}} = \left(\frac{1}{2\pi}\right)^n \int_V f(z)\overline{g(z)}e^{-|z|^2/2}dz.$$

Here "dz" denotes Lebesgue measure on $V_{\mathbb{R}} \cong \mathbb{R}^{2n}$. The holomorphic polynomials $\mathbb{C}[V]$ form a dense subspace in $\mathcal{F}$. One has an irreducible unitary representation π of H_n on $\mathcal{F}$ defined as

$$(\pi(z,t)f)(w) = e^{it-\frac{1}{2}<w,z>-\frac{1}{4}|z|^2}f(w+z).$$

For $\alpha \in \Lambda$ let

$$\phi_\alpha(z,t) := \frac{1}{d_\alpha}\sum_{j=1}^{d_\alpha}\langle \pi(z,t)v_j, v_j\rangle_{\mathcal{F}}, \tag{2.7}$$

where $\{v_1, \ldots, v_{d_\alpha}\}$ is an orthonormal basis for P_α. This description of ϕ_α does not depend on our choice of basis $\{v_j\}$. Define $\phi_{\alpha,\lambda}$ for $\lambda \in \mathbb{R}^\times$ and $\alpha \in \Lambda$ by

$$\phi_{\alpha,\lambda}(z,t) := \phi_\alpha\left(\sqrt{|\lambda|}z, \lambda t\right), \tag{2.8}$$

so that $\phi_\alpha = \phi_{\alpha,1}$. The $\phi_{\alpha,\lambda}$'s are distinct bounded K-spherical functions. We refer to these elements of $\Delta(K, H_n)$ as the spherical functions of *type 1*. From Equation 2.7 one can show that ϕ_α has the general form

$$\phi_\alpha(z,t) = e^{it}q_\alpha(z)e^{-\frac{1}{4}|z|^2} = e^{it}q_\alpha(z)e^{-\gamma(z)/2}$$

where q_α is a K-invariant polynomial on $V_{\mathbb{R}}$ with homogeneous component of highest degree given by $(-1)^{|\alpha|}p_\alpha$. As for the p_α's, the set $\{q_\alpha \mid \alpha \in \Lambda\}$ is a basis for the vector space $\mathbb{C}[V_{\mathbb{R}}]^K$.

The eigenvalues of the Heisenberg sub-Laplacian $\mathcal{L}$ on the type 1 K-spherical functions are given by

$$\widehat{\mathcal{L}}\left(\phi_{\alpha,\lambda}\right) = -|\lambda|(2|\alpha| + n). \tag{2.9}$$

This follows from Proposition 3.20 in [BJR92] together with Lemma 3.4 in [BJRW]. The key point here is that the *quantum harmonic oscillator* $\pi(\mathcal{L})$ on Fock space $\mathcal{F}$ acts via the scalar $-(2m + n)$ on $\mathcal{P}_m(V)$.

In addition to the K-spherical functions of type 1, there are K-spherical functions which arise from the one dimensional representations of H_n. For $w \in V$, let

$$\eta_w(z,t) := \int_K e^{iRe\langle w,kz\rangle}dk = \int_K e^{iRe\langle z,kw\rangle}dk \tag{2.10}$$

where "dk" denotes normalized Haar measure on K. These are the bounded K-spherical functions of *type 2.* Note that η_0 is the constant function 1 and $\eta_w = \eta_{w'}$ if and only if $Kw = Kw'$. Thus we have one K-spherical function for each K-orbit in V and sometimes write "η_{Kw}" in place of "η_w".

It is shown in [BJR92] that every bounded K-spherical function is of type 1 or type 2. Thus we have:

Theorem 2.1. *The bounded K-spherical functions on H_n are parameterized by the set $\big(\Lambda \times \mathbb{R}^\times\big) \cup (V/K)$ via $\Delta(K,H_n) = \Delta_1(K,H_n) \cup \Delta_2(K,H_n)$ where*

$$\Delta_1(K,H_n) = \big\{\phi_{\alpha,\lambda} \mid \alpha \in \Lambda, \lambda \in \mathbb{R}^\times\big\} \quad \textit{and} \quad \Delta_2(K,H_n) = \big\{\eta_{Kw} \mid w \in V\big\}.$$

2.5. **Topology on $\Delta(K,H_n)$.** We give $\Delta(K,H_n)$ the (compact-open) topology of uniform convergence on compact sets. A detailed discussion of this topology can be found in [BJRW]. A key fact is that $\Delta_1(K,H_n)$ is dense in $\Delta(K,H_n)$. Moreover, the space $\Delta(K,H_n)$ can be embedded in $(\mathbb{R}^+)^d \times \mathbb{R}$. Indeed, let $L_{\gamma_1}, \dots, L_{\gamma_d} \in \mathbb{D}_K(H_n)$ be differential operators obtained from the fundamental invariants via symmetrization and let $T = \frac{\partial}{\partial t}$.

Theorem 2.2 (cf. [BJRW])**.** *The map $E : \Delta(K,H_n) \to (\mathbb{R}^+)^d \times \mathbb{R}$ defined by*

$$E(\psi) = \big(\big|\widehat{L_{\gamma_1}}(\psi)\big|, \dots, \big|\widehat{L_{\gamma_d}}(\psi)\big|, -i\widehat{T}(\psi)\big)$$

is a homeomorphism onto its image.

The eigenvalues $\widehat{L_{\gamma_j}}(\psi)$ for the L_{γ_j}'s on $\Delta(K,H_n)$ are real numbers with constant sign and one can describe the map E more explicitly as follows. Let $e_j(\alpha) = \big|\widehat{L_{\gamma_j}}(\phi_\alpha)\big|$ and $2m_j$ be the degree of γ_j. Then

$$E(\phi_{\alpha,\lambda}) = \big(e_1(\alpha)|\lambda|^{m_1}, \dots, e_d(\alpha)|\lambda|^{m_d}, \lambda\big) \quad \text{and} \quad E(\eta_w) = \big(\gamma_1(w), \dots, \gamma_d(w), 0\big).$$

We can use the map E and a metric on $(\mathbb{R}^+)^d \times \mathbb{R}$ to produce a metric on $\Delta(K,H_n)$ which induces the compact-open topology. For example,

$$d(\psi_1,\psi_2) = \sum_{j=1}^{d} \big(\big|\widehat{L_{\gamma_1}}(\psi_j) - \widehat{L_{\gamma_1}}(\psi_j)\big| + \big|\widehat{T}(\psi_1) - \widehat{T}(\psi_2)\big| \tag{2.11}$$

is one such metric on $\Delta(K,H_n)$.

2.6. **The K-spherical transform.** The K-spherical transform for $f \in L^1_K(H_n)$ is the function $\widehat{f}$ on $\Delta(K,H_n)$ defined by

$$\widehat{f}(\psi) := \int_{H_n} f(z,t)\overline{\psi(z,t)}\, dzdt.$$

Here "$dzdt$" denotes Haar measure for the group H_n, which is simply Euclidean measure on $V_\mathbb{R} \times \mathbb{R}$. $\widehat{f}$ is a bounded function with

$$||\widehat{f}||_\infty \leq ||f||_1$$

for $f \in L^1_K(H_n)$. This follows immediately from the fact that for $\psi \in \Delta(K, H_n)$ one has $|\psi(z,t)| \le \psi(0,0) = 1$, since ψ is positive definite.

The compact-open topology on $\Delta(K, H_n)$ is the smallest topology which makes all of the maps $\{\widehat{f} \mid f \in L^1_K(H_n)\}$ continuous. Since $L^1_K(H_n)$ is a Banach $\star$-algebra (with involution defined via $f^*(z,t) := \overline{f(-z,-t)}$), $\widehat{f}$ belongs to the space $C_\circ(\Delta(K, H_n))$ of continuous functions on $\Delta(K, H_n)$ that vanish at infinity.

2.7. **Godement-Plancherel measure.** Godement's Plancherel Theory for Gelfand pairs (G, K) (cf. [God62], or section 1.6 in [GV88]) ensures that there exists a unique positive Borel measure $d\mu$ on the space $\Delta(K, H_n)$ for which

$$\int_{H_n} |f(z,t)|^2 dz dt = \int_{\Delta(K,H_n)} |\widehat{f}(\psi)|^2 d\mu(\psi) \tag{2.12}$$

for all continuous functions $f \in L^1_K(H_n) \cap L^2_K(H_n)$. If $f \in L^1_K(H_n) \cap L^2_K(H_n)$ is continuous and $\widehat{f}$ is integrable with respect to $d\mu$ then one has the *Inversion Formula*:

$$f(z,t) = \int_{\Delta(K,H_n)} \widehat{f}(\psi)\psi(z,t)\, d\mu(\psi). \tag{2.13}$$

The spherical transform $f \mapsto \widehat{f}$ extends uniquely to an isomorphism between $L^2_K(H_n)$ and $L^2(\Delta(K, H_n), d\mu)$.

The following result makes the Godement-Plancherel measure on $\Delta(K, H_n)$ explicit. Given $F : \Delta(K, H_n) \to \mathbb{C}$, we write $F(\alpha, \lambda)$ in place of $F(\phi_{\alpha,\lambda})$. The reader can find proofs of Theorem 2.3 in [BJRW] and [Yan]. The result for $K = U(n)$ is also discussed in [Far87] and [Str91].

Theorem 2.3. *The Godement-Plancherel measure $d\mu$ on $\Delta(K, H_n)$ is given by*

$$\int_{\Delta(K,H_n)} F(\psi)\, d\mu(\psi) = \left(\frac{1}{2\pi}\right)^{n+1} \int_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} d_\alpha F\,(\alpha, \lambda)\, |\lambda|^n d\lambda.$$

Note that $\Delta_2(K, H_n)$ is a set of measure zero in $\Delta(K, H_n)$.

2.8. **Fourier transforms.** We define the Fourier transform $\mathcal{F}_V(f) : V \to \mathbb{C}$ of $f \in L^1(V)$ by

$$\mathcal{F}_V(f)(w) := \int_V f(z) e^{-iRe\langle z,w\rangle} dz$$

where "dz" denotes Euclidean measure on $V_{\mathbb{R}}$. With this normalization one has

$$||\mathcal{F}_V(f)||_2 = (2\pi)^n ||f||_2$$

and the inversion formula reads

$$\int_V \mathcal{F}_V(f)(w) e^{iRe\langle z,w\rangle} dw = (2\pi)^{2n} f(z)$$

for suitable $f : V \to \mathbb{C}$. Similarly, the Euclidean Fourier transform $\mathcal{F}_H(f)$ of $f \in L^1(H_n)$ is defined by

$$\mathcal{F}_H(f)(w,s) = \int_{H_n} f(z,t)e^{-iRe\langle z,w\rangle}e^{-its}dzdt. \tag{2.14}$$

If $f \in L^1_K(H_n)$ then the Fourier and spherical transforms are related via

$$\widehat{f}(\eta_w) = \mathcal{F}_H(f)(w,0). \tag{2.15}$$

Indeed, for $w \in V$, one has

$$\begin{aligned}\widehat{f}(\eta_w) &= \int_{H_n}\int_K f(z,t)e^{-iRe\langle kz,w\rangle}dkdzdt \\ &= \int_{H_n}\int_K f(kz,t)e^{-iRe\langle z,w\rangle}dkdzdt \\ &= \int_{H_n} f(z,t)e^{-iRe\langle z,w\rangle}dzdt \\ &= \mathcal{F}_H(f)(w,0).\end{aligned}$$

3. K-spherical series expansions on V

The dependence of a K-spherical function $\psi(z,t)$ on the central variable t is rather trivial. Thus, it is natural to suppress this direction and consider functions on V. The Godement-Plancherel Theory in section 2.6 leads to orthogonal eigenfunction expansions for K-invariant functions on V. The ideas here are quite standard and well known. For $f : H_n \to \mathbb{C}$, we define $f^\circ : V \to \mathbb{C}$ by

$$f^\circ(z) := f(z,0).$$

Given $g : V \to \mathbb{C}$ we define $g_1 : H_n \to \mathbb{C}$ by

$$g_1(z,t) := g(z)e^{it}.$$

If f is a K-invariant function on H_n then f° is K-invariant on V. If g is K-invariant on V then g_1 is K-invariant on H_n and $(g_1)^\circ = g$. For $D \in \mathbb{D}(H_n)$ define a differential operator on D' on $V_\mathbb{R}$ by

$$D'(g) := (D\,(g_1))^\circ \quad \text{ for } g \in C^\infty(V_\mathbb{R}).$$

It is easy to verify that if $D \in \mathbb{D}_K(H_n)$ then D' is a K-invariant differential operator on $V_\mathbb{R}$. Moreover

$$D'\left(\phi_\alpha^\circ\right) = \left(D\left((\phi_\alpha^\circ)_1\right)\right)^\circ = \left(D\left(\phi_\alpha\right)\right)^\circ = \widehat{D}\left(\phi_\alpha\right)\phi_\alpha^\circ.$$

Thus $\phi_\alpha^\circ(z) = q_\alpha(z)e^{-\gamma(z)/2}$ is an eigenfunction for D' whenever $D \in \mathbb{D}_K(H_n)$.

