THE ORBIT METHOD AND GELFAND PAIRS ASSOCIATED WITH NILPOTENT LIE GROUPS

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Abstract. Let K be a compact Lie group acting by automorphisms on a nilpotent Lie group N. One calls (K, N) a *Gelfand pair* when the integrable K-invariant functions on N form a commutative algebra under convolution. We prove that in this case the coadjoint orbits for $G := K \ltimes N$ which meet the annihilator $\mathfrak{k}^{\perp}$ of the Lie algebra $\mathfrak{k}$ of K do so in single K-orbits. This generalizes a result of the authors and R. Lipsman concerning Gelfand pairs associated with Heisenberg groups.

Contents

1. Introduction

Suppose that N is a connected and simply connected nilpotent Lie group and that K is a compact, not necessarily connected Lie group acting smoothly by automorphisms on N. We say that (K, N) is a *Gelfand pair* when the algebra $L^1_K(N)$ of K-invariant integrable functions on N is commutative under convolution. Equivalently, the algebra $L^1(G//K)$ of integrable K-bi-invariant functions on the semidirect product

$$G := K \ltimes N$$

of K with N is abelian. It is shown in [3] that if (K, N) is a Gelfand pair then N must be a 2-step nilpotent group (or abelian). Thus, we assume throughout this paper that N is 2-step nilpotent.

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A fundamental result due to I. M. Gelfand [9] implies that (K, N) is a Gelfand pair if and only if each irreducible unitary representation π of G has at most a one dimensional space of K-fixed vectors, so that the multiplicity of the trivial representation 1_K of K in $\pi|_K$ is 0 or 1. We will outline a proof of the converse. Suppose that f and g are K-bi-invariant test functions on G and let $\pi \in \widehat{G}$ be an irreducible unitary representation of G in some Hilbert space $\mathcal{H}_\pi$. One shows that the operators $\pi(f)$, $\pi(g)$ on $\mathcal{H}_\pi$ commute with projection onto the subspace $\mathcal{H}_\pi^K$ of K-fixed vectors. As $\dim(\mathcal{H}_\pi^K) \leq 1$, we see that $\pi(f)\pi(g) = \pi(g)\pi(f)$ and conclude that $f * g = g * f$ by applying the Plancherel Theorem. Note that this reasoning shows that for the converse, it suffices that the space of K-fixed vectors have dimension at most 1 for all π in a set with full Plancherel measure in the unitary dual $\widehat{G}$ of G. An application of Ahn reciprocity [18, pp 56-58] shows that (K, N) is a Gelfand pair if and only if the quasi-regular representation $Ind_K^G(1_K)$ of G on $L^2(G/K) \cong L^2(N)$ is multiplicity free. This observation will, however, play no explicit role in the current work.

The unitary representations for certain classes of Lie groups may be obtained via the *orbit method* (also called *geometric quantization*). This method establishes a correspondence between irreducible unitary representations and integral coadjoint orbits in the dual of the Lie algebra. In particular, the method describes the unitary duals for compact groups [10, 12], nilpotent groups (where it reduces to the usual Kirillov correspondence [17]) and for semidirect products of nilpotent groups by compact groups [19].

Geometric multiplicity formulae express multiplicities for group representations in terms of the orbit method. Given a subgroup K of a Lie group G, let $\pi_\circ \in \widehat{K}$ and $\pi \in \widehat{G}$ correspond to coadjoint orbits $\mathcal{O}_\circ \subset \mathfrak{k}^*$ and $\mathcal{O} \subset \mathfrak{g}^*$ respectively and let $p : \mathfrak{g}^* \to \mathfrak{k}^*$ be the restriction map. One expects that the multiplicity of $\pi_\circ$ in $\pi|_H$ is given by $\#[(\mathcal{O} \cap p^{-1}(\mathcal{O}_\circ))/K]$. Such formulae appear in direct integral decompositions for nilpotent groups [6, 7, 8, 20], completely solvable and exponentially solvable groups [20, 21]. It is also known to hold "asymptotically" for representations of compact groups [10, 12] and to some extent for Riemannian symmetric spaces [20].

Consider the group $G = K \ltimes N$ with Lie algebra $\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{n}$ and restriction map $p : \mathfrak{g}^* \to \mathfrak{k}^*$. The coadjoint orbit corresponding to $1_K \in \widehat{K}$ is $\{0\}$ and $p^{-1}(\{0\}) = \mathfrak{k}^\perp \cong \mathfrak{n}^*$. One expects the multiplicity of 1_K in $\pi|_K$ to be the number of K-orbits in the intersection $\mathcal{O} \cap \mathfrak{k}^\perp$, where $\mathcal{O} \subset \mathfrak{g}^*$ is the coadjoint orbit for π. Denote the coadjoint orbit in $\mathfrak{g}^*$ through $\xi \in \mathfrak{g}^*$ by $\mathcal{O}_\xi^G$. In view of the characterization of Gelfand pairs via representation theory, the preceding discussion motivates our main result:

Theorem 1.1. *If (K, N) is a Gelfand pair then the following condition must hold:*

(OC) *for every $\xi \in \mathfrak{k}^\perp$, $\mathcal{O}_\xi^G \cap \mathfrak{k}^\perp$ is a single K-orbit.*

Theorem 1.1 is proved below in Section 5. Our motivating discussion leads one to conjecture that the converse for Theorem 1.1 should also be true. In Section 6, we discuss a line of attack for proving the converse, and consider a particular example.

Theorem 1.1 and its converse are known to hold when N is a Heisenberg group H_n and K is a connected subgroup of the unitary group $U(n)$. This result is proved in [2] and one can find a slightly weaker result described in [1]. The function of the current paper is thus to extend a portion of [2, 1] to a more general setting.

Our proof of Theorem 1.1 involves reducing the study of a pair (K, N) to that of a family of pairs $\{(K_\nu, H_\nu) \mid \nu \in \mathfrak{n}^*\}$ where each H_ν is a Heisenberg group (or abelian). (K, N) is a Gelfand pair if and only if each (K_ν, H_ν) is a Gelfand pair. This idea, which we call *localization*, is discussed in Section 2. Localization allows us to prove Theorem 1.1 by appealing to the analogous result for Heisenberg groups. This is essentially our result from [2] except that the K_ν's need not be connected subgroups of unitary groups. Thus, we first need to generalize the result from [2] to arbitrary compact subgroups of $Aut(H_n)$. This is done in Section 4 below after making a careful study of Condition (OC) to obtain some needed properties in Section 3.

Localization shows that Gelfand pairs associated with Heisenberg groups play a central role in the study of more general Gelfand pairs associated with nilpotent groups. By building on work of V. Kac [14] one can obtain a complete classification of Gelfand pairs associated with Heisenberg groups. (See [3] and [4].) In contrast, we have no such classification of Gelfand pairs associated with more general nilpotent Lie groups and our knowledge of such pairs is much less detailed. At present we know of relatively few examples where N is not a product of Heisenberg groups and abelian groups. These examples include cases where N is a free 2-step group [3] and certain groups of type-H [15].

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1.1. **Notation.** Throughout this paper, N will denote a connected and simply connected two step nilpotent Lie group with Lie algebra $\mathfrak{n}$. K denotes a compact Lie group with Lie algebra $\mathfrak{k}$ that acts smoothly by automorphisms on N. We write $k \cdot n$ for the action of $k \in K$ on $n \in N$. Similarly, $k \cdot X$ and $U \cdot X$ denote actions of $k \in K$ and $U \in \mathfrak{k}$ on $X \in \mathfrak{n}$. The contragredient actions of K and $\mathfrak{k}$ on $\mathfrak{n}^*$ are $(k \cdot \nu)(X) := \nu(k^{-1} \cdot X)$ and $(U \cdot \nu)(X) := -\nu(U \cdot X)$. $G = K \ltimes N$ denotes the semidirect product of K with N and $\mathfrak{g}$ is the Lie algebra of G. Our convention for the semi-direct product group law is

$$(k_1, n_1)(k_2, n_2) = (k_1 k_2, n_1(k_1 \cdot n_2))$$

for $(k_1, n_1), (k_2, n_2) \in G$. $\mathfrak{k}^\perp$ is the annihilator of $\mathfrak{k}$ in $\mathfrak{g}^*$. $\mathcal{O}_\nu^N$, $\mathcal{O}_\mu^K$ and $\mathcal{O}_\xi^G$ denote coadjoint orbits through $\nu \in \mathfrak{n}^*$, $\mu \in \mathfrak{k}^*$ and $\xi \in \mathfrak{g}^*$ for the groups N, K and G.

