The action (Sp(6)xU(3)):$M_{6,3}$(C)

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This Maple worksheet provides details concerning an example presented in section 4.6.2 of our paper *Spaces of Bounded Spherical Functions on Heisenberg Groups: Part I*. At issue is the (multiplicity free) action of the compact group Sp(6)xU(3) on the space V=$M_{6,3}$(C) of 6x3 complex matrices via **(k1, k2). Z = k1 Z k2^t**. We will show that this action is well-behaved as defined in Section 2 of the paper.

```
> restart:with(linalg):
```

Functions **dot** and **sym** below implement the inner product of 6x3 matrices and the symplectic product of vectors in C^6. In the inner product complex conjugation should be applied to the matrix entries of the second input. But for our purposes it will suffice to restrict attention to matrices all of whose entries are real.

```
> dot:=(a,b)->sum(sum(a['u','v']*b['u','v'],'u'=1..6),'v'=1..3);
```

$$dot := (a, b) \to \sum_{'v'=1}^{3} \left(\sum_{'u'=1}^{6} a_{'u', 'v'} b_{'u', 'v'} \right)$$

```
> sym:=(a,b)->sum(a['i']*b['i'+3]-a['i'+3]*b['i'],'i'=1..3);
```

$$sym := (a, b) \to \sum_{'i'=1}^{3} \left(a_{'i'} b_{'i'+3} - a_{'i'+3} b_{'i'} \right)$$

So for example...

```
> z:=matrix(6,3): dot(z,z); sym(col(z,1),col(z,2));
```

$$z_{1,1}^2 + z_{2,1}^2 + z_{3,1}^2 + z_{4,1}^2 + z_{5,1}^2 + z_{6,1}^2 + z_{1,2}^2 + z_{2,2}^2 + z_{3,2}^2 + z_{4,2}^2 + z_{5,2}^2 + z_{6,2}^2 + z_{1,3}^2 + z_{2,3}^2 + z_{3,3}^2 + z_{4,3}^2 + z_{5,3}^2 + z_{6,3}^2$$

$$z_{1,1} z_{4,2} - z_{4,1} z_{1,2} + z_{2,1} z_{5,2} - z_{5,1} z_{2,2} + z_{3,1} z_{6,2} - z_{6,1} z_{3,2}$$

This multiplicity free action has rank 6. Fundamental highest weights and highest weight vectors were given in a paper by Howe and Umeda. We implement these below as A1,...,A6 (highest weights) and h1,...h6 (highest weight vectors).

```
> h1:=z->z[1,1];h1(z);
```

$$h1 := z \to z_{1,1}$$

$$z_{1,1}$$

```
> h2:=z->z[1,1]*z[2,2]-z[1,2]*z[2,1];h2(z);
```

$$h2 := z \to z_{1,1} z_{2,2} - z_{1,2} z_{2,1}$$

$$z_{1,1} z_{2,2} - z_{1,2} z_{2,1}$$

```
> h3:=z->det(delrows(z,4..6));h3(z);
```

$$h3 := z \to linalg{:}\text{-}det(linalg{:}\text{-}delrows(z, 4..6))$$

$$z_{1,1} z_{2,2} z_{3,3} - z_{1,1} z_{2,3} z_{3,2} - z_{2,1} z_{1,2} z_{3,3} + z_{2,1} z_{1,3} z_{3,2} + z_{3,1} z_{1,2} z_{2,3} - z_{3,1} z_{1,3} z_{2,2}$$

```
> A1:=[[1,0,0],[1,0,0]];
```

$$A1 := [[1, 0, 0], [1, 0, 0]]$$

```
> A2:=[[1,1,0],[1,1,0]] ;
```

$$A2 := [[1, 1, 0], [1, 1, 0]]$$

```
> A3:=[[1,1,1],[1,1,1]];
```

$$A3 := [[1, 1, 1], [1, 1, 1]]$$

```
> h4:=z->sym(col(z,1),col(z,2));h4(z);
```

$$h4 := z \to sym(linalg{:}\text{-}col(z, 1), linalg{:}\text{-}col(z, 2))$$

$$z_{1,1} z_{4,2} - z_{4,1} z_{1,2} + z_{2,1} z_{5,2} - z_{5,1} z_{2,2} + z_{3,1} z_{6,2} - z_{6,1} z_{3,2}$$

```
> A4:=[[0,0,0], [1,1,0]] ;
```

$$A4 := [[0, 0, 0], [1, 1, 0]]$$

```
> h5:=z->expand(z[1,1]*sym(col(z,2),col(z,3))-z[1,2]*sym(col(z,1),col
  (z,3))+z[1,3]*sym(col(z,1),col(z,2)));h5(z);
```

$$h5 := z \to expand\left(z_{1,1}\, sym(linalg{:}\text{-}col(z, 2), linalg{:}\text{-}col(z, 3)) - z_{1,2}\, sym(linalg{:}\text{-}col(z, 1), linalg{:}\text{-}col(z, 3)) + z_{1,3}\, sym(linalg{:}\text{-}col(z, 1), linalg{:}\text{-}col(z, 2))\right)$$

$$z_{1,1} z_{2,2} z_{5,3} - z_{1,1} z_{5,2} z_{2,3} + z_{1,1} z_{3,2} z_{6,3} - z_{1,1} z_{6,2} z_{3,3} - z_{1,2} z_{2,1} z_{5,3} + z_{1,2} z_{5,1} z_{2,3} - z_{1,2} z_{3,1} z_{6,3} + z_{1,2} z_{6,1} z_{3,3} + z_{1,3} z_{2,1} z_{5,2} - z_{1,3} z_{5,1} z_{2,2} + z_{1,3} z_{3,1} z_{6,2} - z_{1,3} z_{6,1} z_{3,2}$$

```
> A5:=[[1,0,0], [1,1,1]];
```

$$A5 := [[1, 0, 0], [1, 1, 1]]$$

```
> h6:=z->expand((z[1,1]*z[2,3]-z[2,1]*z[1,3])*sym(col(z,1),col(z,2))-
  (z[1,1]*z[2,2]-z[2,1]*z[1,2])*sym(col(z,1),col(z,3))); h6(z);
```

$$h6 := z \to expand\left(\left(z_{1,1} z_{2,3} - z_{2,1} z_{1,3}\right) sym(linalg{:}\text{-}col(z, 1), linalg{:}\text{-}col(z, 2)) - \left(z_{1,1} z_{2,2} - z_{1,2} z_{2,1}\right) sym(linalg{:}\text{-}col(z, 1), linalg{:}\text{-}col(z, 3))\right)$$

$$z_{1,1}^2 z_{2,3} z_{4,2} - z_{1,1} z_{2,3} z_{4,1} z_{1,2} + z_{1,1} z_{2,3} z_{2,1} z_{5,2} + z_{1,1} z_{2,3} z_{3,1} z_{6,2} - z_{1,1} z_{2,3} z_{6,1} z_{3,2} - z_{2,1} z_{1,3} z_{1,1} z_{4,2} - z_{2,1}^2 z_{1,3} z_{5,2} + z_{2,1} z_{1,3} z_{5,1} z_{2,2} - z_{2,1} z_{1,3} z_{3,1} z_{6,2} + z_{2,1} z_{1,3} z_{6,1} z_{3,2} - z_{1,1}^2 z_{2,2} z_{4,3} + z_{1,1} z_{2,2} z_{4,1} z_{1,3} - z_{1,1} z_{2,2} z_{2,1} z_{5,3} - z_{1,1} z_{2,2} z_{3,1} z_{6,3} + z_{1,1} z_{2,2} z_{6,1} z_{3,3} + z_{1,2} z_{2,1} z_{1,1} z_{4,3} + z_{1,2} z_{2,1}^2 z_{5,3} - z_{1,2} z_{2,1} z_{5,1} z_{2,3} + z_{1,2} z_{2,1} z_{3,1} z_{6,3} - z_{1,2} z_{2,1} z_{6,1} z_{3,3}$$

```
> A6:=[[1,1,0], [2,1,1]];
```

$$A6 := [[1, 1, 0], [2, 1, 1]]$$

```
> A:=[A1,A2,A3,A4,A5,A6];
```

$$A := [[[1, 0, 0], [1, 0, 0]], [[1, 1, 0], [1, 1, 0]], [[1, 1, 1], [1, 1, 1]], [[0, 0, 0], [1, 1, 0]], [[1, 0, 0], [1, 1, 1]], [[1, 1, 0], [2, 1, 1]]]$$