As an example, consider the Heisenberg sub-Laplacian $\mathcal{L}$ given by Equation 2.6. Using Equations 2.5 one computes that

$$\mathcal{L}' = 4\Delta - \gamma/2 + E \tag{3.1}$$

where $\Delta := \sum_{j=1}^n \frac{\partial}{\partial z_j}\frac{\partial}{\partial \overline{z}_j}$ (so that 4Δ is the usual Laplace operator on $V_{\mathbb{R}}$) and $E := \sum_{j=1}^n \left(\overline{z}_j \frac{\partial}{\partial \overline{z}_j} - z_j \frac{\partial}{\partial z_j}\right)$. A homogeneity argument shows that $E(\phi_\alpha^\circ) = 0$ for $\alpha \in \Lambda$. Equation 2.9 thus yields

$$(4\Delta - \gamma/2)\,(\phi_\alpha^\circ) = \mathcal{L}'(\phi_\alpha^\circ) = -(2|\alpha| + n)\phi_\alpha^\circ. \tag{3.2}$$

Proposition 3.1 (cf. [Yan])**.** *$\{\phi_\alpha^\circ \mid \alpha \in \Lambda\}$ is a complete orthogonal system in $L^2_K(V)$ with $||\phi_\alpha^\circ||_2^2 = (2\pi)^n/d_\alpha$.*

Proof. The functional form for $\phi_{\alpha,\lambda}$ shows that $\phi_\alpha^\circ = q_\alpha(z)e^{-\frac{1}{4}|z|^2}$ for some $q_\alpha \in \mathbb{C}[V_{\mathbb{R}}]^K$. Thus ϕ_α° is a K-invariant Schwartz function and hence belongs to $L^2_K(V_{\mathbb{R}})$. For $v \in \mathcal{F}$, define $M_v(z) := \langle \pi(z,0)v, v\rangle_{\mathcal{F}}$. Since π is a square-integrable representation, M_v is square-integrable. Let $\{v_1, \ldots, v_{d_\alpha}\}$, $\{u_1, \ldots, u_{d_\beta}\}$ be orthonormal bases for P_α, P_β. One has $\phi_\alpha^\circ = \frac{1}{d_\alpha}\sum_i M_{v_i}$, $\phi_\beta^\circ = \frac{1}{d_\beta}\sum_j M_{u_j}$ so that

$$\langle \phi_\alpha^\circ, \phi_\beta^\circ \rangle_2 = \frac{1}{d_\alpha d_\beta}\sum_{i,j}\langle M_{v_i}, M_{u_j}\rangle_2.$$

Standard facts concerning square-integrable representations ensure that

$$\langle M_{v_i}, M_{u_j}\rangle_2 = c\left|\langle v_i, u_j\rangle_{\mathcal{F}}\right|^2$$

for some constant c. Thus we have

$$\langle \phi_\alpha^\circ, \phi_\beta^\circ \rangle_2 = \frac{c\delta_{\alpha,\beta}}{d_\alpha},$$

and hence the ϕ_α°'s are pair-wise orthogonal.

To compute the L^2-norm of ϕ_α°, one must determine c. As 1 is a unit vector in $\mathcal{F}$, one has

$$\begin{aligned} c = \langle M_1, M_1\rangle_2 &= ||e^{-\frac{1}{4}|z|^2}||_2^2 \\ &= \int_V e^{-\frac{1}{2}|z|^2}dz \\ &= \frac{2\pi^n}{(n-1)!}\int_0^\infty e^{-r^2/2}r^{2n-1}dr \\ &= (2\pi)^n. \end{aligned}$$

It remains to show that $\{\phi_\alpha^\circ \mid \alpha \in \Lambda\}$ is complete in $L^2_K(V)$. Suppose that $f \in L^2_K(V)$ and that $\langle f, \phi_\alpha^\circ\rangle_2 = 0$ for all $\alpha \in \Lambda$. Since $\phi_\alpha^\circ(z) = q_\alpha(z)e^{-|z|^2/4}$ where $\{q_\alpha \mid \alpha \in \Lambda\}$ is a basis for $\mathbb{C}[V_{\mathbb{R}}]^K$, we see that

$$\int_V f(z)p(z)e^{-|z|^2/4}dz = 0$$

for all polynomials $p \in \mathbb{C}[V_{\mathbb{R}}]$. It follows that $f = 0$ almost everywhere and hence that $\{\phi_\alpha^\circ \mid \alpha \in \Lambda\}$ is complete. $\square$

Note that $|\phi^\circ_\alpha(z)|$ is bounded by $\phi^\circ_\alpha(0) = 1$. For $f \in L^p_K(V)$ and $\alpha \in \Lambda$ let $\widehat{f}(\alpha)$ be defined by

$$\widehat{f}(\alpha) = \langle f, \phi^\circ_\alpha \rangle_2 = \int_V f(z)\phi^\circ_\alpha(z)dz.$$

Corollary 3.2. *If $f \in L^2_K(V)$ then*

$$f = \sum_{\alpha\in\Lambda} \widehat{f}(\alpha)\frac{\phi^\circ_\alpha}{||\phi^\circ_\alpha||_2^2} = \left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha \widehat{f}(\alpha)\phi^\circ_\alpha \quad \text{in } L^2(V) \text{ and}$$

$$||f||_2^2 = \sum_{\alpha\in\Lambda} \frac{|\widehat{f}(\alpha)|^2}{||\phi^\circ_\alpha||_2^2} = \left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha |\widehat{f}(\alpha)|^2.$$

It is shown in [Yan] that the ϕ°_α's are essentially eigenfunctions for $\mathcal{F}_V$. More precisely,

Proposition 3.3 (cf. [Yan]). *The function $\widetilde{\phi_\alpha}(z) := \phi^\circ_\alpha(\sqrt{2}z)$ satisfies*

$$\mathcal{F}_V\left(\widetilde{\phi_\alpha}\right) = (2\pi)^n(-1)^{|\alpha|}\widetilde{\phi_\alpha}.$$

The factor of $\sqrt{2}$ appears above but not in [Yan] because different coordinates on H_n are used there. The key fact used in Yan's proof of Proposition 3.3 is that ϕ°_α can be expressed in terms of the Hermite-Weber transform of p_α. One can also prove Proposition 3.3 by using the fact that a suitable multiple of the Hermite operator $-4\Delta + 2\gamma$ is an infinitesimal generator for the Fourier transform. (See for example, pg 122 in [HT92].) With the notational conventions here, this fact can be written as

$$\mathcal{F}_V = (2\pi i)^n \exp(i\pi(\Delta - \gamma/2)).$$

In view of Equation 3.2, we have

$$(\Delta - \gamma/2)\widetilde{\phi_\alpha} = -\frac{2|\alpha| + n}{2}\widetilde{\phi_\alpha}$$

and hence

$$\mathcal{F}_V\left(\widetilde{\phi_\alpha}\right) = (2\pi i)^n e^{-i\pi(2|\alpha|+n)/2}\widetilde{\phi_\alpha} = (2\pi)^n(-1)^{|\alpha|}\widetilde{\phi_\alpha}$$

as stated.

For $r > 0$, we let $\delta_r : V \to V$ be the dilation

$$\delta_r(z) := rz.$$

Since one has $\mathcal{F}_V(f \circ \delta_r) = \left(\frac{1}{r}\right)^{2n} \mathcal{F}_V(f) \circ \delta_{1/r}$, we obtain the formulas

$$\mathcal{F}_V(\phi^\circ_\alpha) = (4\pi)^n(-1)^{|\alpha|}\phi^\circ_\alpha \circ \delta_2 \quad \text{and more generally} \tag{3.3}$$

$$\mathcal{F}_V(\phi^\circ_\alpha \circ \delta_r) = \left(\frac{4\pi}{r^2}\right)^n (-1)^{|\alpha|}\phi^\circ_\alpha \circ \delta_{2/r}. \tag{3.4}$$

In summary, the set $\{\phi^\circ_\alpha \mid \alpha \in \Lambda\}$ forms a complete orthogonal system in $L^2_K(V)$. Each ϕ°_α is a simultaneous eigenfunction for the differential operators $\{D' \mid D \in \mathbb{D}_K(H_n)\}$ and (modulo dilation) for the Fourier transform $\mathcal{F}_V$.

4. Generalized binomial coefficients

For $\alpha \in \Lambda$, the functions $\{p_\beta \mid |\beta| \leq |\alpha|\}$ form a basis for the space of K-invariant polynomials on $V_{\mathbb{R}}$ of degree at most $2|\alpha|$. Since q_α belongs to this space, we can write:

$$q_\alpha = \sum_{|\beta|\leq|\alpha|} (-1)^{|\beta|} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} p_\beta \tag{4.1}$$

for some well defined numbers $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. We call these values *generalized binomial coefficients* for the action of K on V. They will play a crucial role in the formulation and proof of our characterization of K-invariant Schwartz functions via the spherical transform. The results that we will need concerning generalized binomial coefficients are contained in this section.

Since the functions q_α and p_β are all real valued, the generalized binomial coefficients are real numbers. It is shown in [BR], moreover, that the $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$'s are non-negative. Since $(-1)^{|\alpha|}p_\alpha$ is the homogeneous component of highest degree in q_α, we see that $\begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = 1$ and that $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$ for $\beta \neq \alpha$ with $|\beta| = |\alpha|$. We extend the definition of generalized binomial coefficients by setting $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$ when $|\beta| > |\alpha|$. Since $q_\alpha(0) = 1 = p_0$ and $p_\beta(0) = 0$ for $|\beta| > 0$, we see that $\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = 1$. Here recall that $0 \in \Lambda$ denotes the index with $P_0 = \mathbb{C}$.

The generalized binomial coefficients were introduced by Z. Yan in [Yan], where one finds the important identity

$$\frac{\gamma^k}{k!} p_\beta = \sum_{|\alpha|=|\beta|+k} \frac{d_\alpha}{d_\beta} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} p_\alpha. \tag{4.2}$$

For the case where $K = U(n)$, decomposition 2.1 reads $\mathbb{C}[V] = \sum_{m=0}^\infty \mathcal{P}_m(V)$ and the $U(n)$-invariant polynomial p_m associated with $\mathcal{P}_m(V)$ is $p_m = \frac{(n-1)!}{(m+n-1)!}\gamma^m$ (see Proposition 6.2 in [BJR92]). Equation 4.2 shows that in this case we have

$$\begin{bmatrix} m+k \\ m \end{bmatrix} = \binom{m+k}{k}. \tag{4.3}$$

The generalized binomial coefficients are studied further in [BR], where the reader will find proofs of the following identities:

$$\gamma q_\alpha = -\sum_{|\beta|=|\alpha|+1} \frac{d_\beta}{d_\alpha}\begin{bmatrix}\beta\\ \alpha\end{bmatrix} q_\beta + (2|\alpha|+n)q_\alpha - \sum_{|\beta|=|\alpha|-1}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} q_\beta, \tag{4.4}$$

$$\sum_{|\beta|=\ell}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\begin{bmatrix}\beta\\ \delta\end{bmatrix} = \frac{(|\alpha|-|\delta|)!}{(|\alpha|-\ell)!(\ell-|\delta|)!}\begin{bmatrix}\alpha\\ \delta\end{bmatrix}. \tag{4.5}$$

Letting $\delta = 0$ in Equation 4.5, we see that

$$\sum_{|\beta|=\ell}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \binom{|\alpha|}{\ell}, \tag{4.6}$$

since $\begin{bmatrix}\beta\\ 0\end{bmatrix} = \begin{bmatrix}\alpha\\ 0\end{bmatrix} = 1$. It follows that for fixed β, $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ has polynomial growth in $|\alpha|$. Equation 4.6 gives, in particular,

$$\sum_{|\beta|=|\alpha|-1}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = |\alpha|. \tag{4.7}$$

Evaluating Equation 4.4 at $z = 0$ and using Equation 4.7 yields also

$$\sum_{|\beta|=|\alpha|+1}\frac{d_\beta}{d_\alpha}\begin{bmatrix}\beta\\ \alpha\end{bmatrix} = |\alpha| + n. \tag{4.8}$$

Our characterization of $\mathcal{S}_K(H_n)^\wedge$, presented below in Section 6, involves the application of *difference operators* $\mathcal{D}^+$, $\mathcal{D}^-$ defined as follows.

Definition 4.1. Given a function g on Λ, $\mathcal{D}^+g$ and $\mathcal{D}^-g$ are the functions on Λ defined by

$$\mathcal{D}^+g(\alpha) = \sum_{|\beta|=|\alpha|+1}\frac{d_\beta}{d_\alpha}\begin{bmatrix}\beta\\ \alpha\end{bmatrix} g(\beta) - (|\alpha|+n)g(\alpha) = \sum_{|\beta|=|\alpha|+1}\frac{d_\beta}{d_\alpha}\begin{bmatrix}\beta\\ \alpha\end{bmatrix}(g(\beta)-g(\alpha))$$

$$\mathcal{D}^-g(\alpha) = |\alpha|g(\alpha) - \sum_{|\beta|=|\alpha|-1}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} g(\beta) = \sum_{|\beta|=|\alpha|-1}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(g(\alpha)-g(\beta))$$

for $|\alpha| > 0$ and $\mathcal{D}^-g(0) = 0$.

The two formulae presented for $\mathcal{D}^\pm g$ agree in view of Equations 4.8 and 4.7.

Using this notation, Equation 4.4 can be rewritten as

$$\gamma q_\alpha = -(\mathcal{D}^+ - \mathcal{D}^-)q_\alpha. \tag{4.9}$$

Since $\phi^\circ_\alpha(z) = q_\alpha(z)e^{-\gamma(z)/2}$ and $\phi_{\alpha,\lambda}(z,t) = \phi^\circ_\alpha(\sqrt{|\lambda|}z)e^{i\lambda t}$, we obtain also

$$\gamma\phi^\circ_\alpha = -(\mathcal{D}^+ - \mathcal{D}^-)\phi^\circ_\alpha, \quad \gamma\phi_{\alpha,\lambda} = -\frac{1}{|\lambda|}(\mathcal{D}^+ - \mathcal{D}^-)\phi_{\alpha,\lambda}. \tag{4.10}$$

The right sides in these equations denote the functions on V and H_n obtained by applying the difference operators to the α-index. Thus, for example, $\mathcal{D}^-\phi_{\alpha,\lambda}$ denotes the function on H_n defined by $\mathcal{D}^-\phi_{\alpha,\lambda}(z,t) = |\alpha|\phi_{\alpha,\lambda}(z,t) - \sum_{|\beta|=|\alpha|-1} \left[\begin{array}{c}\alpha\\ \beta\end{array}\right]\phi_{\beta,\lambda}(z,t)$.