2. Localization for Gelfand Pairs

We begin by recalling how Gelfand's representation-theoretic criterion for Gelfand pairs, [9], specializes to pairs of the form (K, N). For $\nu \in \mathfrak{n}^*$, let π_ν denote the irreducible unitary representation of N associated with the coadjoint orbit $\mathcal{O}_\nu^N$. For $k \in K$, let

$$\pi_\nu^k(n) := \pi(k \cdot n),$$

so that π_ν^k is another irreducible unitary representation of N in the Hilbert space $\mathcal{H}_\nu$ for π_ν. The stabilizer of π_ν is

$$\begin{aligned} K_\nu &= \{k \in K \mid \pi_\nu^k \text{ is unitarily equivalent to } \pi_\nu\} \\ &= \{k \in K \mid k \cdot \nu \in \mathcal{O}_\nu^N\} \end{aligned} \tag{2.1}$$

Note that K_ν depends only on $\mathcal{O}_\nu^N$. (A notation such as $K_{\mathcal{O}_\nu^N}$ or K_{π_ν} would be more proper but seems cumbersome.) For each $k \in K_\nu$, one has a unitary operator, $W_\nu(k) : \mathcal{H}_\nu \to \mathcal{H}_\nu$, unique up to scalar multiples, that intertwines π_ν with π_ν^k:

$$\pi_\nu^k(n) = W_\nu(k)\pi_\nu(n)W_\nu(k)^{-1} \quad \text{for } n \in N,\ k \in K_\nu$$

A priori, one obtains a *projective* unitary representation W_ν of K_ν in $\mathcal{H}_\nu$. In this setting one can, however, always choose scalars to ensure that $W_\nu : K_\nu \to U(\mathcal{H}_\nu)$ is a (non-projective) representation. We will prove this below in Lemma 2.3. For $\sigma \in \widehat{K_\nu}$, we obtain a representation $\rho_{\sigma,\nu}$ of $K_\nu \ltimes N$ on $\mathcal{H}_\sigma \hat{\otimes} \mathcal{H}_\nu$ defined as

$$\rho_{\sigma,\nu}(k,n) := \sigma(k) \otimes \pi_\nu(n)W_\nu(k)$$

and $\widetilde{\rho_{\sigma,\nu}} := Ind_{K_\nu \ltimes N}^{K \ltimes N}(\rho_{\sigma.\nu})$ is irreducible. The "Mackey machine" guarantees that all irreducible unitary representations of $G = K \ltimes N$ are obtained in this way. An application of Frobenius reciprocity yields:

$$\begin{aligned} mult(1_K, \widetilde{\rho_{\sigma,\nu}}|_K) &= mult\left(1_K, Ind_{K_\nu}^K(\sigma \otimes W_\nu)\right) \\ &= mult(1_{K_\nu}, \sigma \otimes W_\nu) \\ &= mult(\overline{\sigma}, W_\nu). \end{aligned}$$

Here, "$mult(\alpha,\beta)$" denotes the multiplicity of α in β and $\overline{\sigma}$ is the conjugate representation for σ. One has that (K,N) is a Gelfand pair if and only if $mult(1_K, \psi)$ is 0 or 1 for each $\psi \in \widehat{G}$. Thus we have established the following result, due to G. Carcanno, [5]. (See also [3].)

Theorem 2.1. *Let N be a connected and simply connected two step nilpotent Lie group and K be any compact subgroup of $Aut(N)$. (K,N) is a Gelfand pair if and only if W_ν is a multiplicity free representation of K_ν for each $\nu \in \mathfrak{n}^*$.*

Continuing this discussion, we let $\mathfrak{z}$ denote the center of $\mathfrak{n}$ and $\mathfrak{z}_\nu := Ker(\nu|_\mathfrak{z})$. Since $\mathcal{O}_\nu^N$ is constant on $\mathfrak{z}$, $\mathfrak{z}_\nu$ depends only on $\mathcal{O}_\nu^N$ (equivalently π_ν). N_ν will denote the 2-step group with Lie algebra $\mathfrak{n}_\nu := \mathfrak{n}/\mathfrak{z}_\nu$. The action of K_ν on $\mathfrak{n}$ preserves $\mathfrak{z}_\nu$ and hence one obtains an action of K_ν on N_ν. Indeed, if $k \in K_\nu$ then $k^{-1} \cdot \nu \in \mathcal{O}_\nu^N$, so that $k^{-1} \cdot \nu = \nu + \nu[X, -]$ for some $X \in \mathfrak{n}$. Thus, if $Z \in \mathfrak{z}_\nu$ then $\nu(k \cdot Z) = (k^{-1} \cdot \nu)(Z) = \nu(Z) + \nu[X,Z] = 0$. Hence $k \cdot Z \in \mathfrak{z}_\nu$ for $k \in K_\nu$, $Z \in \mathfrak{z}_\nu$.

Fix a K-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}$ and write

$$\mathfrak{n} = \mathfrak{w} \oplus \mathfrak{z} = \mathfrak{a}_\nu \oplus \mathfrak{b}_\nu \oplus \mathfrak{z}_\nu \oplus \mathfrak{z}'_\nu, \tag{2.2}$$

where $\mathfrak{w}$ denotes the $\langle\cdot,\cdot\rangle$-orthogonal complement of $\mathfrak{z}$, $\mathfrak{z}'_\nu$ is the orthogonal complement of $\mathfrak{z}_\nu$ in $\mathfrak{z}$, $\mathfrak{a}_\nu := \{X \in \mathfrak{w} \mid ad(X)(\mathfrak{w}) \subset \mathfrak{z}_\nu\} = \{X \in \mathfrak{w} \mid \nu[X,\mathfrak{n}] = 0\}$ and $\mathfrak{b}_\nu$ is the orthogonal complement of $\mathfrak{a}_\nu$ in $\mathfrak{w}$. $\mathfrak{n}_\nu$ can be identified with the vector space $\mathfrak{a}_\nu \oplus \mathfrak{h}_\nu$ where

$$\mathfrak{h}_\nu := \mathfrak{b}_\nu \oplus \mathfrak{z}'_\nu. \tag{2.3}$$

This becomes a Lie algebra isomorphism when we equip $\mathfrak{a}_\nu$ with a trivial Lie bracket and $\mathfrak{h}_\nu$ with the bracket given by

$$[X,Y]_\nu := q_\nu[X,Y] \tag{2.4}$$

for $X, Y \in \mathfrak{b}_\nu$ where $q_\nu : \mathfrak{z} \to \mathfrak{z}'_\nu$ is orthogonal projection. Here $\mathfrak{a}_\nu$ is abelian and $\mathfrak{h}_\nu$ is either abelian ($\mathfrak{h}_\nu = \{0\}$ or $\mathfrak{h}_\nu \cong \mathbb{R}$) or is a Heisenberg Lie algebra. In the cases of interest, $\mathfrak{h}_\nu$ will be a Heisenberg Lie algebra as the notation suggests.