Matrix X will be an arbitrary element of the Lie algebra sp(6,C)....

```
> X:=matrix(6,6): X[1,5]:=X[2,4]: X[1,6]:=X[3,4]: X[4,4]:=-X[1,1]: X
  [5,4]:=-X[1,2]: X[6,4]:=-X[1,3]: X[2,6]:=X[3,5]: X[4,5]:=-X[2,1]: X
  [5,5]:=-X[2,2]: X[6,5]:=-X[2,3]: X[1,6]:=X[3,4]: X[4,6]:=-X[3,1]: X
  [5,6]:=-X[3,2]: X[6,6]:=-X[3,3]: X[4,4]:=-X[1,1]: X[4,2]:=X[5,1]: X
  [4,3]:=X[6,1]: X[5,3]:=X[6,2]: evalm(X);
```

$$\begin{bmatrix} X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} & X_{2,4} & X_{3,4} \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} & X_{2,5} & X_{3,5} \\ X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} & X_{3,5} & X_{3,6} \\ X_{4,1} & X_{5,1} & X_{6,1} & -X_{1,1} & -X_{2,1} & -X_{3,1} \\ X_{5,1} & X_{5,2} & X_{6,2} & -X_{1,2} & -X_{2,2} & -X_{3,2} \\ X_{6,1} & X_{6,2} & X_{6,3} & -X_{1,3} & -X_{2,3} & -X_{3,3} \end{bmatrix}$$

Matrix Y will be an arbitrary element of gl(3,C)....

```
> Y:=matrix(3,3):
```

The moment map takes V to the dual of sp(6) x gl(3). We implement this below as function **mom**.

```
> mom:=z->  simplify(dot(evalm(X&*z),z) ) + simplify(dot(evalm(z&*
  transpose(Y)),z)) ;
```

$$mom := z \rightarrow simplify(dot(evalm(X \&* z), z)) + simplify(dot(evalm(z \&* linalg:-transpose(Y)), z))$$

To illustrate the use of function **mom** we give below spherical points for each of A1, A2, A3, A4, A5, A6....

```
> z:=matrix(6,3,0): z[1,1]:=1: evalm(z); mom(z); A1;
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X_{1,1} + Y_{1,1}$$

$$[[1, 0, 0], [1, 0, 0]]$$

```
> z:=matrix(6,3,0): z[1,1]:=1: z[2,2]:=1: evalm(z); mom(z); A2;
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X_{1,1} + X_{2,2} + Y_{1,1} + Y_{2,2}$$

$$[[1, 1, 0], [1, 1, 0]]$$

```
> z:=matrix(6,3,0): z[1,1]:=1: z[2,2]:=1: z[3,3]:=1: evalm(z); mom(z)
  ; A3;
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X_{1,1} + X_{2,2} + X_{3,3} + Y_{1,1} + Y_{2,2} + Y_{3,3}$$

$$[[1, 1, 1], [1, 1, 1]]$$

```
> z:=matrix(6,3,0): z[1,1]:=1: z[4,2]:=1: evalm(z); mom(z); A4;
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Y_{1,1} + Y_{2,2}$$

$$[[0, 0, 0], [1, 1, 0]]$$

```
> z:=matrix(6,3,0): z[1,1]:=1: z[2,2]:=1: z[5,3]:=1: evalm(z); mom(z)
  ; A5;
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X_{1,1} + Y_{1,1} + Y_{2,2} + Y_{3,3}$$

$$[[1, 0, 0], [1, 1, 1]]$$

```
> z:=matrix(6,3,0): z[1,1]:=sqrt(2): z[2,2]:=1: z[4,3]:=1: evalm(z);
  mom(z); A6;
```

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X_{1,1} + X_{2,2} + 2\, Y_{1,1} + Y_{2,2} + Y_{3,3}$$

$$[[1, 1, 0], [2, 1, 1]]$$

%%%%%%%%%%%%%%%%%%%%%%%%%

Using numerical methods, we found the points in V which map to diagonals under the moment map. We code the results of this investigation as procedure **spt** below. We will show that this produces a spherical point which maps to the weight aA1+bA2+cA3+dA4+eA5+fA6.

```
> spt:=proc(a,b,c,d,e,f)
  local Z11,Z12,Z13,Z21,Z22,Z23,Z31,Z32,Z33,Z41,Z42,Z43,Z51,Z52,Z53,
  Z61,Z62,Z63;
  Z11:=a*(a+e+f)*(a+b+f)*(a+b+e+d+f)*(a+b+2*c+e+2*f)*(a+2*b+2*c+d+
  e+2*f)*(c+e+f+a+b)
  /(a+e)/(a+f)/(a+b+2*c+e+f)/(a+b+d+f)/(a+2*b+2*c+e+2*f)/(e+f+a+b):
  Z12:=b*e*f*(a+2*b+d+2*c+e+2*f)*(b+2*c+e+f)*(a+b+d+e+f)*(a+b+c+e+f)
  /(a+2*b+2*c+e+2*f)/(a+b+2*c+e+f)/(a+e)/(a+f)/(a+b+e+f)/(b+d):
  Z13:=e*d*(2*c+2*f+e+a+b)*(b+d+f)*(c+e+f+a+b)*(2*c+e+f+b)*(e+f+a)
  /(a+2*b+2*c+2*f+e)/(a+e)/(b+d)/(a+b+2*c+e+f)/(a+b+d+f)/(e+f+a+b):
  Z21:=e*f*(e+2*c+2*f+a+2*b+d)*(e+2*c+2*f+a+b)*(a+b+f)*(b+d+f)*(c+f+
  b)
  /(2*c+2*f+e+a+2*b)/(e+a)/(f+a)/(b+f)/(a+b+d+f)/(2*c+f+b):
  Z22:=(2*c+2*f+e+a+2*b+d)*a*b*(b+d+f)*(e+2*c+f+b)*(e+f+a)*(c+f+b)
  /(2*c+2*f+e+a+2*b)/(a+e)/(b+d)/(b+f)/(a+f)/(2*c+f+b):
  Z23:=a*d*f*(e+2*c+2*f+a+b)*(e+f+2*c+b)*(e+f+a+b+d)*(c+f+b)
  /(e+2*c+2*f+a+2*b)/(b+d)/(a+e)/(b+f)/(a+b+d+f)/(f+2*c+b):
  Z31:=a*c*(2*c+e+2*f+a+2*b+d)*e*(2*c+2*f+e+a+b)
  /(e+2*c+f+a+b)/(f+2*c+b)/(e+f+a+b)/(f+a+b+d)/(f+a)
  *b/(b+f)*d:
  Z32:=(e+2*c+2*f+a+2*b+d)*(e+2*c+f+b)*c*f*(f+a+b)*(e+f+a)
```