Lemma 4.1. $\partial_\lambda q_\alpha\left(\sqrt{|\lambda|}z\right) = \frac{1}{\lambda}\mathcal{D}^- q_\alpha\left(\sqrt{|\lambda|}z\right)$ *for* $\lambda \neq 0$.

Proof. Equation 4.1 shows that

$$q_\alpha\left(\sqrt{|\lambda|}z\right) = \sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](-1)^{|\beta|}p_\beta\left(\sqrt{|\lambda|}z\right) = \sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](-|\lambda|)^{|\beta|}p_\beta(z).$$

Suppose here that $\lambda > 0$. We compute that

$$\begin{aligned}
\partial_\lambda q_\alpha\left(\sqrt{|\lambda|}z\right) &= \sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](-1)^{|\beta|}|\beta|\lambda^{|\beta|-1}p_\beta(z)\\
&= \frac{1}{\lambda}\sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](-1)^{|\beta|}|\beta|\lambda^{|\beta|}p_\beta(z)\\
&= \frac{|\alpha|}{\lambda}\sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](-\lambda)^{|\beta|}p_\beta(z) - \frac{1}{\lambda}\sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](|\alpha|-|\beta|)(-\lambda)^{|\beta|}p_\beta(z)\\
&= \frac{|\alpha|}{\lambda}q_\alpha\left(\sqrt{|\lambda|}z\right) - \frac{1}{\lambda}\sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](|\alpha|-|\beta|)(-1)^{|\beta|}p_\beta\left(\sqrt{|\lambda|}z\right)
\end{aligned}$$

Equation 4.5 shows that $\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](|\alpha|-|\beta|) = \sum_{|\delta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\ \delta\end{array}\right]\left[\begin{array}{c}\delta\\ \beta\end{array}\right]$. Hence

$$\begin{aligned}
\partial_\lambda q_\alpha\left(\sqrt{|\lambda|}z\right) &= \frac{|\alpha|}{\lambda}q_\alpha\left(\sqrt{|\lambda|}z\right) - \frac{1}{\lambda}\sum_{|\beta|\le|\alpha|}\sum_{|\delta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\ \delta\end{array}\right]\left[\begin{array}{c}\delta\\ \beta\end{array}\right](-1)^{|\beta|}p_\beta\left(\sqrt{|\lambda|}z\right)\\
&= \frac{|\alpha|}{\lambda}q_\alpha\left(\sqrt{|\lambda|}z\right) - \frac{1}{\lambda}\sum_{|\delta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\ \delta\end{array}\right]\left(\sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\delta\\ \beta\end{array}\right](-1)^{|\beta|}p_\beta\left(\sqrt{|\lambda|}z\right)\right)\\
&= \frac{|\alpha|}{\lambda}q_\alpha\left(\sqrt{|\lambda|}z\right) - \frac{1}{\lambda}\sum_{|\delta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\ \delta\end{array}\right]q_\delta\left(\sqrt{|\lambda|}z\right)\\
&= \frac{1}{\lambda}\mathcal{D}^- q_\alpha\left(\sqrt{|\lambda|}z\right).
\end{aligned}$$

A similar analysis applies when $\lambda < 0$. □

Using Lemma 4.1 we obtain that for $\lambda > 0$,

$$\partial_\lambda\phi_{\alpha,\lambda}(z,t) = \partial_\lambda\left[q_\alpha\left(\sqrt{|\lambda|}z\right)e^{-\lambda\gamma(z)/2}e^{i\lambda t}\right] = \frac{1}{\lambda}\mathcal{D}^-\phi_{\alpha,\lambda}(z,t) - \frac{\gamma(z)}{2}\phi_{\alpha,\lambda}(z,t) + it\phi_{\alpha,\lambda}(z,t).$$

In view of Equation 4.10 we can also write

$$\partial_\lambda \phi_{\alpha,\lambda}(z,t) = \frac{1}{\lambda}\mathcal{D}^+\phi_{\alpha,\lambda}(z,t) + \frac{\gamma(z)}{2}\phi_{\alpha,\lambda}(z,t) + it\phi_{\alpha,\lambda}(z,t).$$

Thus we have

$$\partial_\lambda \phi_{\alpha,\lambda} = \left\{ \begin{array}{l} \frac{1}{\lambda}\mathcal{D}^-\phi_{\alpha,\lambda} - \frac{\gamma}{2}\phi_{\alpha,\lambda} + it\phi_{\alpha,\lambda} \\ \frac{1}{\lambda}\mathcal{D}^+\phi_{\alpha,\lambda} + \frac{\gamma}{2}\phi_{\alpha,\lambda} + it\phi_{\alpha,\lambda} \end{array} \right\} \quad \text{for } \lambda > 0, \tag{4.11}$$

and similarly

$$\partial_\lambda \phi_{\alpha,\lambda} = \left\{ \begin{array}{l} \frac{1}{\lambda}\mathcal{D}^-\phi_{\alpha,\lambda} + \frac{\gamma}{2}\phi_{\alpha,\lambda} + it\phi_{\alpha,\lambda} \\ \frac{1}{\lambda}\mathcal{D}^+\phi_{\alpha,\lambda} - \frac{\gamma}{2}\phi_{\alpha,\lambda} + it\phi_{\alpha,\lambda} \end{array} \right\} \quad \text{for } \lambda < 0. \tag{4.12}$$

Equivalently

$$\left(\frac{\gamma}{2} + it\right)\phi_{\alpha,\lambda} = \left\{ \begin{array}{ll} \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^+\right)\phi_{\alpha,\lambda} & \text{for } \lambda > 0 \\ \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^-\right)\phi_{\alpha,\lambda} & \text{for } \lambda < 0 \end{array} \right\} \quad \text{and} \tag{4.13}$$

$$\left(\frac{\gamma}{2} - it\right)\phi_{\alpha,\lambda} = \left\{ \begin{array}{ll} -\left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^-\right)\phi_{\alpha,\lambda} & \text{for } \lambda > 0 \\ -\left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^+\right)\phi_{\alpha,\lambda} & \text{for } \lambda < 0 \end{array} \right\} \quad . \tag{4.14}$$

5. K-Invariant Schwartz functions on V

We let $\mathcal{S}(V)$ denote the space of Schwartz functions on V and $\mathcal{S}(H_n)$ the Schwartz functions on $H_n = V \times \mathbb{R}$. As usual, $\mathcal{S}_K(V)$ and $\mathcal{S}_K(H_n)$ will denote the K-invariant elements in $\mathcal{S}(V)$ and $\mathcal{S}(H_n)$ respectively. Note that the ϕ_α°'s belong to $\mathcal{S}_K(V)$. We would like to characterize the spaces $\mathcal{S}_K(V)$ and $\mathcal{S}_K(H_n)$ in terms of the spherical transform. The problem for $\mathcal{S}_K(V)$ is related to that for $\mathcal{S}_K(H_n)$ as follows. Since $U(n)$ acts trivially on the center of H_n, we have that $\mathcal{S}_K(H_n) \cong \mathcal{S}_K(V) \otimes \mathcal{S}(\mathbb{R})$. Moreover, for $g \in \mathcal{S}_K(V)$, $h \in \mathcal{S}(\mathbb{R})$ and $(\alpha, \lambda) \in \Lambda \times \mathbb{R}^\times$ one computes easily that

$$(g \otimes h)^\wedge(\alpha,\lambda) = \left(\frac{1}{|\lambda|}\right)^n \left(g \circ \delta_{\frac{1}{\sqrt{|\lambda|}}}\right)^\wedge(\alpha)\mathcal{F}(h)(\lambda) \tag{5.1}$$

where $\mathcal{F}(h)(\lambda) = \int_{-\infty}^{\infty} h(t)e^{-i\lambda t}dt$ is the one dimensional Fourier transform in the central direction. Thus the problem of characterizing $\mathcal{S}_K(H_n)$ via the spherical transform leads us to seek conditions on maps $\widehat{f} : \Lambda \to \mathbb{C}$ that are necessary and sufficient to ensure that $f \in L^1_K(V)$ is in fact a Schwartz function. This problem, which is of interest in its own right, is answered cleanly by Theorem 5.1 below. In Section 6 we will return to the problem of characterizing $\mathcal{S}_K(H_n)$ via the spherical transform.

Definition 5.1. We say that a function $F : \Lambda \to \mathbb{C}$ is *rapidly decreasing* if for every sequence $(\alpha_m)_{m=1}^\infty$ in Λ with $\lim_{m\to\infty} |\alpha_m| = \infty$ and every $N \in \mathbb{Z}^+$ one has

$$\lim_{m\to\infty} F(\alpha_m)|\alpha_m|^N = 0.$$

Equivalently, for each $N \in \mathbb{Z}^+$, there is a constant C_N for which

$$|F(\alpha)| \leq \frac{C_N}{(2|\alpha|+n)^N}.$$

Theorem 5.1. *If $f \in \mathcal{S}_K(V)$ then $\widehat{f}$ is rapidly decreasing on Λ. Conversely, if F is rapidly decreasing on Λ then $F = \widehat{f}$ for some $f \in \mathcal{S}_K(V)$. Moreover, the map*

$$\wedge : \mathcal{S}_K(V) \to \{F \mid F \textit{ is rapidly decreasing on } \Lambda\}$$

is a bijection.

Lemma 5.2. *If F is rapidly decreasing on Λ then so are $\mathcal{D}^+F$ and $\mathcal{D}^-F$.*

Proof. Let $c_m := \max_{|\alpha|=m} |F(\alpha)|$. The function $m \mapsto c_m$ is rapidly decreasing on $\mathbb{Z}^+$. Using Equation 4.8 and the fact that the generalized binomial coefficients are non-negative (proved in [BR]) we obtain

$$\begin{aligned} |\mathcal{D}^+F(\alpha)| &\leq \sum_{|\beta|=|\alpha|+1} \frac{d_\beta}{d_\alpha} \left[\begin{array}{c} \beta \\ \alpha \end{array} \right] |F(\beta)| + (|\alpha|+n)|F(\alpha)| \\ &\leq c_{|\alpha|+1} \sum_{|\beta|=|\alpha|+1} \frac{d_\beta}{d_\alpha} \left[\begin{array}{c} \beta \\ \alpha \end{array} \right] + c_{|\alpha|}(|\alpha|+n) \\ &= (|\alpha|+n)\left(c_{|\alpha|} + c_{|\alpha|+1}\right), \end{aligned}$$

which shows that $\mathcal{D}^+F$ is rapidly decreasing. Using Equation 4.7 in a similar fashion yields

$$|\mathcal{D}^-F(\alpha)| \leq |\alpha| \left(c_{|\alpha|-1} + c_{|\alpha|}\right),$$

showing that $\mathcal{D}^-F$ is rapidly decreasing. □

Lemma 5.3. *Let F be a rapidly decreasing function on Λ and G be a bounded function on Λ. Then*

$$\begin{aligned} \sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\mathcal{D}^+G(\alpha) &= -\sum_{\alpha\in\Lambda} d_\alpha \left(\mathcal{D}^- + n\right) F(\alpha)G(\alpha), \\ \sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\mathcal{D}^-G(\alpha) &= -\sum_{\alpha\in\Lambda} d_\alpha \left(\mathcal{D}^+ + n\right) F(\alpha)G(\alpha). \end{aligned}$$

Here all four series converge absolutely.

Proof. We compute formally that

$$\begin{aligned}
\sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\mathcal{D}^+G(\alpha) &= \sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\left(\sum_{|\beta|=|\alpha|+1}\frac{d_\beta}{d_\alpha}\left[\begin{array}{c}\beta\\ \alpha\end{array}\right]G(\beta)-(|\alpha|+n)G(\alpha)\right)\\
&= \sum_{\alpha\in\Lambda}\sum_{|\beta|=|\alpha|+1} d_\beta\left[\begin{array}{c}\beta\\ \alpha\end{array}\right]F(\alpha)G(\beta)-\sum_{\alpha\in\Lambda} d_\alpha(|\alpha|+n)F(\alpha)G(\alpha)\\
&= \sum_{\beta\in\Lambda} d_\beta\left(\sum_{|\alpha|=|\beta|-1}\left[\begin{array}{c}\beta\\ \alpha\end{array}\right]F(\alpha)\right)G(\beta)-\sum_{\alpha\in\Lambda} d_\alpha(|\alpha|+n)F(\alpha)G(\alpha)\\
&= \sum_{\alpha\in\Lambda} d_\alpha\left(-\mathcal{D}^- - n\right)F(\alpha)G(\alpha).
\end{aligned}$$

The hypotheses on F and G ensure that the terms in each series above are products of rapidly decreasing functions with functions of polynomial growth in $|\alpha|$. Thus all of these series converge absolutely and the above rearrangements are justified. Indeed, suppose that $|G(\alpha)| \leq M$ for all $\alpha \in \Lambda$. Equations 4.7 and 4.8 show that

$$\sum_{|\beta|=|\alpha|+1}\frac{d_\beta}{d_\alpha}\left[\begin{array}{c}\beta\\ \alpha\end{array}\right]G(\beta)-(|\alpha|+n)G(\alpha)\leq 2M(|\alpha|+n).$$

Moreover, since $P_\alpha \subset \mathcal{P}_{|\alpha|}(V)$, we have that

$$d_\alpha \leq \dim\left(\mathcal{P}_{|\alpha|}(V)\right) = \binom{|\alpha|+n-1}{|\alpha|} \leq \frac{(|\alpha|+n-1)^{n-1}}{(n-1)!}, \tag{5.2}$$

a polynomial bound on d_α. Hence $\alpha \mapsto d_\alpha(|\alpha|+n)G(\alpha)$ is bounded by $M(|\alpha|+n)^n$ and $\beta \mapsto d_\beta G(\beta)$ by $M(|\beta|+n-1)^{n-1}$. Finally note that $\beta \mapsto \sum_{|\alpha|=|\beta|-1}\left[\begin{array}{c}\beta\\ \alpha\end{array}\right]F(\alpha)$ is bounded by

$$\dim\left(\mathcal{P}_{|\beta|-1}(V)\right)|\beta|H(\beta) \leq (|\beta|+n-2)^{n-1}|\beta|H(\beta)$$

where $H(\beta) = \max\{|F(\alpha)| \;:\; |\alpha| = |\beta|-1\}$ is rapidly decreasing in β.