The action of K_ν on $\mathfrak{n}$ preserves both $\mathfrak{z}_\nu$ and $\langle\cdot,\cdot\rangle$ and hence also $\mathfrak{z}'_\nu$. Moreover, K_ν preserves $\mathfrak{a}_\nu$ and hence also $\mathfrak{b}_\nu$. Indeed, for $X \in \mathfrak{a}_\nu$, $k \in K_\nu$, one has

$$\begin{aligned} \nu[k\cdot X, \mathfrak{w}] &= (k^{-1}\cdot\nu)[X, k^{-1}\cdot\mathfrak{w}] \\ &= (k^{-1}\cdot\nu)[X,\mathfrak{w}] \\ &\subset (k^{-1}\cdot\nu)(\mathfrak{z}_\nu) = \nu(k\cdot\mathfrak{z}_\nu) = \nu(\mathfrak{z}_\nu) = \{0\}. \end{aligned}$$

Thus we obtain a diagonal action of K_ν on the product group $A_\nu \times H_\nu$ with Lie algebra $\mathfrak{a}_\nu \oplus \mathfrak{h}_\nu \cong \mathfrak{n}_\nu$.

Lemma 2.2. *W_ν is multiplicity free if and only if (K_ν, H_ν) is a Gelfand pair.*

Proof. If $\mathfrak{z}_\nu = \mathfrak{z}$ then $\mathcal{O}^N_\nu = \{\nu\}$ and π_ν is a one-dimensional representation. Since two unitary characters are equivalent if and only if they coincide, we see that $K_\nu = \{k \in K \mid \pi^k_\nu = \pi_\nu\}$ and W_ν is the trivial representation of K_ν on $\mathcal{H}_\nu \cong \mathbb{C}$. Thus, W_ν is multiplicity free when $\mathfrak{z}_\nu = \mathfrak{z}$. We also have $\mathfrak{h}_\nu = \{0\}$ when $\mathfrak{z}_\nu = \mathfrak{z}$, so (K_ν, H_ν) is a Gelfand pair in this situation.

Next suppose that $\mathfrak{z}_\nu \neq \mathfrak{z}$, so that $\mathfrak{z}'_\nu \cong \mathfrak{z}/\mathfrak{z}_\nu$ is one dimensional. If $\mathfrak{b}_\nu = \{0\}$ then $\mathfrak{h}_\nu = \mathfrak{z}'_\nu \cong \mathbb{R}$ is one dimensional abelian and (K_ν, H_ν) is a Gelfand pair. In this case, $\mathcal{O}^N_\nu = \{\nu + \nu[X,-] \mid X \in \mathfrak{w}\} = \{\nu\}$ and W_ν is trivially multiplicity free as above.

Finally, suppose that $\mathfrak{z}_\nu \neq \mathfrak{z}$ and $\mathfrak{b}_\nu \neq \{0\}$. In this case, $\mathfrak{h}_\nu$ is two-step with center $\mathfrak{z}'_\nu \cong \mathbb{R}$ and hence is a Heisenberg Lie algebra. Since $\mathcal{O}^N_\nu$ takes a constant non-zero value on $\mathfrak{z}'_\nu$ and the K_ν-action preserves both $\mathfrak{z}'_\nu$ and $\mathcal{O}^N_\nu$, we conclude that K_ν acts trivially on $\mathfrak{z}'_\nu$. Indeed, if Z is a non-zero element in $\mathfrak{z}'_\nu$ and $k \cdot Z = cZ$, say, for $k \in K_\nu$ and $c \in \mathbb{R}$ then $\nu(Z) = \nu(k\cdot Z) = \nu(cZ) = c\nu(Z)$ implies that $c = 1$. The representation π_ν is trivial on $Z_\nu = exp(\mathfrak{z}_\nu)$ and thus factors through $N_\nu = N/Z_\nu$. Identifying N_ν with the product group $A_\nu \times H_\nu$, we can write

$$\pi_\nu(a,n) = \chi(a)\pi'_\nu(n)$$

where $\chi : A_\nu \to \mathbb{T}$ is a unitary character and π'_ν is an irreducible unitary representation of H_ν on $\mathcal{H}_\nu$. For $k \in K_\nu$, $W_\nu(k) : \mathcal{H}_\nu \to \mathcal{H}_\nu$ is a unitary operator intertwining the representations π'_ν and $(\pi'_\nu)^k = (\pi^k_\nu)'$ of H_ν on $\mathcal{H}_\nu$. As K_ν acts trivially on $\mathfrak{z}'_\nu$ and $\mathfrak{z}'_\nu$ is the center of $\mathfrak{h}_\nu$, we see that K_ν preserves *all* coadjoint orbits in $\mathfrak{h}^*_\nu$ that assume non-zero values on $\mathfrak{z}'_\nu$. π'_ν corresponds to the orbit whose restriction to $\mathfrak{z}'_\nu$ is $\nu|\mathfrak{z}'_\nu(\neq 0)$. Since $\mathfrak{b}_\nu$ is the orthocomplement on $\mathfrak{z}'_\nu$ for a K_ν-invariant inner product on $\mathfrak{h}_\nu$, there is a family of dilating automorphisms $\{A_\lambda \mid \lambda \neq 0\}$ that commute with the action of K_ν and permute the orbits with non-zero restriction to $\mathfrak{z}'_\nu$. If we consider the representations $\pi_\ell \in \widehat{N}_\nu$ that arise from the $\ell \in \mathfrak{h}^*_\nu$ that are nonzero on $\mathfrak{z}'_\nu$ then it is evident that $\pi^k_\ell \cong \pi_\ell$ for all k. By examining the effect of the dilating automorphisms we see that the associated representation $W_\ell : K_\nu \to U(\mathcal{H}_{\pi_\ell})$ can be realized using precisely the operators $W_\nu(k)$. That is, we may take $W_\ell(k) = W_\nu(k)$ for all k, independent of the particular generic orbit considered. The remaining elements in $\widehat{H}_\nu$ are one dimensional representations and correspond to single point coadjoint orbits. As discussed above, the intertwining representation for a unitary character is always multiplicity free. Theorem 2.1 now shows that if W_ν is multiplicity free then (K_ν, H_ν) is a Gelfand pair. Conversely, if (K_ν, H_ν) is a Gelfand pair then $W_\ell : K_\nu \to U(\mathcal{H}_{\pi_\ell})$ is multiplicity free for every ℓ as above and hence W_ν is also multiplicity free. □

Lemma 2.3. *W_ν is a (non-projective) representation for all $\nu \in \mathfrak{n}$.*

Proof. The proof of Lemma 2.2 shows that the projective representation $W_\nu : K_\nu \to U(\mathcal{H}_\nu)$ is trivial in the cases where $\mathcal{O}^N_\nu = \{\nu\}$. When $\mathcal{O}^N_\nu \neq \{\nu\}$, H_ν is a Heisenberg group with Lie algebra $\mathfrak{h}_\nu = \mathfrak{b}_\nu \oplus \mathfrak{z}'_\nu$. The group K_ν acts unitarily on $(\mathfrak{b}_\nu, \langle\cdot,\cdot\rangle)$ and fixes $\mathfrak{z}'_\nu$. The proof of Lemma 2.2 shows that W_ν is the intertwining representation for a representation π'_ν of H_ν with non-trivial central character. We write $2n = \dim(\mathfrak{b}_\nu)$ and identify H_ν with the set $\mathbb{C}^n \times \mathbb{R}$ by using an orthonormal basis for $\mathfrak{b}_\nu$. K_ν can then be regarded as a subgroup of $U(n)$ acting on $\mathbb{C}^n \times \mathbb{R}$ as $k \cdot (z,t) = (kz, t)$. As explained in [3], π'_ν can be realized in a *Fock space* $\mathcal{H}_\nu$ consisting of entire functions on $\mathbb{C}^n$ that are square integrable with respect to a Gaussian weight. One has a unitary representation W' of $U(n)$ on $\mathcal{H}_\nu$ given by $W'(k)f(z) = f(k^{-1}z)$ for $k \in U(n)$, $f \in \mathcal{H}_\nu$ and $w \in \mathbb{C}^n$. Equation 4.5 in [3] shows that

$$(\pi'_\nu)^k(z,t) = W'(k)\pi'_\nu(z,t)W'(k)^{-1}$$

for all $k \in U(n)$. Thus, W' intertwines π'_ν with $(\pi'_\nu)^k$ for all $k \in U(n)$. We see that W_ν can be realized as the restriction of W' to a subgroup K_ν of $U(n)$. As W' is a non-projective representation, so is W_ν. □