```
/(f+a)*(f+b)/(e+2*c+f+a+b)/(f+2*c+b)/(e+f+a+b)
/(b+f)/(b+f)*d/(b+d):
Z33:=(e+2*c+2*f+a+b)*(e+2*c+f+b)*(e+f+a+b+d)*c*(f+a+b)*(f+b+d)
/(e+2*c+f+a+b)/(f+2*c+b)/(e+f+a+b)/(f+a+b+d)/(f+b)
*b/(b+d):
Z41:=e*f*b*d*(e+f+2*c+b)*(b+d+f)*(e+f+c+a+b)
/(e+2*c+f+a+b)/(2*c+2*f+e+a+2*b)/(a+e)/(a+f)/(a+b+d+f)/(e+f+a+b):
Z42:=(e+2*c+2*f+a+b)*(c+e+f+a+b)*a*d*(a+b+f)*(e+f+a)*(b+d+f)
/(2*c+f+e+a+b)/(2*c+2*f+e+a+2*b)/(a+e)/(e+f+a+b)/(b+d)/(a+f):
Z43:=a*b*f*(e+2*c+2*f+a+2*b+d)*(e+f+c+a+b)*(e+f+a+b+d)*(a+b+f)
/(2*c+f+e+a+b)/(2*c+2*f+e+a+2*b)/(a+e)/(b+d)/(e+f+a+b)/(a+b+d+f):
Z51:=a*b*d*(e+f+2*c+b)*(f+c+b)*(e+f+a+b+d)*(e+f+a)
/(2*c+2*f+e+a+2*b)/(f+2*c+b)/(a+e)/(a+f)/(b+f)/(a+b+d+f):
Z52:=e*d*f*(2*c+2*f+e+a+b)*(c+f+b)*(e+f+a+b+d)*(a+b+f)
/(a+e)/(a+f)/(b+d)/(b+f)/(2*c+e+2*f+a+2*b)/(f+2*c+b):
Z53:=b*e*(a+b+f)*(a+e+f)*(b+c+f)*(b+d+f)*(a+2*b+2*c+d+e+2*f)
/(a+e)/(b+d)/(b+f)/(a+b+d+f)/(b+2*c+f)/(a+2*b+2*c+e+2*f):
Z61:=(e+2*c+f+b)*(e+f+a+b+d)*c*f*(e+f+a)*(f+a+b)*(f+b+d)
/(e+2*c+f+a+b)/(f+2*c+b)/(e+f+a+b)/(f+a)/(f+a+b+d)/(b+f):
Z62:=(e+2*c+2*f+a+b)*(e+f+a+b+d)*c*e
/(e+2*c+f+a+b)/(e+f+a+b)/(f+2*c+b)/(f+a)
*(b+d+f)*b/(b+f)/(b+d)*a:
Z63:=a*(e+2*c+2*f+a+2*b+d)*c*e*f*(e+f+a)
/(e+2*c+f+a+b)/(f+2*c+b)/(e+f+a+b)/(f+b)/(f+a+b+d)
/(b+d)*d:
matrix([[sqrt(Z11),-sqrt(Z12),sqrt(Z13)],
[sqrt(Z21),sqrt(Z22),-sqrt(Z23)],
[-sqrt(Z31),sqrt(Z32),sqrt(Z33)],
[sqrt(Z41),sqrt(Z42),sqrt(Z43)],
[-sqrt(Z51),sqrt(Z52),sqrt(Z53)],
[-sqrt(Z61),-sqrt(Z62),sqrt(Z63)]])
end:
```

Below we apply procedure **spt** to 6 random integer inputs in the range 1..10. Applying the moment map yields a "diagonal functional" as required. Re-executing these lines will produce a new numerical example each time.

```
> r:=rand(1..10):
> z:=spt(r(),r(),r(),r(),r(),r());
```

$$z := \begin{bmatrix} \frac{1}{715}\sqrt{16588957} & -\frac{8}{429}\sqrt{12122} & \frac{8}{715}\sqrt{4862} \\ \frac{27}{10010}\sqrt{874874} & \frac{3}{143}\sqrt{46189} & -\frac{3}{55}\sqrt{319} \\ -\frac{1}{26}\sqrt{38} & \frac{1}{273}\sqrt{52003} & \frac{1}{91}\sqrt{60697} \\ \frac{32}{715}\sqrt{22} & \frac{1}{143}\sqrt{30107} & \frac{1}{2145}\sqrt{12685673} \\ -\frac{2}{715}\sqrt{70499} & \frac{3}{4004}\sqrt{1335334} & \frac{3}{1540}\sqrt{1144066} \\ -\frac{2}{455}\sqrt{79373} & -\frac{5}{52}\sqrt{58} & \frac{1}{780}\sqrt{8398} \end{bmatrix}$$

```
> mom(z);
```

$$33\,X_{1,1} + 22\,X_{2,2} + 6\,X_{3,3} + 41\,Y_{1,1} + 28\,Y_{2,2} + 16\,Y_{3,3}$$

Here is our general spherical point:

```
> sp:=spt(a,b,c,d,e,f);
```

$$sp := \Bigg[\Bigg[\Big((a\,(a+e+f)\,(a+b+f)\,(a+b+e+d+f)\,(a+b+2c+e+2f)\,(a+2b+2c+d+e+2f)\,(c+e+f+a+b)) \Big/ ((a+e)\,(a+f)\,(a+b+2c+e+f)\,(a+b+d+f)\,(a+2b+2c+e+2f)\,(e+f+a+b)) \Big)^{1/2},$$

$$-\Big((b\,e\,f\,(a+2b+2c+d+e+2f)\,(b+2c+e+f)\,(a+b+e+d+f)\,(c+e+f+a+b)) \Big/ ((a+2b+2c+e+2f)\,(a+b+2c+e+f)\,(a+e)\,(a+f)\,(e+f+a+b)\,(b+d)) \Big)^{1/2},$$

$$\Big((e\,d\,(a+b+2c+e+2f)\,(b+d+f)\,(c+e+f+a+b)\,(b+2c+e+f)\,(a+e+f)) \Big/ ((a+2b+2c+e+2f)\,(a+e)\,(b+d)\,(a+b+2c+e+f)\,(a+b+d+f)\,(e+f+a+b)) \Big)^{1/2} \Bigg],$$

$$\Bigg[\Big((e\,f\,(a+2b+2c+d+e+2f)\,(a+b+2c+e+2f)\,(a+b+f)\,(b+d+f)\,(c+f+b)) \Big/ ((a+2b+2c+e+2f)\,(a+e)\,(a+f)\,(b+f)\,(a+b+d+f)\,(2c+f+b)) \Big)^{1/2},$$

$$\Big(((a+2b+2c+d+e+2f)\,a\,b\,(b+d+f)\,(b+2c+e+f)\,(a+e+f)\,(c+f+b)) \Big/ ((a+2b+2c+e+2f)\,(a+e)\,(b+d)\,(b+f)\,(a+f)\,(2c$$

$+ f + b)))^{1/2}$,

$-$

$$\sqrt{\frac{a\,d\,f\,(a+b+2\,c+e+2\,f)\,(b+2\,c+e+f)\,(a+b+e+d+f)\,(c+f+b)}{(a+2\,b+2\,c+e+2\,f)\,(b+d)\,(a+e)\,(b+f)\,(a+b+d+f)\,(2\,c+f+b)}}$$

],

[

$-$

$((a\,c\,(a+2\,b+2\,c+d+e+2\,f)\,e\,(a+b+2\,c+e+2\,f)\,b\,d) / ((a$

$+ b + 2\,c + e + f)\,(2\,c+f+b)\,(e+f+a+b)\,(a+b+d+f)\,(a+f)\,(b+f)))$

$^{1/2}$

,

$$\sqrt{\frac{(a+2\,b+2\,c+d+e+2\,f)\,(b+2\,c+e+f)\,c\,f\,(a+b+f)\,(a+e+f)\,d}{(a+f)\,(b+f)\,(a+b+2\,c+e+f)\,(2\,c+f+b)\,(e+f+a+b)\,(b+d)}},$$

$(((a+b+2\,c+e+2\,f)\,(b+2\,c+e+f)\,(a+b+e+d+f)\,c\,(a+b$

$+ f)\,(b+d+f)\,b) / ((a+b+2\,c+e+f)\,(2\,c+f+b)\,(e+f+a+b)\,(a+b+d$

$+ f)\,(b+f)\,(b+d)))^{1/2}$],

[$((e\,f\,b\,d\,(b+2\,c+e+f)\,(b+d+f)\,(c+e+f+a+b)) / ((a+b$

$+ 2\,c + e + f)\,(a+2\,b+2\,c+e+2\,f)\,(a+e)\,(a+f)\,(a+b+d+f)\,(e+f+a$

$+ b)))^{1/2}$,

$(((a+b+2c+e+2f)(c+e+f+a+b)\,a\,d\,(a+b+f)(a+e$

$+f)(b+d+f))/((a+b+2c+e+f)(a+2b+2c+e+2f)(a+e)(e+f$

$+a+b)(b+d)(a+f)))^{1/2},$

$((a\,b\,f\,(a+2b+2c+d+e+2f)(c+e+f+a+b)(a+b+e+d$

$+f)(a+b+f))/((a+b+2c+e+f)(a+2b+2c+e+2f)(a+e)(b$

$+d)(e+f+a+b)(a+b+d+f)))^{1/2}\Big],$

$\Big[$

$-$

$((a\,b\,d\,(b+2c+e+f)(c+f+b)(a+b+e+d+f)(a+e+f))/$

$((a+2b+2c+e+2f)(2c+f+b)(a+e)(a+f)(b+f)(a+b+d+f)))$

$^{1/2}, \sqrt{\frac{e\,d\,f\,(a+b+2c+e+2f)(c+f+b)(a+b+e+d+f)(a+b+f)}{(a+e)(a+f)(b+d)(b+f)(a+2b+2c+e+2f)(2c+f+b)}},$

$((b\,e\,(a+b+f)(a+e+f)(c+f+b)(b+d+f)(a+2b+2c+d$

$+e+2f))/((a+e)(b+d)(b+f)(a+b+d+f)(2c+f+b)(a+2b+2c$

$+e+2f)))^{1/2}\Big],$

$\Big[$

$-$

$(((b+2c+e+f)(a+b+e+d+f)\,c\,f\,(a+e+f)(a+b+f)(b$

$+ d + f))\Big/((a + b + 2c + e + f)(2c + f + b)(e + f + a + b)(a + f)(a + b + d + f)(b + f)))^{1/2}$,

$-\sqrt{\dfrac{(a + b + 2c + e + 2f)(a + b + e + d + f)\, c\, e\, (b + d + f)\, b\, a}{(a + b + 2c + e + f)(e + f + a + b)(2c + f + b)(a + f)(b + f)(b + d)}}$,

$((a(a + 2b + 2c + d + e + 2f)\, c\, e\, f\, (a + e + f)\, d)\Big/((a + b + 2c + e + f)(2c + f + b)(e + f + a + b)(b + f)(a + b + d + f)(b + d)))^{1/2}\Big]\Big]$

To help with simplification, we replace each square root in the matrix above with a single variable, t1.... t24.

```
> getallterms:=proc(sp)
      local ntermsarray, dtermsarray, Trms, i, j;
      ntermsarray:=map(x->{op(convert(numer(x^2), list))}, sp);
      dtermsarray:=map(x->{op(convert(denom(x^2), list))}, sp);
      Trms:={};
      for i from 1 to 6 do
          for j from 1 to 3 do
              Trms:=Trms union ntermsarray[i,j] union dtermsarray[i,
  j]
      od od;
      convert(Trms, list)
  end:
> Terms:=getallterms(sp); nops(Terms);
```

$Terms := [a, b, c, d, e, f, a + e, a + f, b + d, b + f, a + b + f, a + e + f, b + d + f, c + f + b, 2c + f + b, a + b + d + f, b + 2c + e + f, e + f + a + b, a + b + 2c + e + f, a + b + 2c + e + 2f, a + b + e + d + f, a + 2b + 2c + e + 2f, c + e + f + a + b, a + 2b + 2c + d + e + 2f]$

24

```
> rewrite := proc(q)
      local num, den, numr, denr, j:
      global Terms;
      num:={op(convert(numer(q), list))};
      den:={op(convert(denom(q), list))};
      numr:=1; denr:=1;
      for j from 1 to nops(Terms) do
          if Terms[j] in num then numr:=numr*t[j] fi;
          if Terms[j] in den then denr:=denr*t[j] fi
      od;
      numr/denr
  end:
```

The matrix zt is the spherical point expressed in terms of the t's.

```
> zt:=zip((x,y)->sign(x)*(rewrite(y^2)), sp, sp);
```

$$zt := \begin{bmatrix} \frac{t_1 t_{11} t_{12} t_{20} t_{21} t_{23} t_{24}}{t_7 t_8 t_{16} t_{18} t_{19} t_{22}} & -\frac{t_2 t_5 t_6 t_{17} t_{21} t_{23} t_{24}}{t_7 t_8 t_9 t_{18} t_{19} t_{22}} & \frac{t_4 t_5 t_{12} t_{13} t_{17} t_{20} t_{23}}{t_7 t_9 t_{16} t_{18} t_{19} t_{22}} \\ \frac{t_5 t_6 t_{11} t_{13} t_{14} t_{20} t_{24}}{t_7 t_8 t_{10} t_{15} t_{16} t_{22}} & \frac{t_1 t_2 t_{12} t_{13} t_{14} t_{17} t_{24}}{t_7 t_8 t_9 t_{10} t_{15} t_{22}} & -\frac{t_1 t_4 t_6 t_{14} t_{17} t_{20} t_{21}}{t_7 t_9 t_{10} t_{15} t_{16} t_{22}} \\ -\frac{t_1 t_2 t_3 t_4 t_5 t_{20} t_{24}}{t_8 t_{10} t_{15} t_{16} t_{18} t_{19}} & \frac{t_3 t_4 t_6 t_{11} t_{12} t_{17} t_{24}}{t_8 t_9 t_{10} t_{15} t_{18} t_{19}} & \frac{t_2 t_3 t_{11} t_{13} t_{17} t_{20} t_{21}}{t_9 t_{10} t_{15} t_{16} t_{18} t_{19}} \\ \frac{t_2 t_4 t_5 t_6 t_{13} t_{17} t_{23}}{t_7 t_8 t_{16} t_{18} t_{19} t_{22}} & \frac{t_1 t_4 t_{11} t_{12} t_{13} t_{20} t_{23}}{t_7 t_8 t_9 t_{18} t_{19} t_{22}} & \frac{t_1 t_2 t_6 t_{11} t_{21} t_{23} t_{24}}{t_7 t_9 t_{16} t_{18} t_{19} t_{22}} \\ -\frac{t_1 t_2 t_4 t_{12} t_{14} t_{17} t_{21}}{t_7 t_8 t_{10} t_{15} t_{16} t_{22}} & \frac{t_4 t_5 t_6 t_{11} t_{14} t_{20} t_{21}}{t_7 t_8 t_9 t_{10} t_{15} t_{22}} & \frac{t_2 t_5 t_{11} t_{12} t_{13} t_{14} t_{24}}{t_7 t_9 t_{10} t_{15} t_{16} t_{22}} \\ -\frac{t_3 t_6 t_{11} t_{12} t_{13} t_{17} t_{21}}{t_8 t_{10} t_{15} t_{16} t_{18} t_{19}} & -\frac{t_1 t_2 t_3 t_5 t_{13} t_{20} t_{21}}{t_8 t_9 t_{10} t_{15} t_{18} t_{19}} & \frac{t_1 t_3 t_4 t_5 t_6 t_{12} t_{24}}{t_9 t_{10} t_{15} t_{16} t_{18} t_{19}} \end{bmatrix}$$

To convert back from "t-variables" to parameters (a,b,...) use this substitution......