The second identity follows from the first by interchanging the roles of F and G. □

Proof of Theorem 5.1. First suppose that $f \in \mathcal{S}_K(V)$ and let $\mathcal{L}$ denote the Heisenberg sub-Laplacian given by Equation 2.6. The associated differential operator $\mathcal{L}'$ on V, given in Equation 3.1, is self-adjoint and preserves $\mathcal{S}_K(V)$. In view of Equation 3.2, we have that for $N \in \mathbb{Z}^+$,

$$\begin{aligned}
(2|\alpha|+n)^N\left|\widehat{f}(\alpha)\right| &= \left|(-(2|\alpha|+n))^N\langle f,\phi_\alpha^\circ\rangle_2\right|\\
&= \left|\langle f,(\mathcal{L}')^N\phi_\alpha^\circ\rangle_2\right| = \left|\langle(\mathcal{L}')^N f,\phi_\alpha^\circ\rangle_2\right|\\
&\leq ||(\mathcal{L}')^N f||_1
\end{aligned}$$

since $|\phi_\alpha^\circ(z)| \leq 1$ for all $z \in V$. Hence we obtain a bound of the form

$$|\widehat{f}(\alpha)| \leq \frac{C_N}{(2|\alpha|+n)^N}$$

for each $N \in \mathbb{Z}^+$ where $C_N = ||(\mathcal{L}')^N f||_1$. It follows immediately that $\widehat{f}$ is a rapidly decreasing function on Λ.

Conversely, suppose that F is a rapidly decreasing function on Λ. The estimate given above in Equation 5.2 shows that d_α is bounded by a polynomial function of $|\alpha|$. It follows that the series $\sum_{\alpha\in\Lambda} d_\alpha (F(\alpha))^p$ converges absolutely for all $p \in \mathbb{Z}^+$. As $|\phi_\alpha^\circ(z)| \leq 1$, we conclude that the series $\sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\phi_\alpha^\circ(z)$ converges absolutely and uniformly in z. Define a function f on V via

$$f(z) = \left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\phi_\alpha^\circ(z). \tag{5.3}$$

Since $\sum_{\alpha\in\Lambda} d_\alpha |F(\alpha)|^2$ converges, Corollary 3.2 shows that $f \in L^2_K(V)$ with $\widehat{f} = F$. Moreover, as

- $\sum_{\alpha\in\Lambda} d_\alpha |F(\alpha)|$ converges, and
- each ϕ_α° is a Schwartz function, and
- the ϕ_α°'s are uniformly bounded by 1,

it follows easily that $\lim_{|z|\to\infty} f(z) = 0$. To prove that f is a Schwartz function, it suffices to show that f is smooth and that $\lim_{|z|\to\infty} \gamma(z)^a(\Delta^b f)(z) = 0$ for all non-negative integers a, b. In view of the preceding analysis, this follows by induction from Lemma 5.4 below.

To complete the proof, note that if $f \in \mathcal{S}_K(V)$ and $F(\alpha) = \widehat{f}(\alpha) = 0$ for all $\alpha \in \Lambda$ then Equation 5.3 implies that $f = 0$ in $L^2_K(V)$. As f is continuous, it follows that $f(z) = 0$ for all $z \in V$. This shows that $f \mapsto \widehat{f}$ is injective on $\mathcal{S}_K(V)$. □

Lemma 5.4. *Let $f = \left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\phi_\alpha^\circ$ where F is rapidly decreasing. Then*

1. *γf can be written in the form $\left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha G(\alpha)\phi_\alpha^\circ$ where G is rapidly decreasing, and*
2. *f is twice differentiable and Δf can be written in the form $\left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha H(\alpha)\phi_\alpha^\circ$ where H is rapidly decreasing.*

Proof. We use Equation 4.10 together with Lemma 5.3 to write

$$\begin{aligned}
\gamma(z)f(z) &= \left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\gamma(z)\phi_\alpha^\circ(z) \\
&= -\left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\left(\mathcal{D}^+ - \mathcal{D}^-\right)\phi_\alpha^\circ(z) \\
&= \left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha \left(\mathcal{D}^- - \mathcal{D}^+\right) F(\alpha)\phi_\alpha^\circ(z).
\end{aligned}$$

In view of Lemma 5.2, $G = (\mathcal{D}^- - \mathcal{D}^+)\, F$ is rapidly decreasing. This establishes (1) in the statement of Lemma 5.4.

Equation 3.2 shows that $(4\Delta - \gamma/2)\phi^\circ_\alpha = -(2|\alpha| + n)\phi^\circ_\alpha$ and hence

$$\Delta\phi^\circ_\alpha = \frac{\gamma}{8}\phi^\circ_\alpha - \frac{2|\alpha| + n}{4}\phi^\circ_\alpha.$$

Thus formal application of Δ term-wise to the series $\left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\phi^\circ_\alpha$ for f yields the series $\left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha H(\alpha)\phi^\circ_\alpha$ where

$$H(\alpha) = \frac{G(\alpha)}{8} - \frac{2|\alpha| + n}{4}F(\alpha)$$

and $\gamma f = \left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha G(\alpha)\phi^\circ_\alpha$ as above. As both F and G are rapidly decreasing, so is H. As both $\sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\phi^\circ_\alpha(z)$ and $\sum_{\alpha\in\Lambda} d_\alpha H(\alpha)\phi^\circ_\alpha(z)$ converge uniformly in z, we conclude that f is twice differentiable with $\Delta f = \left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha H(\alpha)\phi^\circ_\alpha$. This establishes (2) in the statement of Lemma 5.4. □

6. K-Invariant Schwartz functions on H_n

In this section we return to the problem of characterizing the space $\mathcal{S}_K(H_n)$ via the spherical transform. Theorem 6.1 solves this problem and is our main result. Several definitions are required in the formulation of this theorem.

Definition 6.1. Let F be a function on $\Delta(K, H_n)$. We say that F is *rapidly decreasing* on $\Delta(K, H_n)$ if

- F is continuous on $\Delta(K, H_n)$,
- the function $F_\circ$ on V defined by $F_\circ(w) = F(\eta_w)$ belongs to $\mathcal{S}_K(V)$,
- the map $\lambda \mapsto F(\alpha, \lambda)$ is smooth on $\mathbb{R}^\times = (-\infty, 0) \cup (0, \infty)$ for each fixed $\alpha \in \Lambda$,
- for each $m, N \geq 0$ there exists a constant $C_{m,N}$ for which
$$|\partial^m_\lambda F(\alpha, \lambda)| \leq \frac{C_{m,N}}{|\lambda|^{m+N}(2|\alpha| + n)^N}$$
for all $(\alpha, \lambda) \in \Lambda \times \mathbb{R}^\times$.

We say that a continuous function on $\Delta_1(K, H_n)$ is rapidly decreasing if it extends to a rapidly decreasing function on $\Delta(K, H_n) = \Delta_1(K, H_n) \cup \Delta_2(K, H_n)$. Since $\Delta_1(K, H_n)$ is dense in $\Delta(K, H_n)$, such an extension is necessarily unique.

Note that if F is rapidly decreasing on $\Delta(K, H_n)$ then $\alpha \mapsto F(\alpha, \lambda)$ is rapidly decreasing on Λ, in the sense of Definition 5.1, for each $\lambda \neq 0$. We see that F is bounded by letting $m = N = 0$ and one can show, moreover, that F vanishes at infinity by letting $m = 0$ and $N = 1$. We remark that the functions $\partial^m_\lambda F(\alpha, \lambda)$ defined on $\Delta_1(K, H_n)$ need not extend continuously across $\Delta_2(K, H_n)$. Example 6.1 below illustrates this behavior.

Definition 6.2. Let F be a function on $\Delta_1(K,H_n)$ which is smooth in λ. M^+F and M^-F are the functions on $\Delta_1(K,H_n)$ defined by

$$M^+F(\alpha,\lambda) = \left\{ \begin{array}{ll} \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^+\right)F(\alpha,\lambda) & \text{for } \lambda > 0 \\ \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^-\right)F(\alpha,\lambda) & \text{for } \lambda < 0 \end{array} \right\} \quad \text{and}$$

$$M^-F(\alpha,\lambda) = \left\{ \begin{array}{ll} \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^-\right)F(\alpha,\lambda) & \text{for } \lambda > 0 \\ \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^+\right)F(\alpha,\lambda) & \text{for } \lambda < 0 \end{array} \right\}.$$

We remind the reader that the difference operators $\mathcal{D}^\pm$ are defined by

$$\mathcal{D}^+F(\alpha,\lambda) = \sum_{|\beta|=|\alpha|+1} \frac{d_\beta}{d_\alpha}\left[\begin{array}{c}\beta\\ \alpha\end{array}\right]F(\beta,\lambda) - (|\alpha|+n)F(\alpha,\lambda),$$

$$\mathcal{D}^-F(\alpha,\lambda) = |\alpha|F(\alpha,\lambda) - \sum_{|\beta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right]F(\beta,\lambda).$$

Definition 6.3. $\widehat{\mathcal{S}}(K,H_n)$ is the set of all functions $F:\Delta(K,H_n)\to\mathbb{C}$ for which $(M^+)^\ell(M^-)^mF$ is rapidly decreasing for all $\ell,m\geq 0$.

If F is rapidly decreasing on $\Delta(K,H_n)$ then $\lambda\mapsto F(\alpha,\lambda)$ is smooth on $\mathbb{R}^\times$ and we have well defined functions $(M^+)^\ell(M^-)^mF$ on $\Delta_1(K,H_n)$. F belongs to $\widehat{\mathcal{S}}(K,H_n)$ if and only if these functions extend continuously to rapidly decreasing functions on $\Delta(K,H_n)$.

Theorem 6.1. *If $f\in\mathcal{S}_K(H_n)$ then $\widehat{f}\in\widehat{\mathcal{S}}(K,H_n)$. Conversely, if $F\in\widehat{\mathcal{S}}(K,H_n)$ then $F=\widehat{f}$ for some $f\in\mathcal{S}_K(H_n)$. Moreover, the map $^\wedge:\mathcal{S}_K(H_n)\to\widehat{\mathcal{S}}(K,H_n)$ is a bijection.*

If $f\in\mathcal{S}_K(H_n)$ and $\widehat{f}=0$ then the inversion formula for the spherical transform (Equation 2.13) yields that $f=0$. Thus the spherical transform is injective on $\mathcal{S}_K(H_n)$. To prove Theorem 6.1, it remains to show that $\mathcal{S}_K(H_n)^\wedge\subset\widehat{\mathcal{S}}(K,H_n)$ and that $\widehat{\mathcal{S}}(K,H_n)\subset\mathcal{S}_K(H_n)^\wedge$. This will require most of the remainder of this section. First, however, we will present an example.

Example 6.1. Consider the case where $K=U(n)$. Decomposition 2.1 reads $\mathbb{C}[V]=\sum_{m=0}^\infty\mathcal{P}_m(V)$ and one has $\Delta(U(n),H_n)\cong\left(\mathbb{R}^\times\times\mathbb{Z}^+\right)\cup\mathbb{R}^+$. Using Equation 4.3, one computes that the difference operators $\mathcal{D}^\pm$ appearing in the definition of the set $\widehat{\mathcal{S}}(U(n),H_n)$ are given by

$$\mathcal{D}^+g(m)=(m+n)(g(m+1)-g(m)),\qquad \mathcal{D}^-g(m)=m(g(m)-g(m-1))$$

for functions $g:\mathbb{Z}^+\to\mathbb{C}$.

Let $f\in\mathcal{S}_K(H_n)$ be defined by $f(z,t)=g(z)h(t)$ where $g(z)=e^{-|z|^2}$ and $h\in\mathcal{S}(\mathbb{R})$. We will compute the spherical transform $\widehat{f}:\Delta(U(n),H_n)\to\mathbb{C}$.