Theorem 2.1 together with Lemma 2.2 shows that (K, N) is a Gelfand pair if and only if (K_ν, H_ν) is a Gelfand pair for each $\nu \in \mathfrak{n}^*$. In fact, for the converse it suffices that (K_ν, H_ν) be a Gelfand pair for any family of functionals $\nu \in \mathfrak{n}^*$ that yield a set of representations π_ν with full Plancherel measure in $\widehat{N}$. Here K_ν is given by 2.1 and H_ν is the Lie group with Lie algebra given by 2.3 and 2.4. The pair (K_ν, H_ν)

depends only on $\mathcal{O}_\nu^N$. In the trivial cases where $H_\nu = \{0\}$ or $\mathbb{R}$, (K_ν, H_ν) is always a Gelfand pair. In the cases of interest, H_ν is a Heisenberg group. By "localizing" at each coadjoint orbit $\mathcal{O}_\nu^N \subset \mathfrak{n}^*$, we reduce the study of (K, N) to that of a family of pairs (K_ν, H_ν) associated with Heisenberg groups. An alternative treatment of localization that involves free 2-step groups can be found in [3]. A somewhat weaker result appeared in [16].

Lemma 2.4 (Localization Lemma). *If (K, N) is a Gelfand pair then (K_ν, H_ν) is a Gelfand pair for all $\nu \in \mathfrak{n}^*$. Conversely, if (K_ν, H_ν) is a Gelfand pair for a.e. $\pi_\nu \in \widehat{N}$ then (K, N) is a Gelfand pair.*

Proof. As explained above, we need only prove the converse. Let $f, g \in L^1_K(N)$ and suppose that (K_ν, H_ν) is a Gelfand pair. Lemma 2.2 ensures that W_ν is multiplicity free. Since $\pi_\nu(f)$ and $\pi_\nu(g)$ are $W_\nu(K_\nu)$-invariant operators on $\mathcal{H}_\nu$, they are simultaneously diagonalized by the decomposition of $\mathcal{H}_\nu$ into $W_\nu(K_\nu)$-irreducible subspaces. Thus, $\pi_\nu(f * g) = \pi_\nu(f)\pi_\nu(g) = \pi_\nu(g)\pi_\nu(f) = \pi_\nu(g * f)$. If (K_ν, H_ν) is a Gelfand pair for a.e. $\pi_\nu \in \widehat{N}$ then an application of the Plancherel Theorem shows that $f * g = g * f$. □

Remark 2.1. The interplay between "every" and "almost every" is an interesting feature of the Localization Lemma. We see that if (K_ν, H_ν) is a Gelfand pair for a.e. $\nu \in \mathfrak{n}^*$ then in fact (K_ν, H_ν) is a Gelfand pair for every $\nu \in \mathfrak{n}^*$. Equivalently, if W_ν is multiplicity free for a.e. $\nu \in \mathfrak{n}^*$ then W_ν is multiplicity free for every $\nu \in \mathfrak{n}^*$.

We conclude this section with an application of Lemma 2.4. K° will denote the identity component of K. In [2], it is shown that if N is a Heisenberg group then (K, N) is a Gelfand pair if and only if (K°, N) is a Gelfand pair. Localization allows one to easily generalize this result to 2-step groups.

Proposition 2.5. *(K, N) is a Gelfand pair if and only if (K°, N) is a Gelfand pair.*

Proof. Since $L^1_K(N) \subset L^1_{K^\circ}(N)$, it's clear that (K, N) is a Gelfand pair whenever (K°, N) is a Gelfand pair. Suppose that (K, N) is a Gelfand pair and $\nu \in \mathfrak{n}^*$. Lemma 2.4 ensures that (K_ν, H_ν) is a Gelfand pair and hence so is $((K_\nu)^\circ, H_\nu)$ by Proposition 2.2 in [2]. Since $(K_\nu)^\circ \subset (K^\circ)_\nu$, we have that $((K^\circ)_\nu, H_\nu)$ is a Gelfand pair. As this holds for each $\nu \in \mathfrak{n}^*$, Lemma 2.4 implies that (K°, N) is a Gelfand pair as desired. □

3. More on Condition (OC)

Our goal here is to recast Condition (OC) algebraically. This is done below in Lemma 3.2. We begin by computing the coadjoint orbits for $G = K \ltimes N$. The exponential map will be used to identify N with its Lie algebra $\mathfrak{n}$. The BCH formula shows that the group product and Lie bracket are related by

$$XY = X + Y + \frac{1}{2}[X, Y]$$

for $X, Y \in \mathfrak{n}$. The inverse for X is $X^{-1} = -X$. The derived action of K on $\mathfrak{n}$ coincides with the action of K on N under this identification. Elements in G will be written in the form (k, X) where $k \in K$ and $X \in (\mathfrak{n} = N)$. The group law in G is given by

$$(k_1, X_1)(k_2, X_2) = (k_1 k_2, X_1(k_1 \cdot X_2))$$

and one has $(k, X)^{-1} = (k^{-1}, k^{-1} \cdot X^{-1}) = (k^{-1}, -k^{-1} \cdot X)$. Elements of $\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{n}$ will be written as as (U, X) or as $U + X$ where $U \in \mathfrak{k}$ and $X \in \mathfrak{n}$. The Lie bracket in $\mathfrak{g}$ is given by

$$[(U_1, X_1), (U_2, X_2)] = ([U_1, U_2], [X_1, X_2] + U_1 \cdot X_2 - U_2 \cdot X_1)$$

where $U \cdot X$ denotes the derived action of $\mathfrak{k}$ on $\mathfrak{n}$. In particular, $[U, X] = U \cdot X \in \mathfrak{n}$.

For $k \in K$, $U \in \mathfrak{k}$ and $X, Y \in \mathfrak{n}$ one has

$$Ad_G(k)U = Ad_K(k)U, \tag{3.1}$$

$$Ad_G(Y)X = Ad_N(Y)X = X + [Y, X], \tag{3.2}$$

$$Ad_G(k)X = k \cdot X, \tag{3.3}$$

$$Ad_G(Y)U = U - U \cdot Y - \frac{1}{2}[Y, U \cdot Y]. \tag{3.4}$$

Equations 3.1 and 3.2 follow from the facts that K and N are subgroups of G and that N is 2-step. Equation 3.3 holds since $(k, 0)(e, X)(k, 0)^{-1} = (e, k \cdot X)$. To establish Equation 3.4, we use the fact that $\mathfrak{n}$ is an ideal in $\mathfrak{g}$ together with the fact that $\mathfrak{n}$ in 2-step. Indeed

$$Ad_G(Y)U = e^{ad_G(Y)}U = U + [Y, U] + \frac{1}{2}[Y, [Y, U]] = U - U \cdot Y - \frac{1}{2}[Y, U \cdot Y].$$

Using these equations we now compute

$$\begin{aligned} Ad_G(k, Y)(U + X) &= Ad_G((e, Y)(k, 0))(U + X) = Ad_G(Y)(Ad_G(k)(U + X)) \\ &= Ad_G(Y)(Ad_K(k)U + k \cdot X) \\ &= Ad_G(Y)(Ad_K(k)U) + Ad_G(Y)(k \cdot X) \\ &= Ad_K(k)(U) - (Ad_K(k)U) \cdot Y - \frac{1}{2}[Y, (Ad_K(k)U) \cdot Y] + k \cdot X + [Y, k \cdot X]. \end{aligned}$$