```
> Sub:={seq(t[j]=sqrt(Terms[j]), j=1..nops(Terms))};
```

$$Sub := \{t_1 = \sqrt{a}, t_2 = \sqrt{b}, t_3 = \sqrt{c}, t_4 = \sqrt{d}, t_5 = \sqrt{e}, t_6 = \sqrt{f}, t_7 = \sqrt{a+e}, t_8 = \sqrt{a+f}, t_9 = \sqrt{b+d}, t_{10} = \sqrt{b+f}, t_{11} = \sqrt{a+b+f}, t_{12} = \sqrt{a+e+f}, t_{13} = \sqrt{b+d+f}, t_{14} = \sqrt{c+f+b}, t_{15} = \sqrt{2c+f+b}, t_{16} = \sqrt{a+b+d+f}, t_{17} = \sqrt{b+2c+e+f}, t_{18} = \sqrt{e+f+a+b}, t_{19} = \sqrt{a+b+2c+e+f}, t_{20} = \sqrt{a+b+2c+e+2f}, t_{21} = \sqrt{a+b+e+d+f}, t_{22} = \sqrt{a+2b+2c+e+2f}, t_{23} = \sqrt{c+e+f+a+b}, t_{24} = \sqrt{a+2b+2c+d+e+2f}\}$$

We apply the moment map to zt, convert back to parameters (a,...,f) , simplify and collect terms...

```
> mom(zt):
  subs(Sub,%):
  simplify(%):
  collect(%,[X[1,1],X[2,2],X[3,3],Y[1,1],Y[2,2],Y[3,3]]);
```

$$(c+e+f+a+b)\,X_{1,1} + (c+f+b)\,X_{2,2} + c\,X_{3,3} + (e+2f+a+b+d+c)\,Y_{1,1} + (f+d+e+c+b)\,Y_{2,2} + (c+e+f)\,Y_{3,3}$$

So this is a "diagonal functional" and the weight for aA1+bA2+cA3+dA4+eA5+fA6 is:

```
> evalm(a*A1+b*A2+c*A3+d*A4+e*A5+f*A6);
```

$$\begin{bmatrix} c+e+f+a+b & c+f+b & c \\ e+2f+a+b+d+c & f+d+e+c+b & c+e+f \end{bmatrix}$$

This completes the justification that z:=spt(a,b,c,d,e,f) is indeed a (generalized) spherical point for the weight aA1+bA2+cA3+dA4+eA5+fA6.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

Next we evaluate the highest weight vectors h1,...,h6 at our general spherical point sp. For this we work in terms of the t's and simplify.

```
> h1(zt):h1n:=subs(Sub,%);
```

$$h1n := \left(\sqrt{a}\,\sqrt{a+b+f}\,\sqrt{a+e+f}\,\sqrt{a+b+2c+e+2f}\,\sqrt{a+b+e+d+f}\,\sqrt{c+e+f+a+b}\,\sqrt{a+2b+2c+d+e+2f}\right) \Big/ \left(\sqrt{a+e}\,\sqrt{a+f}\,\sqrt{a+b+d+f}\,\sqrt{e+f+a+b}\,\sqrt{a+b+2c+e+f}\,\sqrt{a+2b+2c+e+2f}\right)$$

```
> h2(zt):factor(%):subs(Sub,%):h2n:=factor(%);
```

$$h2n := \left(\sqrt{b+2c+e+f}\,\sqrt{c+f+b}\,\sqrt{b+d+f}\,\sqrt{b}\,(a+2b+2c+d+e+2f)\,\sqrt{c+e+f+a+b}\,\sqrt{a+b+e+d+f}\,\sqrt{a+b+2c+e+2f}\,\sqrt{a+b+f}\right) \Big/ \left(\sqrt{2c+f+b}\,\sqrt{b+f}\,\sqrt{b+d}\,(a+2b+2c+e+2f)\,\sqrt{a+b+2c+e+f}\,\sqrt{e+f+a+b}\,\sqrt{a+b+d+f}\right)$$

```
> h3(zt):factor(%):subs(Sub,%):h3n:=factor(%);
```

$$h3n := \left(\sqrt{c}\,(b+2c+e+f)\,\sqrt{c+f+b}\,(a+2b+2c+d+e+2f)\,\sqrt{c+e+f+a+b}\,(a+b+2c+e+2f)\right) \Big/ \left((2c+f+b)\,(a+b+2c+e+f)\,(a+2b+2c+e+2f)\right)$$

```
> h4(zt):factor(%):subs(Sub,%):h4n:=factor(%);
```

$$h4n := \frac{\sqrt{b+d+f}\,\sqrt{d}\,\sqrt{a+2b+2c+d+e+2f}\,\sqrt{a+b+e+d+f}}{\sqrt{b+d}\,\sqrt{a+b+d+f}}$$

```
> factor(h5(zt)):subs(Sub,%):h5n:=factor(%);
```

$$h5n := \left(\sqrt{e}\,\sqrt{b+2c+e+f}\,\sqrt{a+2b+2c+d+e+2f}\,\sqrt{c+e+f+a+b}\,\sqrt{a+b+e+d+f}\,\sqrt{a+b+2c+e+2f}\,\sqrt{a+e+f}\right) \Big/ \left(\sqrt{a+e}\,\sqrt{e+f+a+b}\,\sqrt{a+b+2c+e+f}\,\sqrt{a+2b+2c+e+2f}\right)$$

```
> factor(h6(zt)):subs(Sub,%):h6n:=factor(%);
```

$$h6n := -\left(\sqrt{b+d+f}\,\sqrt{b+2c+e+f}\,\sqrt{c+f+b}\,\sqrt{f}\,(a+2b+2c+d+e+2f)\,\sqrt{c+e+f+a+b}\,\sqrt{a+b+e+d+f}\,(a+b+2c+e+2f)\,\sqrt{a+e+f}\,\sqrt{a+b+f}\right) \Big/$$

$$\left(\sqrt{b+f}\,\sqrt{2\,c+f+b}\,\sqrt{a+f}\,\sqrt{e+f+a+b}\,\sqrt{a+b+2\,c+e+f}\,\sqrt{a+b+d+f}\right.$$
$$\left.(a+2\,b+2\,c+e+2\,f)\right)$$

These are the formulas given in section 4.8 of our paper. They show that for positive real parameters (a,b, c,d,e,f) each fundamental highest weight vector $h_j(z)$ takes a non-zero value at z=sp=spt(a,b,c,d,e,f). It follows that sp lies in the open Borel orbit in V and Corollary 3.4 in our paper implies that (for positive integer values of the parameters a,...,f) the spherical point sp is well-adapted to the highest weight vector $h_1^{\,a}\, h_2^{\,b}\, h_3^{\,c}\, h_4^{\,d}\, h_5^{\,e}\, h_6^{\,f}$. But we can also demonstrate this via direct computation as follows...

The well-adapted property uses directional derivatives of the HWV's evaluated at the spherical points. We simplify using the t-substitution. The i,j'th entry of dk is the i,j'th derivative of h_k evaluated at the spherical point sp.