The polynomial q_m associated with $\mathcal{P}_m(V)$ is a suitably normalized generalized Laguerre polynomial which can be written explicitly as (see eg. [BJR92])

$$q_m(z) = (n-1)! \sum_{j=0}^{m} \binom{m}{j} \frac{1}{(n+j-1)!} \left(-\frac{|z|^2}{2} \right)^j .$$

Thus, for $\lambda \neq 0$ we have

$$\begin{aligned} \left(g \circ \delta_{1/\sqrt{|\lambda|}} \right)^{\wedge} (m) &= \int_V e^{-|z|^2/|\lambda|} q_m(z) e^{-|z|^2/4} dz \\ &= (n-1)! \sum_{j=0}^{m} \binom{m}{j} \frac{(-1)^j}{2^j (n+j-1)!} \int_V |z|^{2j} e^{-K|z|^2} dz \end{aligned}$$

where $K = \frac{1}{|\lambda|} + \frac{1}{4} = \frac{4+|\lambda|}{4|\lambda|}$. One has

$$\int_V |z|^{2j} e^{-K|z|^2} dz = \frac{2\pi^n}{(n-1)!} \int_0^\infty e^{-Kr^2} r^{2(n+j)-1} dr = \frac{\pi^n (n+j-1)!}{(n-1)! K^{n+j}}$$

and hence

$$\begin{aligned} \left(g \circ \delta_{1/\sqrt{|\lambda|}} \right)^{\wedge} (m) &= \left(\frac{\pi}{K} \right)^n \sum_{j=0}^{m} \binom{m}{j} \left(-\frac{1}{2K} \right)^j \\ &= \left(\frac{\pi}{K} \right)^n \left(1 - \frac{1}{2K} \right)^m = \left(\frac{4\pi|\lambda|}{4+|\lambda|} \right)^n \left(\frac{4-|\lambda|}{4+|\lambda|} \right)^m . \end{aligned}$$

Thus we have (see Equation 5.1)

$$\begin{aligned} \widehat{f}(m,\lambda) &= \left(\frac{1}{|\lambda|} \right)^n \left(g \circ \delta_{\frac{1}{\sqrt{|\lambda|}}} \right)^{\wedge} (m) \mathcal{F}(h)(\lambda) \\ &= \left(\frac{4\pi}{4+|\lambda|} \right)^n \left(\frac{4-|\lambda|}{4+|\lambda|} \right)^m \mathcal{F}(h)(\lambda). \end{aligned}$$

For $w \in V$ one has (see Equation 2.15)

$$\begin{aligned} \widehat{f}(w) &= \mathcal{F}_H(f)(w,0) = \mathcal{F}_V(g)(w)\mathcal{F}(h)(0) \\ &= \left(\pi^n \int_{-\infty}^{\infty} h(t) dt \right) e^{-|w|^2/4}. \end{aligned}$$

One can verify directly that $\widehat{f} \in \widehat{\mathcal{S}}(U(n), H_n)$, but we will not do this here. In particular, the functions $\lambda \mapsto \widehat{f}(\lambda, m)$ are differentiable on $\mathbb{R}^\times$ for each $m \in \mathbb{Z}^+$, and the derivatives $\partial_\lambda \widehat{f}(\lambda, m)$ satisfy estimates as in Definition 6.1. Note, however, that these derivatives do not agree as $\lambda \to 0$. Specifically, $\lim_{\lambda \to 0} \partial_\lambda \widehat{f}(\lambda, m)$ depends on m. Hence there is no function $g \in L^1_{U(n)}(H_n)$ whose $U(n)$-spherical transform satisfies $\widehat{g}(\lambda, m) = \partial_\lambda \widehat{f}(\lambda, m)$ on $\mathbb{R}^\times \times \mathbb{Z}^+$.

This example illustrates why the characterization of $\mathcal{S}_K(H_n)^\wedge$ is somewhat complicated. Although the behavior of the derivatives of the functions $\lambda \mapsto \widehat{f}(\lambda, m)$ does come into play, as expected, the space $\mathcal{S}_K(H_n)^\wedge$ is not "closed under λ-derivatives". Indeed, one must "replace" ∂_λ by the operators $M^\pm$ which involve both ∂_λ and the difference operators $\mathcal{D}^\pm$.

Proof that $\mathcal{S}_K(H_n)^\wedge \subset \widehat{\mathcal{S}}(K, H_n)$. Suppose that $f \in \mathcal{S}_K(H_n)$ and let $F := \widehat{f}$. We begin by showing that F is rapidly decreasing. F is continuous on $\Delta(K, H_n)$, as is the spherical transform of any integrable K-invariant function. Moreover, Equation 2.15 shows that $F_\circ(w) = \mathcal{F}_H(f)(w, 0)$. Since f is a Schwartz function, so is $F_\circ(w)$. Thus F satisfies the first two conditions in Definition 6.1.

Next we will show that F satisfies the estimates in Definition 6.1 for $m = 0$. The argument is similar to that in the first part of the proof for Theorem 5.1. Recall that the Heisenberg sub-Laplacian $\mathcal{L}$ is a self-adjoint operator on $L^2(H_n)$ with $\mathcal{L}(\phi_{\alpha,\lambda}) = -|\lambda|(2|\alpha| + n)\phi_{\alpha,\lambda}$. Thus we have

$$\begin{aligned} |\lambda|^N (2|\alpha| + n)^N \, |F(\alpha, \lambda)| &= \left|(-|\lambda|(2|\alpha| + n))^N \langle f, \phi_{\alpha,\lambda}\rangle_2\right| \\ &= \left|\langle f, \mathcal{L}^N \phi_{\alpha,\lambda}\rangle_2\right| = \left|\langle \mathcal{L}^N f, \phi_{\alpha,\lambda}\rangle_2\right| \\ &\le ||\mathcal{L}^N f||_1 \end{aligned}$$

since $|\phi_{\alpha,\lambda}(z,t)| \le 1$ for all $(z,t) \in H_n$. Letting $C_{0,N} := ||\mathcal{L}^N f||_1$, we see that the inequalities in Definition 6.1 hold for $m = 0$.

Since $\phi_{\alpha,\lambda}(z,t)$ is smooth in $\lambda \in \mathbb{R}^\times$ for fixed (z,t) and $f\overline{\phi_{\alpha,\lambda}}$ is a Schwartz function, $F(\alpha, \lambda) = \widehat{f}(\alpha, \lambda)$ is smooth in $\lambda \in \mathbb{R}^\times$ with

$$\partial_\lambda F(\alpha, \lambda) = \int f(z,t)\partial_\lambda \overline{\phi_{\alpha,\lambda}(z,t)} dz dt. \tag{6.1}$$

Equations 4.11 and 4.12 provide formulae for $\partial_\lambda \overline{\phi_{\alpha,\lambda}} = \partial_\lambda \phi_{\alpha,-\lambda}$ but we require a different approach here. We write $\phi_\alpha(z,t) = \phi_\alpha^\circ(z)e^{it}$ as $\phi_\alpha^\circ(z, \overline{z})e^{it}$ so that

$$\overline{\phi_{\alpha,\lambda}(z,t)} = \phi_\alpha^\circ(|\lambda| z, \overline{z})e^{-i\lambda t}.$$

We see that

$$\partial_\lambda \overline{\phi_{\alpha,\lambda}(z,t)} = \frac{1}{\lambda}\left[\left(\sum_{j=1}^n z_j \frac{\partial}{\partial z_j}\right)\phi_\alpha^\circ\right](|\lambda| z, \overline{z}) - it\overline{\phi_{\alpha,\lambda}(z,t)}.$$

Substituting this expression into Equation 6.1 and integrating by parts gives

$$\partial_\lambda F(\alpha, \lambda) = \frac{1}{\lambda}(Df)^\wedge(\alpha, \lambda) - i(tf)^\wedge(\alpha, \lambda)$$

where $Df := -\sum_{j=1}^n \frac{\partial}{\partial z_j}(z_j f)$. Since tf and Df are both Schwartz functions on H_n, they satisfy estimates as above. Thus, given $N \ge 0$, one can find constants A and B

with

$$|\partial_\lambda F(\alpha,\lambda)| \le \frac{A}{|\lambda|^{N+1}(2|\alpha|+n)^N} + \frac{B}{|\lambda|^{N+1}(2|\alpha|+n)^{N+1}} \le \frac{C_{1,N}}{|\lambda|^{N+1}(2|\alpha|+n)^N}$$

where $C_{1,N} = A+B$. By induction on m we see that $|\partial_\lambda^m F(\alpha,\lambda)|$ satisfies an estimate as in Definition 6.1. This completes the proof that F is rapidly decreasing.

Equations 4.13 and 4.13 show that

$$M^+F = \left(\left(\frac{\gamma}{2}+it\right)f\right)^\wedge\Big|_{\Delta_1(K,H_n)} \quad \text{and} \quad M^-F = -\left(\left(\frac{\gamma}{2}-it\right)f\right)^\wedge\Big|_{\Delta_1(K,H_n)}$$

Thus $(M^+)^\ell(M^-)^m F$ is the restriction of $\widehat{g}$ to $\Delta_1(K,H_n)$ where

$$g = (-1)^m(\gamma/2+it)^\ell(\gamma/2-it)^m f.$$

Since $g \in \mathcal{S}_K(H_n)$, it now follows that $(M^+)^\ell(M^-)^m F$ is rapidly decreasing. Thus $F \in \widehat{\mathcal{S}}(K,H_n)$ as desired. □

The following proposition is required to complete the proof of Theorem 6.1.

Proposition 6.2. *Let F be a bounded measurable function on $\Delta(K,H_n)$ with*

$$|F(\alpha,\lambda)| \le \frac{C}{|\lambda|^N(2|\alpha|+n)^N}$$

for some $N \ge n+2$ and some constant C. Then

1. *$F \in L^p(\Delta(K,H_n))$ for all $p \ge 1$, and*
2. *$F = \widehat{f}$ for some bounded continuous function $f \in L^2_K(H_n)$.*

Suppose, for example, that f is a continuous function in $L^1_K(H_n)$ with $F = \widehat{f}$ rapidly decreasing. Proposition 6.2 shows that f is square integrable and that F is integrable. Thus, the inversion formula 2.13 applies and we can recover f from F via

$$f(z,t) = \left(\frac{1}{2\pi}\right)^{n+1}\int_{\mathbb{R}^\times}\sum_{\alpha\in\Lambda} d_\alpha F(\alpha,\lambda)\phi_{\alpha,\lambda}(z,t)|\lambda|^n d\lambda. \tag{6.2}$$

In particular, we see that this formula certainly holds for any function $f \in \mathcal{S}_K(H_n)$.

Proof of Proposition 6.2. To establish the first assertion, it suffices to prove that $F \in L^1(\Delta(K,H_n))$. Indeed, $|F(\alpha,\lambda)|^p$ satisfies an inequality as in the statement of the proposition with N replaced by pN.

Let $A_1 := \{\phi_{\alpha,\lambda} \mid |\lambda|(2|\alpha|+n) \le 1\}$ and $A_2 := \{\phi_{\alpha,\lambda} \mid |\lambda|(2|\alpha|+n) > 1\}$. Let M be a constant for which $|F(\psi)| \le M$ for all $\psi \in \Delta(K,H_n)$ and let

$$d_m := \dim(\mathcal{P}_m(V)) = \binom{m+n-1}{m}. \tag{6.3}$$

Note that

$$\sum_{|\alpha|=m} d_\alpha = d_m \le (m+n-1)^{n-1}.$$

We compute

$$\begin{aligned}\int_{A_1}|F(\psi)|d\mu(\psi) &= \left(\frac{1}{2\pi}\right)\sum_{\alpha\in\Lambda} d_\alpha \int_{0<|\lambda|<\frac{1}{2|\alpha|+n}} |F(\alpha,\lambda)||\lambda|^n d\lambda \\ &\le \frac{2M}{(2\pi)^{n+1}}\sum_{m=0}^{\infty} d_m \int_0^{\frac{1}{2m+n}} \lambda^n d\lambda \\ &= \frac{2M}{(n+1)(2\pi)^{n+1}}\sum_{m=0}^{\infty} d_m \left(\frac{1}{2m+n}\right)^{n+1}.\end{aligned}$$

Since $d_m = O(m^{n-1})$, we see that the last series converges. Hence $|F|$ is integrable over A_1. Next we compute

$$\begin{aligned}\int_{A_2}|F(\psi)|d\mu(\psi) &= \left(\frac{1}{2\pi}\right)\sum_{\alpha\in\Lambda} d_\alpha \int_{|\lambda|>\frac{1}{2|\alpha|+n}} |F(\alpha,\lambda)||\lambda|^n d\lambda \\ &\le \frac{2C}{(2\pi)^{n+1}}\sum_{m=0}^{\infty} d_m \int_{\frac{1}{2m+n}}^{\infty} \frac{\lambda^n}{\lambda^N(2m+n)^N} d\lambda \\ &\le \frac{2C}{(N-n-1)(2\pi)^{n+1}}\sum_{m=0}^{\infty} \frac{d_m}{(2m+n)^{n+1}}.\end{aligned}$$

The hypothesis that $N-n \ge 2$ was used above to evaluate the integral of $1/\lambda^{N-n}$ over $\frac{1}{2m+n} < \lambda < \infty$. Since $d_m = O(m^{n-1})$, we see that the series in the last expression converges. Hence $|F|$ is integrable over A_1. As $A_1 \cup A_2 = \Delta_1(K, H_n)$ is a set of full measure in $\Delta(K, H_n)$, it follows that $F \in L^1(\Delta(K, H_n))$.

Next let f be the function on H_n defined by

$$f(z,t) = \int_{\Delta(K,H_n)} F(\psi)\psi(z,t)d\mu(\psi).$$

Since $F \in L^1(\Delta(K, H_n))$ and the spherical functions are continuous and bounded by 1, we see that f is well defined, continuous and bounded by $||F||_{L^1(\Delta(K,H_n))}$. Moreover, since $F \in L^2(\Delta(K, H_n))$ and $^\wedge : L^2(\Delta(K, H_n)) \to L^2_K(H_n)$ is an isometry, we have that $f \in L^2_K(H_n)$ with $||f||_2^2 = \int_{\Delta(K,H_n)} |F(\psi)|^2 d\mu(\psi)$ and $\widehat{f} = F$. This establishes the second assertion in Proposition 6.2. □

Remark 6.1. One can show that the set A_1 used in the proof of Proposition 6.2 is compact in $\Delta(K, H_n)$. Thus any bounded measurable function is integrable over A_1. This observation motivates the decomposition used in the proof.