Thus

$$\begin{aligned} &(Ad_G(k, Y))(U, X) = \\ &\left(Ad_K(k)(U), k \cdot X - (Ad_K(k)U) \cdot Y + [Y, k \cdot X] - \frac{1}{2}[Y, (Ad_K(k)U) \cdot Y] \right). \end{aligned} \tag{3.5}$$

The coadjoint actions of N and $\mathfrak{n}$ on $\mathfrak{n}^*$ are defined by

$$\begin{aligned} (Ad_N^*(X)\nu)(Y) &= \nu(Ad_N(X^{-1})Y), \\ (ad_N^*(X)\nu)(Y) &= \nu(ad_N(-X)Y) = -\nu([X, Y]). \end{aligned}$$

Since N is 2-step, our identification of N with $\mathfrak{n}$ allows one to write

$$Ad_N^*(X)\nu = \nu + ad_N^*(X)\nu = \nu - \nu([X,-]). \tag{3.6}$$

One checks easily that the coadjoint actions are related to the action of K on N by the following identities:

$$k \cdot (ad_N^*(X)\nu) = ad_N^*(k \cdot X)(k \cdot \nu) \tag{3.7}$$

$$k \cdot (Ad_N^*(X)\nu) = Ad_N^*(k \cdot X)(k \cdot \nu) \tag{3.8}$$

Write points $\xi \in \mathfrak{g}^*$ as $\xi = (\mu, \nu)$, where $\mu \in \mathfrak{k}^*$ and $\nu \in \mathfrak{n}^*$. That is, $\xi(U, X) = \mu(U) + \nu(X)$. Equation 3.5 yields:

$$\begin{aligned}(\mu,\nu)(Ad_G(k,Y)(U,X)) &= \mu(Ad_K(k)U) + \nu(k \cdot X) - \nu((Ad_K(k)U)\cdot Y) \\ &\quad +\nu([Y, k\cdot X]) - \frac{1}{2}\nu([Y,(Ad_K(k)U)\cdot Y]) \\ &= (Ad_K^*(k^{-1})\mu)(U) + (k^{-1}\cdot\nu)(X) - (Ad_K^*(k^{-1})(Y\times\nu))(U) \\ &\quad -(k^{-1}\cdot ad_N^*(Y)\nu)(X) + \frac{1}{2}Ad_K^*(k^{-1})(Y \times ad_N^*(Y)\nu)(U).\end{aligned} \tag{3.9}$$

Here, $Y \times \nu \in \mathfrak{k}^*$ is defined for $Y \in \mathfrak{n}$ and $\nu \in \mathfrak{n}^*$ by

$$(Y \times \nu)(U) := \nu(U \cdot Y) \tag{3.10}$$

for $U \in \mathfrak{k}$. The map $\times : \mathfrak{n} \times \mathfrak{n}^* \to \mathfrak{k}^*$ satisfies a fundamental equivariance property:

$$Ad_K^*(k)(Y \times \nu) = (k \cdot Y) \times (k \cdot \nu). \tag{3.11}$$

Using Equations 3.6, 3.7, 3.8 and 3.11, Equation 3.9 can be written as:

$$\begin{aligned}&Ad_G^*\left((k,Y)^{-1}\right)(\mu,\nu) \\ &\quad = \Bigg(Ad_K^*(k^{-1})\mu - Ad_K^*(k^{-1})(Y\times\nu) + \frac{1}{2}Ad_K^*(k^{-1})(Y \times ad_N^*(Y)\nu), \\ &\qquad\qquad k^{-1}\cdot\nu - k^{-1}\cdot ad_N^*(Y)\nu\Bigg) \\ &\quad = \Bigg(Ad_K^*(k^{-1})\mu - (k^{-1}\cdot Y)\times(k^{-1}\cdot\nu) + \frac{1}{2}(k^{-1}\cdot Y \times ad_N^*(k^{-1}\cdot Y)(k^{-1}\cdot\nu)), \\ &\qquad\qquad Ad_N^*(k^{-1}\cdot Y^{-1})(k^{-1}\cdot\nu)\Bigg).\end{aligned}$$

Since $(k,Y) = ((k,Y)^{-1})^{-1} = (k^{-1}, k^{-1}\cdot Y^{-1})^{-1} = (k^{-1}, -k^{-1}\cdot Y)^{-1}$, this yields

$$\begin{aligned}Ad_G^*(k,Y)(\mu,\nu) &= \left(Ad_K^*(k)\mu + Y\times(k\cdot\nu) + \frac{1}{2}Y\times ad_N^*(Y)(k\cdot\nu), Ad_N^*(Y)(k\cdot\nu)\right) \\ &= k\cdot\left(\mu + (k^{-1}\cdot Y)\times\nu + \frac{1}{2}(k^{-1}\cdot Y)\times ad_N^*(k^{-1}\cdot Y)\nu, Ad_N^*(k^{-1}\cdot Y)\nu\right)\end{aligned}$$

where $k \cdot (\mu, \nu) := (Ad^*_K(k)\mu, k \cdot \nu) = Ad^*_G(k, 0)(\mu, \nu)$. By letting k and Y range over K and $\mathfrak{n}$ in this expression, one obtains the following description of the coadjoint orbits for G.

Lemma 3.1.
$$\mathcal{O}^G_{(\mu,\nu)} = \left\{ k \cdot \left(\mu + X \times \nu + \frac{1}{2} X \times ad^*_N(X)\nu, Ad^*_N(X)\nu \right) \;\middle|\; k \in K, X \in \mathfrak{n} \right\}.$$

We identify $\mathfrak{k}^\perp = \{\xi \in \mathfrak{g}^* \mid \xi(\mathfrak{k}) = \{0\}\} = \{(0, \nu) \mid \nu \in \mathfrak{n}^*\}$ with $\mathfrak{n}^*$. In view of Lemma 3.1, we have

$$\mathcal{O}^G_{(\mu,\nu)} \cap \mathfrak{k}^\perp = \left\{ k \cdot Ad^*_N(X)\nu \mid \mu + X \times \nu + \frac{1}{2} X \times ad^*_N(X)\nu = 0 \right\}. \tag{3.12}$$

We see that $\mathcal{O}^G_{(\mu,\nu)} \cap \mathfrak{k}^\perp$ is K-saturated and contained in $K \cdot \mathcal{O}^N_\nu$. The orbits $\mathcal{O}^G_\xi$ through points $\xi \in \mathfrak{k}^\perp$ are precisely the orbits $\mathcal{O}^G_{(0,\nu)}$ as ν ranges over $\mathfrak{n}^*$. Condition (OC) states that $\mathcal{O}^G_{(0,\nu)} \cap \mathfrak{k}^\perp$ must equal $K \cdot (0, \nu)$ for each $\nu \in \mathfrak{n}^*$. Setting $\mu = 0$ in Equation 3.12 produces an algebraic reformulation of this condition.

Lemma 3.2. *(K, N) satisfies Condition (OC) if and only if, for all $\nu \in \mathfrak{n}^*$ and $X \in \mathfrak{n}$:*
$$\text{If } X \times \nu + \frac{1}{2} X \times ad^*_N(X)\nu = 0, \text{ then } Ad^*_N(X)\nu \in K \cdot \nu.$$

Condition (OC) has properties in common with the Gelfand pair condition. We conclude this section by presenting three such properties. These are required for our proof of Theorem 1.1.

Lemma 3.3. *If $K_1 \subset K_2$ and (K_1, N) satisfies condition (OC) then so does (K_2, N).*

Proof. This is clear from Lemma 3.2 since the $\times$-map is compatible with restriction. That is, if $\times_1 : \mathfrak{n} \times \mathfrak{n}^* \to \mathfrak{k}_1^*$ and $\times_2 : \mathfrak{n} \times \mathfrak{n}^* \to \mathfrak{k}_2^*$ are the $\times$-maps for the actions of K_1 and K_2 on N, then one has $(X \times_1 \nu)(U) = (X \times_2 \nu)(U)$ for $U \in \mathfrak{k}_1$. □

The following result parallels Proposition 2.5.