```
> u:=matrix(6,3):w:=matrix(6,3):
> d1:=matrix(6,3,(i,j)->coeff(coeff(h1(evalm(zt+p*w)),p),w[i,j])):
> matrix(6,3,(i,j)->coeff(coeff(h2(evalm(zt+p*w)),p),w[i,j])):
  d2:=map(factor,subs(Sub,%)):
> matrix(6,3,(i,j)->coeff(coeff(h3(evalm(zt+p*w)),p),w[i,j])):
  d3:=subs(Sub,%):
> matrix(6,3,(i,j)->coeff(coeff(h4(evalm(zt+p*w)),p),w[i,j])):
  d4:=map(factor,subs(Sub,%)):
> matrix(6,3,(i,j)->coeff(coeff(h5(evalm(zt+p*w)),p),w[i,j])):map
  (factor,%):
  d5:=map(factor,subs(Sub,%)):
> matrix(6,3,(i,j)->coeff(coeff(h6(evalm(zt+p*w)),p),w[i,j])):map
  (factor,%):
  d6:=subs(Sub,%):
```

The well-adapted condition says that the sum of the following six terms is the spherical point sp.

```
> term1:=map(factor,evalm((a)/h1n*d1)):
> term2:=map(factor,evalm(b/h2n*d2)):
> term3:=map(factor,evalm(c/h3n*d3)):
> term4:=map(simplify,evalm(d/h4n*d4)):
> term5:=map(factor,evalm(e/h5n*d5)):
> term6:=map(factor,evalm(f/h6n*d6)):
> ans:=map(factor,evalm(term1+term2+term3+term4+term5+term6));
```

$$ans := \Bigg[$$

$$\Bigg[\left(\sqrt{a}\,\sqrt{a+b+f}\,\sqrt{a+e+f}\,\sqrt{a+b+2\,c+e+2\,f}\,\sqrt{a+b+e+d+f}\right.$$
$$\left.\sqrt{c+e+f+a+b}\,\sqrt{a+2\,b+2\,c+d+e+2\,f}\right)\Big/$$

$$\left(\sqrt{a+e}\,\sqrt{a+f}\,\sqrt{a+b+d+f}\,\sqrt{e+f+a+b}\,\sqrt{a+b+2\,c+e+f}\right.$$

$$\sqrt{a+2b+2c+e+2f}\Big),$$

$$-\Big(\sqrt{b}\,\sqrt{e}\,\sqrt{f}\,\sqrt{b+2c+e+f}\,\sqrt{a+b+e+d+f}\,\sqrt{c+e+f+a+b}$$

$$\sqrt{a+2b+2c+d+e+2f}\Big)\Big/$$

$$\Big(\sqrt{a+e}\,\sqrt{a+f}\,\sqrt{b+d}\,\sqrt{e+f+a+b}\,\sqrt{a+b+2c+e+f}$$

$$\sqrt{a+2b+2c+e+2f}\Big),$$

$$\Big(\sqrt{d}\,\sqrt{b+d+f}\,\sqrt{e}\,\sqrt{b+2c+e+f}\,\sqrt{a+e+f}\,\sqrt{a+b+2c+e+2f}$$
$$\sqrt{c+e+f+a+b}\Big)\Big/$$
$$\Big(\sqrt{a+e}\,\sqrt{e+f+a+b}\,\sqrt{a+b+2c+e+f}\,\sqrt{a+2b+2c+e+2f}\,\sqrt{b+d}$$
$$\sqrt{a+b+d+f}\Big)\Big],$$

$$\Big[\Big(\sqrt{e}\,\sqrt{f}\,\sqrt{a+b+f}\,\sqrt{b+d+f}\,\sqrt{c+f+b}\,\sqrt{a+b+2c+e+2f}$$

$$\sqrt{a+2b+2c+d+e+2f}\Big)\Big/$$

$$\Big(\sqrt{a+e}\,\sqrt{a+f}\,\sqrt{b+f}\,\sqrt{2c+f+b}\,\sqrt{a+b+d+f}$$

$$\sqrt{a+2b+2c+e+2f}\Big),$$

$$\Big(\sqrt{a}\,\sqrt{b}\,\sqrt{a+e+f}\,\sqrt{b+d+f}\,\sqrt{c+f+b}\,\sqrt{b+2c+e+f}$$

$$\sqrt{a+2b+2c+d+e+2f}\Big)\Big/$$

$$\Big(\sqrt{a+e}\,\sqrt{a+f}\,\sqrt{b+d}\,\sqrt{b+f}\,\sqrt{2c+f+b}\,\sqrt{a+2b+2c+e+2f}\Big),$$

$$-\Big(\sqrt{c+f+b}\,\sqrt{b+2c+e+f}\,\sqrt{a+b+2c+e+2f}\,\sqrt{d}\,\sqrt{a}\,\sqrt{f}$$
$$\sqrt{a+b+e+d+f}\Big)\Big/$$

$$\left(\sqrt{a+e}\,\sqrt{b+d}\,\sqrt{a+2b+2c+e+2f}\,\sqrt{b+f}\,\sqrt{2c+f+b}\,\sqrt{a+b+d+f}\right)\Big],$$

$$\Bigg[$$

$$-\left(\sqrt{a}\,\sqrt{b}\,\sqrt{c}\,\sqrt{d}\,\sqrt{e}\,\sqrt{a+b+2c+e+2f}\,\sqrt{a+2b+2c+d+e+2f}\right)\Big/$$

$$\left(\sqrt{a+f}\,\sqrt{b+f}\,\sqrt{2c+f+b}\,\sqrt{a+b+d+f}\,\sqrt{e+f+a+b}\right.$$

$$\left.\sqrt{a+b+2c+e+f}\right),$$

$$\left(\sqrt{c}\,\sqrt{d}\,\sqrt{f}\,\sqrt{a+b+f}\,\sqrt{a+e+f}\,\sqrt{b+2c+e+f}\right.$$

$$\left.\sqrt{a+2b+2c+d+e+2f}\right)\Big/$$

$$\left(\sqrt{a+f}\,\sqrt{b+d}\,\sqrt{b+f}\,\sqrt{2c+f+b}\,\sqrt{e+f+a+b}\,\sqrt{a+b+2c+e+f}\right),$$

$$\left(\sqrt{b+d+f}\,\sqrt{b}\,\sqrt{a+b+e+d+f}\,\sqrt{a+b+f}\,\sqrt{c}\,\sqrt{b+2c+e+f}\right.$$

$$\left.\sqrt{a+b+2c+e+2f}\right)\Big/$$

$$\left(\sqrt{e+f+a+b}\,\sqrt{a+b+d+f}\,\sqrt{b+f}\,\sqrt{b+d}\,\sqrt{2c+f+b}\right.$$

$$\left.\sqrt{a+b+2c+e+f}\right)\Big],$$

$$\Big[\left(\sqrt{b}\,\sqrt{d}\,\sqrt{e}\,\sqrt{f}\,\sqrt{b+d+f}\,\sqrt{b+2c+e+f}\,\sqrt{c+e+f+a+b}\right)\Big/$$

$$\left(\sqrt{a+e}\,\sqrt{a+f}\,\sqrt{a+b+d+f}\,\sqrt{e+f+a+b}\,\sqrt{a+b+2c+e+f}\right.$$

$$\left.\sqrt{a+2b+2c+e+2f}\right),$$

$$\left(\sqrt{a}\,\sqrt{d}\,\sqrt{a+b+f}\,\sqrt{a+e+f}\,\sqrt{b+d+f}\,\sqrt{a+b+2c+e+2f}\right.$$

$$\left.\sqrt{c+e+f+a+b}\right)\Big/$$

$$\left(\sqrt{a+e}\,\sqrt{a+f}\,\sqrt{b+d}\,\sqrt{e+f+a+b}\,\sqrt{a+b+2c+e+f}\right.$$

$$\left.\sqrt{a+2b+2c+e+2f}\right),$$

$$\left(\sqrt{b}\,\sqrt{a}\,\sqrt{a+b+f}\,\sqrt{a+b+e+d+f}\,\sqrt{c+e+f+a+b}\,\sqrt{f}\right.$$
$$\left.\sqrt{a+2b+2c+d+e+2f}\right)\Big/$$
$$\left(\sqrt{a+e}\,\sqrt{b+d}\,\sqrt{a+2b+2c+e+2f}\,\sqrt{a+b+d+f}\,\sqrt{a+b+2c+e+f}\right.$$
$$\left.\sqrt{e+f+a+b}\right)\Big],$$

$$\Big[-\frac{\sqrt{a}\,\sqrt{b}\,\sqrt{d}\,\sqrt{a+e+f}\,\sqrt{c+f+b}\,\sqrt{b+2c+e+f}\,\sqrt{a+b+e+d+f}}{\sqrt{a+e}\,\sqrt{a+f}\,\sqrt{b+f}\,\sqrt{2c+f+b}\,\sqrt{a+b+d+f}\,\sqrt{a+2b+2c+e+2f}},$$

$$\frac{\sqrt{d}\,\sqrt{e}\,\sqrt{f}\,\sqrt{a+b+f}\,\sqrt{c+f+b}\,\sqrt{a+b+2c+e+2f}\,\sqrt{a+b+e+d+f}}{\sqrt{a+e}\,\sqrt{a+f}\,\sqrt{b+d}\,\sqrt{b+f}\,\sqrt{2c+f+b}\,\sqrt{a+2b+2c+e+2f}},$$

$$\left(\sqrt{e}\,\sqrt{a+2b+2c+d+e+2f}\,\sqrt{c+f+b}\,\sqrt{b+d+f}\,\sqrt{b}\,\sqrt{a+b+f}\right.$$
$$\left.\sqrt{a+e+f}\right)\Big/$$
$$\left(\sqrt{a+e}\,\sqrt{b+d}\,\sqrt{a+2b+2c+e+2f}\,\sqrt{b+f}\,\sqrt{2c+f+b}\,\sqrt{a+b+d+f}\right)\Big],$$

$$\Big[$$

$$-\left(\sqrt{c}\,\sqrt{f}\,\sqrt{a+b+f}\,\sqrt{a+e+f}\,\sqrt{b+d+f}\,\sqrt{b+2c+e+f}\right.$$

$$\left.\sqrt{a+b+e+d+f}\right)\Big/$$

$$\left(\sqrt{a+f}\,\sqrt{b+f}\,\sqrt{2c+f+b}\,\sqrt{a+b+d+f}\,\sqrt{e+f+a+b}\right.$$

$$\left.\sqrt{a+b+2c+e+f}\right),$$

$$-\frac{\sqrt{a}\,\sqrt{b}\,\sqrt{c}\,\sqrt{e}\,\sqrt{b+d+f}\,\sqrt{a+b+2c+e+2f}\,\sqrt{a+b+e+d+f}}{\sqrt{a+f}\,\sqrt{b+d}\,\sqrt{b+f}\,\sqrt{2c+f+b}\,\sqrt{e+f+a+b}\,\sqrt{a+b+2c+e+f}},$$

$$\frac{\sqrt{a+e+f}\,\sqrt{f}\,\sqrt{d}\,\sqrt{c}\,\sqrt{a}\,\sqrt{a+2b+2c+d+e+2f}\,\sqrt{e}}{\sqrt{b+d}\,\sqrt{a+b+d+f}\,\sqrt{a+b+2c+e+f}\,\sqrt{e+f+a+b}\,\sqrt{b+f}\,\sqrt{2c+f+b}}$$

]]

The following code checks that this **ans** does equal the spherical point **sp** by comparing the squares and signs of each entry in **ans** and **sp**.