Proof that $\widehat{\mathcal{S}}(K, H_n) \subset \mathcal{S}_K(H_n)^\wedge$. Suppose now that $F \in \widehat{\mathcal{S}}(K, H_n)$. Proposition 6.2 shows that $F = \widehat{f}$ where

$$f(z,t) = \left(\frac{1}{2\pi}\right)^{n+1} \int_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} d_\alpha F(\alpha,\lambda)\phi_{\alpha,\lambda}(z,t)|\lambda|^n d\lambda$$

is K-invariant, continuous, bounded and square integrable. To show that $f \in \mathcal{S}_K(H_n)$, we will show that f is smooth and that

$$\left(\frac{\gamma^2}{4}+t^2\right)^a \left(\frac{\partial}{\partial t}\right)^b \Delta^c f \in L^2_K(H_n)$$

for all $a, b, c \geq 0$. This will follow from the facts

1. $\Delta f \in L^2_K(H_n)$ with $(\Delta f)^\wedge \in \widehat{\mathcal{S}}(K, H_n)$,
2. $\frac{\partial f}{\partial t} \in L^2_K(H_n)$ with $\left(\frac{\partial f}{\partial t}\right)^\wedge \in \widehat{\mathcal{S}}(K, H_n)$, and
3. $\left(\frac{\gamma}{2} \pm it\right) f \in L^2_K(H_n)$ with $\left(\left(\frac{\gamma}{2} \pm it\right) f\right)^\wedge \in \widehat{\mathcal{S}}(K, H_n)$,

which we prove below. Here $\Delta = \sum_{j=1}^n \frac{\partial}{\partial z_j}\frac{\partial}{\partial \overline{z_j}}$ as before.

Using Equation 2.9 for the eigenvalues of the Heisenberg sub-Laplacian, one obtains

$$\begin{aligned} -|\lambda|(2|\alpha|+n)\phi_{\alpha,\lambda} = \mathcal{L}\phi_{\alpha,\lambda} &= \left[4\Delta - \frac{\gamma}{2}\left(\frac{\partial}{\partial t}\right)^2\right]\phi_{\alpha,\lambda} \\ &= 4\Delta\phi_{\alpha,\lambda} - \frac{\lambda^2}{2}\gamma\phi_{\alpha,\lambda}. \end{aligned}$$

Using Equation 4.10 for $\gamma\phi_{\alpha,\lambda}$ gives

$$\begin{aligned} 4\Delta\phi_{\alpha,\lambda} &= -\frac{|\lambda|}{2}(\mathcal{D}^+ - \mathcal{D}^-)\phi_{\alpha,\lambda} - |\lambda|(2|\alpha|+n)\phi_{\alpha,\lambda} \\ &= -\frac{|\lambda|}{2}\left[\sum_{|\beta|=|\alpha|+1} \frac{d_\beta}{d_\alpha}\left[\begin{array}{c}\beta\\ \alpha\end{array}\right]\phi_{\beta,\lambda} + (2|\alpha|+n)\phi_{\alpha,\lambda} + \sum_{|\beta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right]\phi_{\beta,\lambda}\right]. \end{aligned}$$

Define a function F_Δ on $\Delta_1(K, H_n)$ by

$$\begin{aligned} F_\Delta(\alpha,\lambda) &= -\frac{|\lambda|}{2}(\mathcal{D}^+ - \mathcal{D}^-)F(\alpha,\lambda) - |\lambda|(2|\alpha|+n)F(\alpha,\lambda) \\ &= -\frac{|\lambda|}{2}\left[\sum_{|\beta|=|\alpha|+1} \frac{d_\beta}{d_\alpha}\left[\begin{array}{c}\beta\\ \alpha\end{array}\right]F(\beta,\lambda) + (2|\alpha|+n)F(\alpha,\lambda) + \sum_{|\beta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right]F(\beta,\lambda)\right]. \end{aligned}$$

It is not hard to show that $F_\Delta \in \widehat{\mathcal{S}}(K, H_n)$. In particular, note that Equations 4.7 and 4.8 give

$$|F_\Delta(\alpha,\lambda)| \leq \frac{|\lambda|}{2}\left[(|\alpha|+n)\sum_{|\beta|=|\alpha|+1}|F(\beta,\lambda)| + (2|\alpha|+n)|F(\alpha,\lambda)| + |\alpha|\sum_{|\beta|=|\alpha|-1}|F(\beta,\lambda)|\right].$$

One uses this to show that F_Δ satisfies estimates as in Definition 6.1. Moreover, Lemma 5.3 shows that for each $\lambda \neq 0$

$$\sum_{\alpha\in\Lambda} d_\alpha F(\alpha,\lambda)(\mathcal{D}^+ - \mathcal{D}^-)\phi_{\alpha,\lambda} = \sum_{\alpha\in\Lambda} d_\alpha(\mathcal{D}^+ - \mathcal{D}^-)F(\alpha,\lambda)\phi_{\alpha,\lambda}$$

and hence also

$$\int_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} d_\alpha F_\Delta(\alpha,\lambda)|\lambda|^n d\lambda = 4\int_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} d_\alpha F(\alpha,\lambda)\left(\Delta\phi_{\alpha,\lambda}\right)|\lambda|^n d\lambda.$$

We conclude that $\Delta f \in L^2_K(H_n)$ with $4(\Delta f)^\wedge = F_\Delta \in \widehat{\mathcal{S}}(K,H_n)$ This proves item (1) above.

Next note that the function defined on $\Delta_1(K,H_n)$ by $\lambda F(\alpha,\lambda)$ belongs to $\widehat{\mathcal{S}}(K,H_n)$. Since $\frac{\partial\phi_{\alpha,\lambda}}{\partial t} = i\lambda\phi_{\alpha,\lambda}$, we see that $\frac{\partial f}{\partial t} \in L^2_K(H_n)$ with $\left(\frac{\partial f}{\partial t}\right)^\wedge = i\lambda F \in \widehat{\mathcal{S}}(K,H_n)$. This establishes item (2) above.

We begin the proof of item (3) by setting

$$\widetilde{F}(z,\lambda) = \sum_{\alpha\in\Lambda} d_\alpha |\lambda|^n F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) \tag{6.4}$$

for each $\lambda \neq 0$, so that

$$f(z,t) = \left(\frac{1}{2\pi}\right)^{n+1}\int_{\mathbb{R}^\times} \widetilde{F}(z,\lambda)e^{i\lambda t}d\lambda.$$

Note that we can compute $\partial_\lambda \widetilde{F}$ by taking derivatives term-wise in Equation 6.4. For $\lambda > 0$ we have

$$\begin{aligned}\partial_\lambda \widetilde{F}(z,\lambda) = \sum_{\alpha\in\Lambda} d_\alpha n\lambda^{n-1}F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) + \sum_{\alpha\in\Lambda} d_\alpha\lambda^n\partial_\lambda F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z)\\ + \sum_{\alpha\in\Lambda} d_\alpha\lambda^n F(\alpha,\lambda)\partial_\lambda\phi^\circ_{\alpha,\lambda}(z).\end{aligned}$$

Since $F \in \widehat{\mathcal{S}}(K,H_n)$, the estimates in Definition 6.1 can be applied to show that the first two sums converge absolutely for each $\lambda > 0$. For the third sum, we use Equations 4.11 for $\partial_\lambda\phi^\circ_{\alpha,\lambda}(z) = \partial_\lambda\phi_{\alpha,\lambda}(z,0)$ together with Lemma 5.3 to derive two identities:

$$\begin{aligned}&\sum_{\alpha\in\Lambda} d_\alpha\lambda^n F(\alpha,\lambda)\partial_\lambda\phi^\circ_{\alpha,\lambda}(z)\\ &\quad= \left\{\begin{array}{l} -\frac{\gamma}{2}\sum_{\alpha\in\Lambda} d_\alpha\lambda^n F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) + \sum_{\alpha\in\Lambda} d_\alpha\lambda^n F(\alpha,\lambda)\frac{1}{\lambda}\mathcal{D}^-\phi^\circ_{\alpha,\lambda}(z) \\ \frac{\gamma}{2}\sum_{\alpha\in\Lambda} d_\alpha\lambda^n F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) + \sum_{\alpha\in\Lambda} d_\alpha\lambda^n F(\alpha,\lambda)\frac{1}{\lambda}\mathcal{D}^+\phi^\circ_{\alpha,\lambda}(z)\end{array}\right\}\\ &\quad= \left\{\begin{array}{l} -\frac{\gamma}{2}\widetilde{F}(z,\lambda) - \sum_{\alpha\in\Lambda} d_\alpha\lambda^{n-1}(\mathcal{D}^+ + n)F(\alpha,\lambda) \\ \frac{\gamma}{2}\widetilde{F}(z,\lambda) - \sum_{\alpha\in\Lambda} d_\alpha\lambda^{n-1}(\mathcal{D}^- + n)F(\alpha,\lambda)\end{array}\right\}\end{aligned}$$

Substituting these identities in the expression for $\partial_\lambda\widetilde{F}(z,\lambda)$ gives

$$\partial_\lambda\widetilde{F}(z,\lambda) = \left\{\begin{array}{l} -\frac{\gamma}{2}\widetilde{F}(z,\lambda) + \sum_{\alpha\in\Lambda} d_\alpha\lambda^n\left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^+\right)F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) \\ \frac{\gamma}{2}\widetilde{F}(z,\lambda) + \sum_{\alpha\in\Lambda} d_\alpha\lambda^n\left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^-\right)F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z)\end{array}\right\} \tag{6.5}$$

both valid for $\lambda > 0$. We have similar identities for $\lambda < 0$:

(6.6)
$$\partial_\lambda \widetilde{F}(z,\lambda) = \left\{ \begin{array}{l} -\frac{\gamma}{2}\widetilde{F}(z,\lambda) + \sum_{\alpha\in\Lambda} d_\alpha \lambda^n \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^-\right) F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) \\ \frac{\gamma}{2}\widetilde{F}(z,\lambda) + \sum_{\alpha\in\Lambda} d_\alpha \lambda^n \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^+\right) F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) \end{array} \right\}.$$

Note that $\left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^\pm\right) F$ is the restriction of $M^\pm F$ to $\Delta_1^+(K,H_n) = \{\phi_{\alpha,\lambda} \mid \alpha \in \Lambda, \lambda > 0\}$ and also of $M^\mp F$ to $\Delta_1^-(K,H_n) = \{\phi_{\alpha,\lambda} \mid \alpha\in\Lambda, \lambda < 0\}$. Since $M^\pm F \in \widehat{\mathcal{S}}(K,H_n)$, $M^\pm F$ is integrable on $\Delta(K,H_n)$ and Equations 6.5 and 6.6 show that $\partial_\lambda \widetilde{F}(z,\lambda)$ is integrable on $\mathbb{R}^\times = \{\lambda \mid \lambda \neq 0\}$. We have,

$$\begin{aligned}(2\pi)^{n+1} it f(z,t) &= \int_{\mathbb{R}^\times} \widetilde{F}(z,\lambda)\partial_\lambda\left(e^{i\lambda t}\right) d\lambda \\ &= \int_0^\infty \widetilde{F}(z,\lambda)\partial_\lambda\left(e^{i\lambda t}\right) d\lambda + \int_{-\infty}^0 \widetilde{F}(z,\lambda)\partial_\lambda\left(e^{i\lambda t}\right) d\lambda \\ &= -\int_0^\infty \partial_\lambda\widetilde{F}(z,\lambda)e^{i\lambda t}d\lambda - \int_{-\infty}^0 \partial_\lambda\widetilde{F}(z,\lambda)e^{i\lambda t}d\lambda \\ &\quad - \lim_{\lambda\to 0^+}\widetilde{F}(z,\lambda) + \lim_{\lambda\to 0^-}\widetilde{F}(z,\lambda).\end{aligned}$$

It can be shown that the limits $\lim_{\lambda\to 0^\pm}\widetilde{F}(z,\lambda)$ exist and are equal. Here one needs to use the hypotheses that F is continuous across $\Delta_2(K,H_n)$ and that $F_\circ$ is a Schwartz function. The proof, which is rather involved, is presented below in Section 7. Using Equations 6.5 and 6.6 for $\partial_\lambda\widetilde{F}(z,\lambda)$ we obtain

$$\begin{aligned}(2\pi)^{n+1}\left(\pm\frac{\gamma}{2} + it\right) f(z,t) &= -\int_0^\infty \sum_{\alpha\in\Lambda} d_\alpha |\lambda|^n \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^\pm\right) F(\alpha,\lambda)\phi_{\alpha,\lambda}(z,t)d\lambda \\ &\quad - \int_{-\infty}^0 \sum_{\alpha\in\Lambda} d_\alpha|\lambda|^n \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^\mp\right) F(\alpha,\lambda)\phi_{\alpha,\lambda}(z,t)d\lambda \\ &= \int_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} d_\alpha M^\pm F(\alpha,\lambda)\phi_{\alpha,\lambda}(z,t)|\lambda|^n d\lambda.\end{aligned}$$

We conclude that $\left(\pm\frac{\gamma}{2} + it\right) f \in L_K(H_n)$ with $(2\pi)^{n+1}\left(\left(\pm\frac{\gamma}{2}+it\right)f\right)^\wedge = M^\pm F \in \widehat{\mathcal{S}}(K,H_n)$. This completes the proof for item (3). □

The characterization of the spherical transform of a Schwartz function given in Theorem 6.1 can be used to construct functions on H_n whose transform has a predetermined support. Recall that Theorem 2.2 asserts the existence of a map $E : \Delta(K,H_n) \to (\mathbb{R}^+)^d \times \mathbb{R}$ which is a homeomorphism onto its image. In Corollary 6.3 below, $K = U(n)$ or $\mathbb{T}^n$, so that $d = 1$ or $d = n$. In the first case, $\Delta(U(n),H_n) \cong$

$\{(m,\lambda) \mid m \in \mathbb{Z}^+, \lambda \in \mathbb{R}\} \coprod \{s \mid s > 0\}$ with

$$E(m,\lambda) = \left(\frac{|\lambda|}{2}(2m+n), \lambda\right) \quad \text{and} \quad E(s) = \left(\frac{s}{2}, 0\right).$$

In the second case, we have $\Delta(\mathbb{T}^n, H_n) \cong \{(m,\lambda) \mid m \in (\mathbb{Z}^+)^n, \lambda \in \mathbb{R}\} \coprod \{s \mid s \in (\mathbb{R}^+)^n\}$ with

$$E(m,\lambda) = \left(\frac{|\lambda|}{2}(2m_1+n), \dots, \frac{|\lambda|}{2}(2m_n+n)\lambda\right) \quad \text{and} \quad E(s) = \left(\frac{s_1}{2}, \dots, \frac{s_n}{2}, 0\right).$$

Corollary 6.3. *Let F be a smooth function of compact support on $\mathbb{R}^2$ (or $\mathbb{R}^{n+1}$). Then $\widetilde{F} = F \circ E$ is the spherical transform of a $U(n)$- (or $\mathbb{T}^n$-) invariant Schwartz function on H_n.*

Proof. Since F has compact support, we see that $\widetilde{F}(m,\lambda) = 0$ if $|\lambda|$ and $|\lambda|(2|m|+n)$ are sufficiently large. Hence our growth conditions are automatically satisfied. The only property left to verify is that $M^\pm \widetilde{F}$ extends continuously across $\Delta_2(K, H_n)$.