Lemma 3.4. *(K, N) satisfies Condition (OC) if and only if (K°, N) satisfies Condition (OC).*

Proof. Lemma 3.3 shows that if (K°, N) satisfies Condition (OC) then so does (K, N). For the converse, let $K^\circ, k_1K^\circ, \ldots, k_nK^\circ$ be the connected components of K. Thus, $G^\circ = K^\circ \ltimes N$ and $G = G^\circ \amalg k_1G^\circ \amalg \cdots \amalg k_nG^\circ$. Note that $\mathfrak{k}^\circ = \mathfrak{k}$ and $\mathfrak{g}^\circ = \mathfrak{g}$. Let $\nu \in \mathfrak{n}^* = \mathfrak{k}^\perp \subset \mathfrak{g}^*$. We have
$$\begin{aligned}
\mathcal{O}^G_\nu &= Ad^*(G)\nu = Ad^*(G^\circ)\nu \cup Ad^*(G^\circ)(k_1 \cdot \nu) \cup \cdots \cup Ad^*(G^\circ)(k_n \cdot \nu) \\
&= Ad^*(G^\circ)\nu \amalg Ad^*(G^\circ)(k_{i_1} \cdot \nu) \amalg \cdots \amalg Ad^*(G^\circ)(k_{i_m} \cdot \nu) \\
&= \mathcal{O}^{G^\circ}_\nu \amalg \mathcal{O}^{G^\circ}_{k_{i_1}\cdot\nu} \amalg \cdots \amalg \mathcal{O}^{G^\circ}_{k_{i_m}\cdot\nu}
\end{aligned}$$

say. Similarly one has

$$\begin{aligned} K\cdot\nu &= K^\circ\cdot\nu\cup K^\circ\cdot(k_1\cdot\nu)\cup\cdots\cup K^\circ\cdot(k_n\cdot\nu)\\ &= K^\circ\cdot\nu\amalg K^\circ\cdot(k_{j_1}\cdot\nu)\amalg\cdots\amalg K^\circ\cdot(k_{j_\ell}\cdot\nu)\end{aligned}$$

say. As (K,N) satisfies Condition (OC) we have that $\mathcal{O}^G_\nu\cap\mathfrak{k}^\perp = K\cdot\nu$. That is,

$$\left(\mathcal{O}^{G^\circ}_\nu\cap\mathfrak{k}^\perp\right)\amalg\cdots\amalg\left(\mathcal{O}^{G^\circ}_{k_{i_m}\cdot\nu}\cap\mathfrak{k}^\perp\right) = (K^\circ\cdot\nu)\amalg\cdots\amalg(K^\circ\cdot(k_{j_\ell}\cdot\nu)).$$

The sets on each side of this equation are connected and disjoint and ν belongs to both $\mathcal{O}^{G^\circ}_\nu\cap\mathfrak{k}^\perp$ and $K^\circ\cdot\nu$. Thus we must have that $\mathcal{O}^{G^\circ}_\nu\cap\mathfrak{k}^\perp = K^\circ\cdot\nu$. This shows that (K°,N) satisfies Condition (OC). □

Lemma 1.3 in [3] shows that if (K_1,N) is a Gelfand pair and K_2 is conjugate to K_1 in $Aut(G)$ then (K_2,N) is also a Gelfand pair. The analogous result concerning Condition (OC) reads:

Lemma 3.5. *Let $K_2=\phi K_1\phi^{-1}$ where K_1 is a compact subgroup of $Aut(N)$ and $\phi\in Aut(N)$. Then (K_1,N) satisfies Condition* (OC) *if and only if (K_2,N) does.*

Proof. Let $\Phi: K_1\to K_2$ be the map given by $\Phi(k) := \phi\circ k\circ\phi^{-1}$ and let $\Phi':\mathfrak{k}_1\to\mathfrak{k}_2$ be the derivative of Φ. For $U\in\mathfrak{k}_1$, $X\in\mathfrak{n}$ we have

$$\begin{aligned}\Phi'(U)\cdot X &= \left.\frac{d}{dt}\right|_{t=0}(exp_{K_2}(\Phi'(tU)))(X)\\ &= \left.\frac{d}{dt}\right|_{t=0}(\phi\circ exp_K(tU)\circ\phi^{-1})(X)\\ &= \phi(U\cdot(\phi^{-1}(X))).\end{aligned}$$

Thus for $X\in\mathfrak{n}$, $\nu\in\mathfrak{n}^*$, $U\in\mathfrak{k}_1$ we have

$$\begin{aligned}(X\times\nu)(\Phi'(U)) &= \nu(\Phi'(U)\cdot X)\\ &= \nu(\phi(U\cdot\phi^{-1}(X)))\\ &= (\nu\circ\phi)(U\cdot\phi^{-1}(X))\\ &= (\phi^{-1}(X)\times(\nu\circ\phi))(U)\end{aligned}$$

so that $(X\times\nu)\circ\Phi' = \phi^{-1}(X)\times(\nu\circ\phi)$ in $\mathfrak{k}_1^*$. In addition, one checks that $(ad^*_N(X)\nu)\circ\phi = ad^*_N(\phi^{-1}(X))(\nu\circ\phi)$. This together with Equation 3.6 yields $(Ad^*_N(X)\nu)\circ\phi = Ad^*_N(\phi^{-1}(X))(\nu\circ\phi)$.

Now suppose that (K_1,N) satisfies Condition (OC) and that $X\times\nu+\frac{1}{2}X\times ad^*_N(X)\nu = 0$ in $\mathfrak{k}_2^*$. Thus also

$$\left(X\times\nu+\frac{1}{2}X\times ad^*_N(X)\nu\right)\circ\Phi' = \phi^{-1}(X)\times(\nu\circ\phi)+\frac{1}{2}\phi^{-1}(X)\times ad^*_N(\phi^{-1}(X))(\nu\circ\phi) = 0$$

in $\mathfrak{k}_1^*$. Lemma 3.2 ensures that $Ad_N^*(\phi^{-1}(X))(\nu \circ \phi) \in K \cdot (\nu \circ \phi)$. Since $Ad_N^*(\phi^{-1}(X))(\nu \circ \phi) = (Ad_N^*(X)\nu) \circ \phi$, this shows that for some $k_1 \in K_1$:

$$Ad_N^*(X)\nu = (k_1 \cdot (\nu \circ \phi)) \circ \phi^{-1}.$$

But

$$\begin{aligned}(k_1 \cdot (\nu \circ \phi) \circ \phi^{-1})(X) &= \nu((\phi \circ k_1^{-1} \circ \phi^{-1})(X)) \\ &= \nu((\phi \circ k_1 \circ \phi^{-1})^{-1}(X)) \\ &= (k_2 \cdot \nu)(X)\end{aligned}$$

where $k_2 := \Phi(k_1) \in K_2$. Hence we have that $Ad_N^*(X)\nu \in K_2 \cdot \nu$. In view of Lemma 3.2, (K_2, N) satisfies Condition (OC). □

4. Heisenberg groups

The Heisenberg group of dimension $2n+1$ can be written as

$$H_n := \mathbb{C}^n \times \mathbb{R}$$

with product $(z,t)(w,s) := (z+w, t+s-Im(z \cdot \overline{w})/2)$. The group $U(n)$ of n by n unitary matrices acts by automorphisms on H_n via $k \cdot (z,t) := (kz, t)$. This yields a maximal compact connected subgroup of $Aut(H_n)$. The main result in [2] asserts that if K is a closed connected subgroup of $U(n)$ then (K, H_n) is a Gelfand pair if and only if (K, H_n) satisfies Condition (OC). We now generalize this result to encompass arbitrary compact Lie subgroups of $Aut(H_n)$.

Theorem 4.1. *Let K be a compact Lie subgroup of $Aut(H_n)$. Then (K, H_n) is a Gelfand pair if and only if (K, H_n) satisfies Condition* (OC).