```
> for i from 1 to 6 do;
      for j from 1 to 3 do;
          simplify(ans[i,j]^2/sp[i,j]^2);
          print(%);
      end do;
  end do;
```

$$1$$
$$1$$
$$1$$
$$1$$
$$1$$
$$1$$
$$1$$
$$1$$
$$1$$
$$1$$
$$1$$
$$1$$
$$1$$
$$1$$
$$1$$
$$1$$
$$1$$
$$1$$

```
> evalm(map(sign,sp)-map(sign,ans));
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

%%%

Our results hold for all positive reals parameters a...f. To complete the verification that our action is well-behaved we must also consider non-generic spherical points for which one or more coefficient a...f is zero. Here we apply Lemma 3.5 from our paper. First we check condition (3) in the Lemma: limits of (generalized) spherical points exist if we take some variables to zero. Since the moment map is continuous, these limits are also (generalized) spherical points....

First, we generate all possible ways the variables can go to zero. We eliminate the empty set. There are 63 non-empty subsets of {a=0, b=0, c=0, d=0, e=0, f=0}.

```
> with(combinat):
> zs:=[a=0,b=0,c=0,d=0,e=0,f=0];
  ch:=[seq(choose(%)[i],i=2..64)]:
  :nops(%);
```

$$zs := [a=0, b=0, c=0, d=0, e=0, f=0]$$

$$63$$

The following code lists each possible setting of parameters a...f to zero and prints the limiting (generalized) spherical point. This verifies condition (3) in Lemma 3.5. In particular no singularities arise as we perfom these limits via setting parameters to zero in succession.

```
> for i from 1 to nops(ch) do:
       print(ch[i]);
       zf:=evalm(sp):
       for j from 1 to nops(ch[i]) do:
            zf:=subs(ch[i][j],evalm(zf)):
       end do:
       print(zf);
  end do:
```

$$[a=0]$$

$$\left[\left[0, -\sqrt{\frac{b\,(2b+2c+d+e+2f)\,(b+e+d+f)\,(c+e+f+b)}{(2b+2c+e+2f)\,(e+f+b)\,(b+d)}},\right.\right.$$
$$\left.\sqrt{\frac{d\,(b+2c+e+2f)\,(c+e+f+b)\,(e+f)}{(2b+2c+e+2f)\,(b+d)\,(e+f+b)}}\right],$$
$$\left[\sqrt{\frac{(2b+2c+d+e+2f)\,(b+2c+e+2f)\,(c+f+b)}{(2b+2c+e+2f)\,(2c+f+b)}}, 0, 0\right],$$
$$\left[0, \sqrt{\frac{(2b+2c+d+e+2f)\,c\,(e+f)\,d}{(2c+f+b)\,(e+f+b)\,(b+d)}},\right.$$
$$\left.\sqrt{\frac{(b+2c+e+2f)\,(b+e+d+f)\,c\,b}{(2c+f+b)\,(e+f+b)\,(b+d)}}\right],$$
$$\left[\sqrt{\frac{b\,d\,(c+e+f+b)}{(2b+2c+e+2f)\,(e+f+b)}}, 0, 0\right],$$
$$\left[0, \sqrt{\frac{d\,(b+2c+e+2f)\,(c+f+b)\,(b+e+d+f)}{(b+d)\,(2b+2c+e+2f)\,(2c+f+b)}},\right.$$

$$\sqrt{\frac{b\,(e+f)\,(c+f+b)\,(2\,b+2\,c+d+e+2\,f)}{(b+d)\,(2\,c+f+b)\,(2\,b+2\,c+e+2\,f)}}\Bigg],$$