For $K = U(n)$, we first consider $M^+\widetilde{F}(m,\lambda)$ with $\lambda \to 0^+$, $|\lambda|(2m+n) \to s$. We have

$$\begin{aligned}
M^+\widetilde{F}(m,\lambda) &= \partial_\lambda \widetilde{F}(m,\lambda) - \frac{1}{\lambda}\mathcal{D}^+\widetilde{F}(m,\lambda) \\
&= \partial_\lambda \left[F\left(\frac{\lambda}{2}(2m+n), \lambda\right)\right] - \frac{m+n}{\lambda}\left[\widetilde{F}(m+1,\lambda) - \widetilde{F}(m,\lambda)\right] \\
&= \frac{2m+n}{2}\partial_1 F\left(\frac{\lambda}{2}(2m+n), \lambda\right) + \partial_2 F\left(\frac{\lambda}{2}(2m+n), \lambda\right) \\
&\quad - \frac{m+n}{\lambda}\left[F\left(\frac{\lambda}{2}(2m+n+2), \lambda\right) - F\left(\frac{\lambda}{2}(2m+n), \lambda\right)\right].
\end{aligned}$$

Now substitute $\lambda(2m+n)/2 \approx s/2$, to obtain

$$\begin{aligned}
M^+\widetilde{F}(m,\lambda) &\approx \frac{s}{2\lambda}\partial_1 F\left(\frac{s}{2}, \lambda\right) + \partial_2 F\left(\frac{s}{2}, \lambda\right) - \frac{s}{2\lambda^2}\left[F\left(\frac{s}{2}+\lambda, \lambda\right) - F\left(\frac{s}{2}, \lambda\right)\right] \\
&\quad - \frac{n}{2\lambda}\left[F\left(\frac{s}{2}+\lambda, \lambda\right) - F\left(\frac{s}{2}, \lambda\right)\right] \\
&= \partial_2 F\left(\frac{s}{2}, \lambda\right) - \frac{s}{2\lambda^2}\left[F\left(\frac{s}{2}+\lambda, \lambda\right) - F\left(\frac{s}{2}, \lambda\right) - \lambda\partial_1 F\left(\frac{s}{2}, \lambda\right)\right] \\
&\quad - \frac{n}{2\lambda}\left[F\left(\frac{s}{2}+\lambda, \lambda\right) - F\left(\frac{s}{2}, \lambda\right)\right],
\end{aligned}$$

which converges to

$$\partial_2 F\left(\frac{s}{2}, 0\right) - \frac{s}{4}\partial_1^2 F\left(\frac{s}{2}, 0\right) - \frac{n}{2}\partial_1 F\left(\frac{s}{2}, 0\right)$$

as $\lambda \to 0^+$. Now for $\lambda \to 0^-$, $\lambda(2m+n)/2 \to -s/2$ and we have:

$$\begin{aligned}
M^+\widetilde{F}(m,\lambda) &= \partial_\lambda \widetilde{F}(m,\lambda) - \frac{1}{\lambda}\mathcal{D}^-\widetilde{F}(m,\lambda) \\
&= \partial_\lambda \left[F\left(-\frac{\lambda}{2}(2m+n),\lambda\right)\right] - \frac{m}{\lambda}\left[\widetilde{F}(m,\lambda) - \widetilde{F}(m-1,\lambda)\right] \\
&\approx -\frac{2m+n}{2}\partial_1 F\left(\frac{s}{2},\lambda\right) + \partial_2 F\left(\frac{s}{2},\lambda\right) - \frac{m}{\lambda}\left[F\left(\frac{s}{2},\lambda\right) - F\left(\frac{s}{2}+\lambda,\lambda\right)\right] \\
&\approx \frac{s}{2\lambda}\partial_1 F\left(\frac{s}{2},\lambda\right) + \partial_2 F\left(\frac{s}{2},\lambda\right) + \frac{s}{2\lambda^2}\left[F\left(\frac{s}{2},\lambda\right) - F\left(\frac{s}{2}+\lambda,\lambda\right)\right] \\
&\quad + \frac{n}{2\lambda}\left[F\left(\frac{s}{2},\lambda\right) - F\left(\frac{s}{2}+\lambda,\lambda\right)\right],
\end{aligned}$$

which converges to

$$\partial_2 F\left(\frac{s}{2},0\right) - \frac{s}{4}\partial_1^2 F\left(\frac{s}{2},0\right) - \frac{n}{2}\partial_1 F\left(\frac{s}{2},0\right)$$

as $\lambda \to 0^-$. A similar calculation shows that, as $\lambda \to 0^\pm$ and $|\lambda|(2m+n)/2 \to s/2$, we get

$$M^-\widetilde{F}(m,\lambda) = \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^\mp\right)\widetilde{F}(m,\lambda) \to \partial_2 F\left(\frac{s}{2},0\right) + \frac{s}{4}\partial_1^2 F\left(\frac{s}{2},0\right) + \frac{n}{2}\partial_1 F\left(\frac{s}{2},0\right).$$

Therefore $M^\pm\widetilde{F}$ extends continuously to a smooth function on $\Delta_2(U(n),H_n)$, and by induction, we see that $(M^+)^\ell(M^-)^m\widetilde{F}$ will also be smooth on $\Delta_2(U(n),H_n)$.

For the case $K=\mathbb{T}^n$, we have:

$$\begin{aligned}
\widetilde{F}(m,\lambda) &= F\left(\frac{|\lambda|}{2}(2m_1+1),\ldots,\frac{|\lambda|}{2}(2m_n+1),\lambda\right) \\
\mathcal{D}^+\widetilde{F}(m,\lambda) &= \sum_{j=1}^n (m_j+1)[\widetilde{F}(m_1,\ldots,m_j+1,\ldots,m_n,\lambda) - \widetilde{F}(m,\lambda)], \\
\mathcal{D}^-\widetilde{F}(m,\lambda) &= \sum_{j=1}^n m_j[\widetilde{F}(m,\lambda) - \widetilde{F}(m_1,\ldots,m_j-1,\ldots,m_n,\lambda)].
\end{aligned}$$

If we approach $\Delta_2(\mathbb{T}^n,H_n)$ by taking $\lambda\to 0$, $|\lambda|(2m_j+1)\to s_j$, then we obtain, with $s=(s_1,\ldots,s_n)$:

$$M^\pm\widetilde{F}(m,\lambda) \to \partial_{n+1}F(s,0) \mp \left[\frac{1}{4}\sum_{j=1}^n s_j\partial_j^2 F(s,0) + \frac{1}{2}\sum_{j=1}^n s_j\partial_j F(s,0)\right].$$

Hence $\widetilde{F}$ is the spherical transform of a $\mathbb{T}^n$-invariant Schwartz function on H_n. □

Equation 6.2 shows how a K-invariant Schwartz function $f \in \mathcal{S}_K(H_n)$ is determined by its K-spherical transform $\widehat{f}$. We conclude this section with a result that shows how the Fourier transform $\mathcal{F}_H(f)$ of f is determined by $\widehat{f}$. The formula in

Proposition 6.4 differs from Equation 6.2 in that no integration is involved. Proposition 6.4 shows, in particular, that the Fourier transform $w \mapsto \mathcal{F}_H f(w,s)$ along a fixed central "level" $s \neq 0$ is completely determined by the values of the spherical transform $\{\widehat{f}(\alpha,s) \mid \alpha \in \Lambda\}$ associated with this level.

Proposition 6.4. *For $f \in \mathcal{S}_K(H_n)$ one has*

$$\mathcal{F}_H(f)(w,s) = \begin{cases} 2^n \sum_{\alpha\in\Lambda}(-1)^{|\alpha|} d_\alpha \widehat{f}(\alpha,s)\phi^\circ_\alpha\left(2w/\sqrt{|s|}\right) & s\neq 0 \\ \widehat{f}(\eta_w) & s = 0 \end{cases}.$$

Proof. The identity $\mathcal{F}_H(f)(w,0) = \widehat{f}(\eta_w)$ is Equation 2.15. Next we use Equation 6.2 to write

$$\begin{aligned}\mathcal{F}_H(f)(w,s) &= \left(\frac{1}{2\pi}\right)^{n+1} \int_{H_n}\int_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} d_\alpha \widehat{f}(\alpha,\lambda)\phi^\circ_\alpha\left(\sqrt{|\lambda|}z\right) e^{i(\lambda-s)t}|\lambda|^n e^{-iRe\langle z,w\rangle} d\lambda dz dt \\ &= \left(\frac{1}{2\pi}\right)^{n+1} \int_{\mathbb{R}}\int_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} d_\alpha \widehat{f}(\alpha,\lambda)\mathcal{F}_V\left(\phi^\circ_\alpha\circ\delta_{\sqrt{|\lambda|}}\right)(w) e^{i(\lambda-s)t}|\lambda|^n d\lambda dt.\end{aligned}$$

Here one can use the estimate given in Definition 6.1 for $m = 0$ to justify rearranging the series and integrals as above. In view of Equation 3.4, we have

$$\mathcal{F}_V\left(\phi^\circ_\alpha\circ\delta_{\sqrt{|\lambda|}}\right)(w) = (-1)^{|\alpha|}\left(\frac{4\pi}{|\lambda|}\right)^n \phi^\circ_\alpha\left(\frac{2w}{\sqrt{|\lambda|}}\right)$$

and hence

$$\begin{aligned}\mathcal{F}_H(f)(w,s) &= \int_{\mathbb{R}}\int_{\mathbb{R}^\times} \frac{2^n}{2\pi}\sum_{\alpha\in\Lambda}(-1)^{|\alpha|} d_\alpha \widehat{f}(\alpha,\lambda)\phi^\circ_\alpha\left(\frac{2w}{\sqrt{|\lambda|}}\right) e^{i(\lambda-s)t} d\lambda dt \\ &= 2^n \sum_{\alpha\in\Lambda}(-1)^{|\alpha|} d_\alpha \widehat{f}(\alpha,s)\phi^\circ_\alpha\left(2w/\sqrt{|s|}\right)\end{aligned}$$

for $s \neq 0$. Again, one can use the estimate in Definition 6.1 to justify the rearrangement of series and integrals employed above. □

7. Analysis of boundary terms

Throughout this section, F will denote a rapidly decreasing function on $\Delta(K,H_n)$ and $F_\circ \in S_K(V)$ is defined by $F_\circ(w) = F(\eta_w)$. (See Definition 6.1.) In the proof of Theorem 6.1, we defined a function $\widetilde{F}(z,\lambda)$ on $V \times \mathbb{R}^\times$ by

$$\widetilde{F}(z,\lambda) = \sum_{\alpha\in\Lambda} d_\alpha |\lambda|^n F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z).$$

In order to complete the proof, we need to show that the limits $\lim_{\lambda\to 0^\pm}\widetilde{F}(z,\lambda)$ exist and are equal. In fact, we will prove the following result.