Proof. Since $U(n)$ is a maximal compact connected subgroup in $Aut(H_n)$, one can find an automorphism ϕ of H_n with $\phi K^\circ \phi^{-1} \subset U(n)$. For this we refer the reader to Theorem 3.1 in Chapter 15 of [13] and remark that $Aut(H_n) \cong Aut(\mathfrak{h}_n)$ is a linear group and hence has only a finite number of connected components. Proposition 2.5 together with Lemma 1.3 in [3] shows that (K, H_n) is a Gelfand pair if and only if $(\phi K^\circ \phi^{-1}, H_n)$ is a Gelfand pair. By Theorem 1.2 in [2], the latter condition is equivalent to $(\phi K^\circ \phi^{-1}, H_n)$ satisfying Condition (OC). Lemmas 3.4 and 3.5 show that $(\phi K^\circ \phi^{-1}, H_n)$ satisfies Condition (OC) if and only if (K, H_n) satisfies (OC). □

5. Proof of Theorem 1.1

We will prove Theorem 1.1 via localization from Theorem 4.1. Let $\nu \in \mathfrak{n}^*$ and form the decomposition given by Equation 2.2. The projection map

$$p_\nu : \mathfrak{n} \to \mathfrak{n}_\nu = \mathfrak{n}/\mathfrak{z}_\nu \cong \mathfrak{a}_\nu \oplus \mathfrak{h}_\nu$$

is a K_ν-equivariant Lie algebra homomorphism. Since $\mathfrak{z}_\nu = Ker(\mathcal{O}_\nu^N|_\mathfrak{z})$, there is a coadjoint orbit in $\mathfrak{a}_\nu^* \times \mathfrak{h}_\nu^*$ that maps diffeomorphically to $\mathcal{O}_\nu^N$ under p_ν^*. This has the form $\{\gamma\} \times \mathcal{O}_{\nu'}^{H_\nu}$, where $\gamma \in \mathfrak{a}_\nu^*$, $\nu' \in \mathfrak{h}_\nu^*$ and $p_\nu^*((\gamma, \nu')) = \nu$. Since the action of K_ν preserves $\{\gamma\} \times \mathcal{O}_{\nu'}^{H_\nu}$ and the factors in the product $(\mathfrak{a}_\nu \times \mathfrak{h}_\nu)^* \cong \mathfrak{a}_\nu^* \times \mathfrak{h}_\nu^*$, we must have that K_ν fixes γ. The following lemma contains the key step in our proof of Theorem 1.1.

Lemma 5.1. *Let $p_\nu(X) = (A, X') \in \mathfrak{a}_\nu \oplus \mathfrak{h}_\nu$ for $X \in \mathfrak{n}$ given. One has*

(a) $$Ad_N^*(X)\nu = p_\nu^*\left(\gamma, Ad_{H_\nu}^*(X')\nu'\right) \quad \textit{and}$$

(b) $$\left(X \times \nu + \frac{1}{2} X \times ad_N^*(X)\nu\right)\Big|_{\mathfrak{k}_\nu} = X' \times \nu' + \frac{1}{2} X' \times ad_{H_\nu}^*(X')\nu'$$

Proof. To establish (a), one computes

$$\begin{aligned} p_\nu^*(\gamma, Ad_{H_\nu}^*(X')\nu') &= p_\nu^*(Ad_{A_\nu \times H_\nu}^*(A, X')(\gamma, \nu')) \\ &= p_\nu^*(Ad_{A_\nu \times H_\nu}^*(p_\nu(X))(\gamma, \nu')) \\ &= Ad_N^*(X) p_\nu^*(\gamma, \nu') \\ &= Ad_N^*(X)\nu. \end{aligned}$$

To prove (b), we will show that $X \times \nu$ agrees with $X' \times \nu'$ on $\mathfrak{k}_\nu$ and that $X \times ad_N^*(X)\nu$ agrees with $X' \times ad_{H_\nu}(X')\nu'$ on $\mathfrak{k}_\nu$. First note that for any $\alpha \in \mathfrak{n}_\nu^*$ we have

$$p_\nu(X) \times \alpha = X \times p_\nu^*(\alpha) \tag{5.1}$$

on $\mathfrak{k}_\nu$. Indeed, for $U \in \mathfrak{k}_\nu$ one has $(X \times p_\nu^*(\alpha))(U) = (p_\nu^*(\alpha))(U \cdot X) = \alpha(p_\nu(U \cdot X)) = \alpha(U \cdot p_\nu(X)) = (p_\nu(X) \times \alpha)(U)$. Thus we have

$$\begin{aligned} X \times \nu &= X \times p_\nu^*(\gamma, \nu') \\ &= p_\nu(X) \times (\gamma, \nu') \\ &= (A, X') \times (\gamma, \nu') \end{aligned}$$

on $\mathfrak{k}_\nu$. Since K_ν fixes γ we have $\gamma(U \cdot A) = 0$ for $U \in \mathfrak{k}_\nu$ and hence

$$\big((A, X') \times (\gamma, \nu')\big)(U) = \gamma(U \cdot A) + \nu'(U \cdot X') = \nu'(U \cdot X') = (X' \times \nu')(U).$$

This shows that $X \times \nu = X' \times \nu'$ on $\mathfrak{k}_\nu$. Similarly we compute

$$\begin{aligned} X \times ad_N^*(X)\nu &= X \times ad_N^*(X) p_\nu^*(\gamma, \nu') \\ &= X \times p_\nu^*(ad_{A_\nu \times H_\nu}^*(p_\nu(X))(\gamma, \nu')) \\ &= p_\nu(X) \times ad_{A_\nu \times H_\nu}^*(p_\nu(X))(\gamma, \nu') \\ &= (A, X') \times ad_{A_\nu \times H_\nu}^*(A, X')(\gamma, \nu') \end{aligned}$$

on $\mathfrak{k}_\nu$. But for $U \in \mathfrak{k}_\nu$ we have

$$\begin{aligned} \big((A, X') \times ad_{A_\nu \times H_\nu}^*(A, X')(\gamma, \nu')\big)(U) &= -(\gamma, \nu')([(A, X'), U \cdot (A, X')]) \\ &= -(\gamma, \nu')(0, [X', U \cdot X']_\nu) \\ &= -\nu'([X', U \cdot X']_\nu) \\ &= (X' \times ad_{H_\nu}^*(X')\nu')(U) \end{aligned}$$

and thus $X \times ad^*_N(X)\nu = X' \times ad^*_{H_\nu}(X')\nu'$ on $\mathfrak{k}_\nu$. $\square$

We can now complete the proof of Theorem 1.1. Suppose that (K,N) is a Gelfand pair and that $\nu \in \mathfrak{n}^*$, $X \in \mathfrak{n}$ satisfy $X \times \nu + \frac{1}{2}X \times ad^*_N(X)\nu = 0$. If H_ν is abelian ($\mathfrak{h}_\nu \cong \mathbb{R}$, or $\mathfrak{h}_\nu = \{0\}$) then $\mathcal{O}^N_\nu = \{\nu\}$ and we trivially have $Ad^*_N(X)\nu = \nu \in K \cdot \nu$. Otherwise, H_ν is a Heisenberg group and (K_ν, H_ν) is a Gelfand pair by Lemma 2.4. Hence (K_ν, H_ν) satisfies Condition (OC) by Theorem 4.1. By (b) in Lemma 5.1 we have that $X' \times \nu' + \frac{1}{2}X' \times ad^*_{H_\nu}(X')\nu' = 0$ on $\mathfrak{k}_\nu$. We conclude that $Ad^*_{H_\nu}(X')\nu' = k \cdot \nu'$ for some $k \in K_\nu$ by Lemma 3.2. But then (a) in Lemma 5.1 ensures that $Ad^*_N(X)\nu = k \cdot \nu$ also holds. Indeed, we have:

$$\begin{aligned}
Ad^*_N(X)\nu &= p^*_\nu(\gamma, Ad^*_{H_\nu}(X')\nu') \\
&= p^*_\nu(\gamma, k \cdot \nu') \\
&= p^*_\nu(k \cdot (\gamma, \nu')) \\
&= k \cdot p^*_\nu(\gamma, \nu'). \\
&= k \cdot \nu
\end{aligned}$$

In view of Lemma 3.2, this implies that (K,N) satisfies Condition (OC).

6. A converse for Theorem 1.1?

We conjecture that the converse for Theorem 1.1 is also true. That is, we expect that if (K,N) satisfies Condition (OC) then (K,N) must be a Gelfand pair. Theorem 4.1 provides strong supporting evidence by showing this to be true when N is a Heisenberg group. One can attempt to use Theorem 4.1 in a proof of the converse via localization as in Section 5. We outline such an argument below in order to show where the difficulty lies in this approach. Our proof of Theorem 1.1 shows that for fixed $\nu \in \mathfrak{n}^*$, the condition in Lemma 3.2 holds for all $X \in \mathfrak{n}$ whenever (K_ν, H_ν) is a Gelfand pair. We expect that the converse also holds. In other words, we conjecture that for $\nu \in \mathfrak{n}^*$, $\mathcal{O}^G_{(0,\nu)} \cap \mathfrak{k}^\perp$ is a single K-orbit if and only if (K_ν, H_ν) is a Gelfand pair.

Suppose that (K,N) is *not* a Gelfand pair. We would like to show that Condition (OC) must fail for (K,N). The Localization Lemma 2.4 ensures that (K_ν, H_ν) is *not* a Gelfand pair for some $\nu \in \mathfrak{n}^*$. Here H_ν is necessarily a Heisenberg group. As in Section 5, let $\gamma \in \mathfrak{a}^*_\nu$, $\nu' \in \mathfrak{h}^*_\nu$ satisfy $p^*_\nu((\gamma, \nu')) = \nu$. In view of Theorem 4.1, Condition (OC) fails for (K_ν, H_ν). As ν' is non-zero on the center $\mathfrak{z}'_\nu$ of $\mathfrak{h}_\nu$, one can argue that for some $X' \in \mathfrak{h}_\nu$, one has $X' \times \nu' + \frac{1}{2}X' \times ad^*_{H_\nu}(X')\nu' = 0$ on $\mathfrak{k}_\nu$ but $Ad^*_{H_\nu}(X')\nu' \notin K_\nu \cdot \nu'$. Examples show that we need *not* necessarily have that $X' \times \nu + \frac{1}{2}X' \times ad^*_N(X')\nu = 0$ on all of $\mathfrak{k}$. However, we conjecture that

> one can always find some $X \in \mathfrak{n}$, $A \in \mathfrak{a}_\nu$ with $p_\nu(X) = (A, X')$ and $X \times \nu + \frac{1}{2}X \times ad^*_N(X)\nu = 0$ on all of $\mathfrak{k}$.

If this is indeed the case, then Condition (OC) must fail for (K,N) since we also have that $Ad^*_N(X)\nu \notin K \cdot \nu$. Indeed, if one had $Ad^*_N(X)\nu = k \cdot \nu$ then k must belong to

K_ν and reasoning as in Section 5, one obtains that $p_\nu^*(\gamma, Ad^*_{H_\nu}(X)\nu') = p^*(\gamma, k\cdot\nu')$. Since p^* is injective, this would contradict $Ad^*_{H_\nu}(X')\nu' \notin K_\nu \cdot \nu'$.

Example 6.1. Let $N := \mathbb{C}\times H_1$ and $K := U(1) = \mathbb{T}$ act via $k\cdot(z,w,t) := (kz, kw, t)$ for $z, w \in \mathbb{C}$ and $t \in \mathbb{R}$. $\mathfrak{n}^*$ has basis $\{\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda\}$ where $\alpha_1(z,w,t) = Re(z)$, $\alpha_2(z,w,t) = Im(z)$, $\beta_1(z,w,t) = Re(w)$, $\beta_2(z,w,t) = Im(w)$, $\lambda(z,w,t) = t$. Let $\nu = \alpha_1 + \lambda$. Then $\mathcal{O}_\nu^N = \nu + Span(\beta_1, \beta_2)$. Since $k\cdot\nu(z,w,t) = Re(z/k) + t$, we see that $k\cdot\nu \in \mathcal{O}_\nu^N$ if and only if $k = 1$. Thus $K_\nu = \{1\}$ and $\mathfrak{k}_\nu = \{0\}$. We use the inner product $\langle (x+iy, u+iv, t), (x'+iy', u'+iv', t')\rangle = xx'+yy'+uu'+vv'+tt'$ to form the decomposition 2.2. We obtain $\mathfrak{z}_\nu = Span((i,0,0),(1,0,-1))$, $\mathfrak{z}'_\nu = Span((1,0,1))$, $\mathfrak{a}_\nu = \{0\}$, $\mathfrak{b}_\nu = \mathfrak{w} = Span((0,1,0),(0,i,0))$. H_ν is the 3-dimensional Heisenberg group with Lie algebra $\mathfrak{h}_\nu = \mathfrak{b}_\nu \oplus \mathfrak{z}'_\nu = \{(t,w,t) \mid w \in \mathbb{C}, t \in \mathbb{R}\}$. The Lie bracket given by Equation 2.4 is $[(t,w,t),(t',w',t')]_\nu = \left(-\frac{1}{2}Im(w\overline{w'}), 0, -\frac{1}{2}Im(w\overline{w'})\right)$. The map $\mathfrak{h}_\nu \to \mathfrak{h}_1$ given by $(t,w,t) \mapsto (w/\sqrt{2}, t)$ is a Lie algebra isomorphism. Thus, $L^1_{K_\nu}(H_\nu) = L^1_{\{1\}}(H_\nu) \cong L^1(H_1)$ is non-abelian. It follows from Lemma 2.4 that (K,N) is *not* a Gelfand pair. We remark that this example is also discussed in [16].

We'll show that Condition (OC) fails for (K,N). In fact, for $\nu = \alpha_1 + \lambda$ the intersection $\mathcal{O}^G_{(0,\nu)} \cap \mathfrak{k}^\perp$ contains more than one K-orbit. This is consistent with the conjectures discussed in this section. A computation shows that for $X = (z,w,t) \in \mathfrak{n}$ and $i\theta \in \mathfrak{k} = i\mathbb{R}$, one has

$$\left(X\times\nu + \frac{1}{2}X\times ad^*_N(X)\nu\right)(i\theta) = -\left(Im(z) + \frac{1}{2}|w|^2\right)\theta.$$

Thus, for $X := (-i/2, 1, 0)$ we have $X\times\nu + \frac{1}{2}X\times ad^*_N(X)\nu = 0$. Moreover, one computes $Ad^*_N(X)\nu = \nu - \beta_2$, so that $(Ad^*_N(X)\nu)(0,i,0) = -1$. On the other hand, $(k\cdot\nu)(0,i,0) = 0$ for all $k \in K$, and hence $Ad^*_N(X)\nu \notin K\cdot\nu$.

In this example, we have $X = X' + Z$ where $X' = (0,1,0) \in \mathfrak{h}_\nu$ and $Z = (-i/2,0,0) \in \mathfrak{z}_\nu$. We have $Ad^*_{H_\nu}(X')\nu' \notin \{\nu'\} = K_\nu\cdot\nu'$, but (trivially) $X'\times\nu' + \frac{1}{2}X'\times ad^*_{H_\nu}(X')\nu' = 0$ on $\mathfrak{k}_\nu$. On the other hand, one can verify that there is *no* choice of $X' \in \mathfrak{h}_\nu$ which satisfies $Ad^*_{H_\nu}(X')\nu' \notin K_\nu\cdot\nu'$ and $X'\times\nu + \frac{1}{2}X'\times ad^*_N(X')\nu = 0$ on all of $\mathfrak{k}$.

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