Proposition 7.1. $\lim_{\lambda\to 0}\sum_{\alpha\in\Lambda} d_\alpha|\lambda|^n F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) = \left(\frac{1}{2\pi}\right)^n \int_V F_\circ(w)\eta_w(z)dw.$

It is easy to prove Proposition 7.1 for the case where $F=\widehat{f}$ for some $f\in S_K(H_n)$. Indeed, the inversion formula for the spherical transform yields

$$(2\pi)^{n+1}\int f(z,t)dt = \int\int \widetilde{F}(z,\lambda)e^{i\lambda t}d\lambda dt = 2\pi \lim_{\lambda\to 0^\pm}\widetilde{F}(z,\lambda).$$

Thus $\lim_{\lambda\to 0}\widetilde{F}(z,\lambda)$ exists and one has

$$\begin{aligned}\lim_{\lambda\to 0}\widetilde{F}(z,\lambda) &= (2\pi)^n\int f(z,t)dt = \frac{(2\pi)^n}{(2\pi)^{2n}}\int\int_V \mathcal{F}_V(f)(w,t)e^{iRe\langle w,z\rangle}dwdt\\ &= \left(\frac{1}{2\pi}\right)^n\int_V \mathcal{F}_H(f)(w,0)e^{iRe\langle w,z\rangle}dw = \left(\frac{1}{2\pi}\right)^n\int_V F_\circ(w)e^{iRe\langle w,z\rangle}dw\\ &= \left(\frac{1}{2\pi}\right)^n\int_V F_\circ(w)\eta_w(z)dw.\end{aligned}$$

Here we have used Equation 2.15 and K-invariance of $F_\circ$. This proves Proposition 7.1 for functions $F\in S_K(H_n)^\wedge$. Since we use Proposition 7.1 to prove that $\widehat{\mathcal{S}}(K,H_n)\subset S_K(H_n)^\wedge$, we can not, however, assume here that $F\in S_K(H_n)^\wedge$.

The case where $K=U(n)$ plays a special role in our proof of Proposition 7.1. We will denote the bounded $U(n)$-spherical functions on H_n by

$$\Delta_1(U(n),H_n)=\{\phi^U_{m,\lambda}\mid m=0,1,2,\dots\},\quad \Delta_2(U(n),H_n)=\{\eta^U_w\mid w\in V\}.$$

The K-spherical functions $\phi_{\alpha,\lambda}$ are related to the $U(n)$-spherical functions $\phi^U_{m,\lambda}$ via

$$d_m\phi^U_{m,\lambda}=\sum_{|\alpha|=m} d_\alpha\phi_{\alpha,\lambda} \tag{7.1}$$

where as before, $d_m := \dim(\mathcal{P}_m(V))$. This follows from the fact that $d_m p_m = \sum_{|\alpha|=m} d_\alpha p_\alpha$.

Lemma 7.2. *For $w\in V$ one has*

$$\lim_{N\to\infty}\frac{1}{d_N}\sum_{|\alpha|=N} d_\alpha F\left(\alpha,\frac{|w|^2}{2N+n}\right)=\int_{U(n)}F_\circ(kw)dk.$$

Proof. Let $s = |w|^2$. It is shown in [BJRW] that $\phi^U_{N,\frac{s}{2N+n}}$ converges to η^U_w uniformly on compact sets. Using this fact together with Equation 7.1 one computes

$$\begin{aligned}\int_{U(n)} F_\circ(kw)dk &= \int_{U(n)} \left(\frac{1}{2\pi}\right)^{2n} \int_V \mathcal{F}_V F_\circ(z) e^{i\langle z,kw\rangle} dz dk \\ &= \left(\frac{1}{2\pi}\right)^{2n} \int_V \mathcal{F}_V F_\circ(z) \eta^U_w(z) dz \\ &= \lim_{N\to\infty} \left(\frac{1}{2\pi}\right)^{2n} \int_V \mathcal{F}_V F_\circ(z) \left(\phi^U_{N,\frac{s}{2N+n}}\right)^\circ (z) dz \\ &= \lim_{N\to\infty} \left(\frac{1}{2\pi}\right)^{2n} \frac{1}{d_N} \sum_{|\alpha|=N} d_\alpha \int_V \mathcal{F}_V F_\circ(z) \phi^\circ_{\alpha,\frac{s}{2N+n}}(z) dz.\end{aligned}$$

Given $\phi_{\alpha,\lambda} \in \Delta_1(K, H_n)$, choose a point $w_{\alpha,\lambda} \in V$ such that

$$d\left(\phi_{\alpha,\lambda}, \eta_{w_{\alpha,\lambda}}\right) \le d\left(\phi_{\alpha,\lambda}, \eta\right)$$

for all $\eta \in \Delta_2(K, H_n)$. Here $d(\cdot,\cdot)$ denotes the metric on $\Delta(K, H_n)$ defined by Equation 2.11. We can now write

$$\begin{aligned}\int_{U(n)} F_\circ(kw)dk &= \lim_{N\to\infty} \left(\frac{1}{2\pi}\right)^{2n} \frac{1}{d_N} \sum_{|\alpha|=N} d_\alpha \int_V \mathcal{F}_V F_\circ(z) \eta_{w_{\alpha,\frac{s}{2N+n}}}(z) dz \\ &= \lim_{N\to\infty} \left(\frac{1}{2\pi}\right)^{2n} \frac{1}{d_N} \sum_{|\alpha|=N} d_\alpha \int_V \mathcal{F}_V F_\circ(z) e^{i\langle w_{\alpha,\frac{s}{2N+n}}, z\rangle} dz \\ &= \lim_{N\to\infty} \frac{1}{d_N} \sum_{|\alpha|=N} d_\alpha F_\circ\left(w_{\alpha,\frac{s}{2N+n}}\right) \\ &= \lim_{N\to\infty} \frac{1}{d_N} \sum_{|\alpha|=N} d_\alpha F\left(\alpha, \frac{s}{2N+n}\right).\end{aligned}$$

Here we have used continuity of F and the fact that $F_\circ$, and hence also $\mathcal{F}_V F_\circ$, are K-invariant. □

Next we define a function ${}^U F$ on $\Delta(U(n), H_n)$ by

$$\begin{aligned}{}^U F\left(\phi^U_{m,\lambda}\right) &= \frac{1}{d_m} \sum_{|\alpha|=m} d_\alpha F(\alpha, \lambda), \\ {}^U F\left(\eta^U_w\right) &= \int_{U(n)} F_\circ(kw) dk.\end{aligned}$$

If $F = \widehat{f}$ for some $f \in \mathcal{S}_K(H_n)$, then one can check that ${}^U F$ is the $U(n)$-spherical transform of the function ${}^U f \in \mathcal{S}_{U(n)}(H_n)$ defined by ${}^U f(z,t) = \int_{U(n)} f(kz,t)dk$. This observation motivates the definition of ${}^U F$. In general one has the following result.

Lemma 7.3. *${}^UF$ is rapidly decreasing on $\Delta(U(n), H_n)$.*

Proof. It is clear that $({}^UF)_\circ$ is a Schwartz function and that ${}^UF$ satisfies the estimates in Definition 6.1. We need, however, to show that ${}^UF$ is continuous across $\Delta_2(U(n), H_n)$. Suppose that (m_N, λ_N) is a sequence in $\mathbb{Z}^+ \times \mathbb{R}^\times$ with $\lim_{N\to\infty} \phi^U_{m_N,\lambda_N} = \eta^U_w$ in $\Delta(U(n), H_n)$. This occurs if and only if $|\lambda_N|(2m_N + n) \to |w|^2$ (see [BJRW]). Since

$$d\left(\left(m_N, \frac{|w|^2}{2m_N+n}\right), (m_N, \lambda_N)\right) = \left|\frac{|w|^2}{2m_N+n} - \lambda_N\right| \to 0$$

we see that one also has

$$\lim_{N\to\infty} \phi^U_{m_N, \frac{|w|^2}{2m_N+n}} = \eta^U_w$$

in $\Delta(U(n), H_n)$. Since $\lim_{N\to\infty} m_N = \infty$, we have

$$\begin{aligned}
\lim_{N\to\infty} {}^UF(m_N, \lambda_N) &= \lim_{N\to\infty} {}^UF\left(m_N, \frac{|w|^2}{2m_N+n}\right) \\
&= \lim_{N\to\infty} \frac{1}{d_{m_N}} \sum_{|\alpha|=N} d_\alpha F\left(\alpha, \frac{|w|^2}{2N+n}\right) \\
&= \int_{U(n)} F_\circ(kw)dk \qquad \text{(by Lemma 7.2)} \\
&= {}^UF(\eta^U_s).
\end{aligned}$$

□

Lemma 7.4 shows that Proposition 7.1 holds for the case where $K = U(n)$ and $z = 0$.

Lemma 7.4. *Let G be a rapidly decreasing function on $\Delta(U(n), H_n)$ and $w \in V$. Then $\lim_{\lambda\to 0} \sum_{m=0}^\infty d_m|\lambda|^n G(m,\lambda) = \left(\frac{1}{2\pi}\right)^n \int_V G_\circ(w)dw$.*

Proof. We will show that, for λ small, the left hand side of the above equation is close to a Riemann sum for the integral on the right hand side. Define a function g on $\mathbb{R}^+$ via

$$g\left(\frac{|w|^2}{2}\right) = G_\circ(w).$$

g is continuous and rapidly decreasing on $\mathbb{R}^+$. Using spherical coordinates on V, one sees that

$$\left(\frac{1}{2\pi}\right)^n \int_V G_\circ(w)dw = \frac{1}{(n-1)!}\int_0^\infty g(s)s^{n-1}ds. \tag{7.2}$$

Choose points $w_{m,\lambda} \in V$ with $|w_{m,\lambda}|^2 = |\lambda|(2m+n)$, and hence that

$$d\left(\phi^U_{m,\lambda}, \eta^U_{w_{m,\lambda}}\right) = |\lambda|.$$

We have

$$\lim_{\lambda\to 0}\sum_{m=0}^{\infty} d_m|\lambda|^n G(m,\lambda) = \lim_{\lambda\to 0}\sum_{m=0}^{\infty} d_m|\lambda|^n G\left(\eta^U_{w_{m,\lambda}}\right)$$
$$= \lim_{\lambda\to 0}\sum_{m=0}^{\infty} d_m|\lambda|^n g\left(\frac{|\lambda|}{2}(2m+n)\right).$$

Comparing this with Equation 7.2, we see that to complete the proof we must show that

$$\lim_{\lambda\to 0}\sum_{m=0}^{\infty} d_m|\lambda|^n g\left(\frac{|\lambda|}{2}(2m+n)\right) = \frac{1}{(n-1)!}\int_0^\infty g(s)s^{n-1}ds. \tag{7.3}$$

In fact, as shown below, Equation 7.3 is valid for any function $g \in L^1(\mathbb{R}^+, s^{n-1}ds)$.

It suffices to establish Equation 7.3 for a characteristic function. Suppose that $g(s) = 1$ for $s \in [a, b]$ and $g(s) = 0$ elsewhere. Then

$$\sum_{m=0}^{\infty} d_m|\lambda|^n g\left(\frac{|\lambda|}{2}(2m+n)\right) = |\lambda|^n \sum_{m=A_\lambda}^{B_\lambda} d_m,$$

where A_λ is the smallest integer with $a \le |\lambda|(2A_\lambda+n)/2$ and B_λ is the largest integer with $b \ge |\lambda|(2B_\lambda + n)/2$. Now

$$\sum_{m=A_\lambda}^{B_\lambda} d_m = \sum_{m=0}^{B_\lambda}\binom{m+n-1}{n-1} - \sum_{m=0}^{A_\lambda}\binom{m+n-1}{n-1} = \binom{B_\lambda+n}{n} - \binom{A_\lambda+n}{n}$$

and thus

$$\lim_{\lambda\to 0}|\lambda|^n\sum_{m=A_\lambda}^{B_\lambda} d_m = \lim_{\lambda\to 0}\left[\frac{|\lambda|^n}{n!}(B_\lambda+1)\cdots(B_\lambda+n) - \frac{|\lambda|^n}{n!}(A_\lambda+1)\cdots(A_\lambda+n)\right]$$
$$= \frac{1}{n!}(b^n - a^n) = \frac{1}{(n-1)!}\int_a^b s^{n-1}ds.$$
$$= \frac{1}{(n-1)!}\int_0^\infty g(s)s^{n-1}ds$$

as desired. □

Lemma 7.5. $\lim_{\lambda\to 0}\sum_{\alpha\in\Lambda} d_\alpha|\lambda|^n F(\alpha,\lambda) = \left(\frac{1}{2\pi}\right)^n \int_V F_\circ(w)dw.$

Proof. Lemma 7.3 ensures that the function ${}^U F$ is rapidly decreasing on $\Delta(U(n), H_n)$. Thus, by Lemma 7.4 we have

$$\lim_{\lambda\to 0}\sum_{m=0}^{\infty} d_m|\lambda|^n\, {}^U F(m,\lambda) = \left(\frac{1}{2\pi}\right)^n \int_V ({}^U F)_\circ(w)dw.$$

But

$$\sum_{\alpha\in\Lambda} d_\alpha|\lambda|^n F(\alpha,\lambda) = \sum_{m=0}^{\infty} d_m|\lambda|^n \frac{1}{d_m}\sum_{|\alpha|=m} d_\alpha F(\alpha,\lambda)$$
$$= \sum_{m=0}^{\infty} d_m|\lambda|^n\, {}^U\!F(m,\lambda),$$

and

$$\int_V F_\circ(w)dw = \int_V\int_{U(n)} F_\circ(kw)dkdw = \int_V ({}^U\!F)_\circ(w)dw.$$

□

Proof of Proposition 7.1. Lemma 7.5 shows that Proposition 7.1 holds for the case where $z = 0$. The proposition now follows by replacing $F(\alpha,\lambda)$ by $F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z)$. For $z \in V$ fixed, we see that $F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z)$ is a rapidly decreasing function on $\Delta_1(K,H_n)$ with continuous extension across $\Delta_2(K,H_n)$ given by $\eta_w \mapsto F_\circ(w)\eta_w(z)$. Note that $w \mapsto F_\circ(w)\eta_w(z)$ belongs to $\mathcal{S}_K(V)$ for fixed z. Thus Lemma 7.5 yields $\lim_{\lambda\to 0}\sum_{\alpha\in\Lambda} d_\alpha|\lambda|^n F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) = \left(\frac{1}{2\pi}\right)^n \int_V F_\circ(w)\eta_w(z)dw$. □

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