$$\left[-\sqrt{\frac{(b+e+d+f)\,c\,(e+f)}{(2\,c+f+b)\,(e+f+b)}},0,0\right]\Bigg]$$

$$[b=0]$$

$$\Bigg[\Bigg[\sqrt{\frac{a\,(a+e+d+f)\,(a+2\,c+d+e+2\,f)\,(c+e+f+a)}{(a+e)\,(a+2\,c+e+f)\,(a+d+f)}},0,$$

$$\sqrt{\frac{e\,(d+f)\,(c+e+f+a)\,(2\,c+e+f)}{(a+e)\,(a+2\,c+e+f)\,(a+d+f)}}\Bigg],$$

$$\Bigg[\sqrt{\frac{e\,(a+2\,c+d+e+2\,f)\,(d+f)\,(c+f)}{(a+e)\,(a+d+f)\,(2\,c+f)}},0,$$

$$-\sqrt{\frac{a\,(2\,c+e+f)\,(a+e+d+f)\,(c+f)}{(a+e)\,(a+d+f)\,(2\,c+f)}}\Bigg],$$

$$\left[0,\sqrt{\frac{(a+2\,c+d+e+2\,f)\,(2\,c+e+f)\,c}{(a+2\,c+e+f)\,(2\,c+f)}},0\right],$$

$$\left[0,\sqrt{\frac{(c+e+f+a)\,a\,(d+f)}{(a+2\,c+e+f)\,(a+e)}},0\right],$$

$$\left[0,\sqrt{\frac{e\,(c+f)\,(a+e+d+f)}{(a+e)\,(2\,c+f)}},0\right],$$

$$\Bigg[-\sqrt{\frac{(2\,c+e+f)\,(a+e+d+f)\,c\,(d+f)}{(a+2\,c+e+f)\,(2\,c+f)\,(a+d+f)}},0,$$

$$\sqrt{\frac{a\,(a+2\,c+d+e+2\,f)\,c\,e}{(a+2\,c+e+f)\,(2\,c+f)\,(a+d+f)}}\Bigg]\Bigg]$$

$$[c=0]$$

$$\Bigg[\Bigg[\big((a\,(a+e+f)\,(a+b+f)\,(a+b+e+d+f)\,(a+b+e+2\,f)\,(a+2\,b+d+e$$

$$+2f))\big/((e+f+a+b)\,(a+e)\,(a+f)\,(a+b+d+f)\,(a+2\,b+e+2\,f))\big)^{1/2},$$

$$-\sqrt{\frac{b\,e\,f\,(a+2\,b+d+e+2\,f)\,(e+f+b)\,(a+b+e+d+f)}{(e+f+a+b)\,(a+2\,b+e+2\,f)\,(a+e)\,(a+f)\,(b+d)}},$$

$$\sqrt{\frac{e\,d\,(a+b+e+2\,f)\,(b+d+f)\,(e+f+b)\,(a+e+f)}{(e+f+a+b)\,(a+2\,b+e+2\,f)\,(a+e)\,(b+d)\,(a+b+d+f)}}\Bigg],$$

$$\left[\sqrt{\frac{ef(a+2b+d+e+2f)(a+b+e+2f)(a+b+f)(b+d+f)}{(b+f)(a+2b+e+2f)(a+e)(a+f)(a+b+d+f)}},\right.$$

$$\sqrt{\frac{(a+2b+d+e+2f)\,a\,b\,(b+d+f)(e+f+b)(a+e+f)}{(b+f)(a+2b+e+2f)(a+e)(b+d)(a+f)}},$$

$$\left.-\sqrt{\frac{a\,d\,f(a+b+e+2f)(e+f+b)(a+b+e+d+f)}{(b+f)(a+2b+e+2f)(b+d)(a+e)(a+b+d+f)}}\right],$$

$$[0, 0, 0],$$

$$\left[\sqrt{\frac{e\,f\,b\,d\,(e+f+b)(b+d+f)}{(e+f+a+b)(a+2b+e+2f)(a+e)(a+f)(a+b+d+f)}},\right.$$

$$\sqrt{\frac{(a+b+e+2f)\,a\,d\,(a+b+f)(a+e+f)(b+d+f)}{(e+f+a+b)(a+2b+e+2f)(a+e)(b+d)(a+f)}},$$

$$\left.\sqrt{\frac{a\,b\,f(a+2b+d+e+2f)(a+b+e+d+f)(a+b+f)}{(e+f+a+b)(a+2b+e+2f)(a+e)(b+d)(a+b+d+f)}}\right],$$

$$\left[-\sqrt{\frac{a\,b\,d\,(e+f+b)(a+b+e+d+f)(a+e+f)}{(b+f)(a+2b+e+2f)(a+e)(a+f)(a+b+d+f)}},\right.$$

$$\sqrt{\frac{e\,d\,f(a+b+e+2f)(a+b+e+d+f)(a+b+f)}{(b+f)(a+e)(a+f)(b+d)(a+2b+e+2f)}},$$

$$\left.\sqrt{\frac{b\,e\,(a+b+f)(a+e+f)(b+d+f)(a+2b+d+e+2f)}{(b+f)(a+e)(b+d)(a+b+d+f)(a+2b+e+2f)}}\right],$$

$$[0, 0, 0]]$$

$$[d = 0]$$

$$\left[\left[\sqrt{\frac{a\,(a+e+f)(a+b+2c+e+2f)(c+e+f+a+b)}{(a+e)(a+f)(a+b+2c+e+f)}},\right.\right.$$

$$\left.-\sqrt{\frac{ef(b+2c+e+f)(c+e+f+a+b)}{(a+b+2c+e+f)(a+e)(a+f)}}, 0\right],$$

$$\left[\sqrt{\frac{ef(a+b+2c+e+2f)(c+f+b)}{(a+e)(a+f)(2c+f+b)}},\right.$$

$$\left.\sqrt{\frac{a\,(b+2c+e+f)(a+e+f)(c+f+b)}{(a+e)(a+f)(2c+f+b)}}, 0\right],$$

$$\left[0, 0, \sqrt{\frac{(a+b+2c+e+2f)(b+2c+e+f)\,c}{(a+b+2c+e+f)(2c+f+b)}}\right],$$

$$\left[0, 0, \sqrt{\frac{a f (c+e+f+a+b)}{(a+b+2c+e+f)(a+e)}}\right],$$

$$\left[0, 0, \sqrt{\frac{e (a+e+f)(c+f+b)}{(a+e)(2c+f+b)}}\right],$$

$$\left[-\sqrt{\frac{(b+2c+e+f) c f (a+e+f)}{(a+b+2c+e+f)(2c+f+b)(a+f)}},\right.$$

$$\left.-\sqrt{\frac{(a+b+2c+e+2f) c e a}{(a+b+2c+e+f)(2c+f+b)(a+f)}}, 0\right]\Bigg]$$

$$[e=0]$$

$$\left[\left[\sqrt{\frac{(a+b+2c+2f)(a+2b+2c+d+2f)(c+f+a+b)}{(a+b+2c+f)(a+2b+2c+2f)}}, 0, 0\right],\right.$$

$$\left[0, \sqrt{\frac{(a+2b+2c+d+2f) b (b+d+f)(c+f+b)}{(a+2b+2c+2f)(b+d)(b+f)}},\right.$$

$$\left.-\sqrt{\frac{d f (a+b+2c+2f)(c+f+b)}{(a+2b+2c+2f)(b+d)(b+f)}}\right],$$

$$\left[0, \sqrt{\frac{(a+2b+2c+d+2f) c f d}{(b+f)(a+b+2c+f)(b+d)}}, \sqrt{\frac{(a+b+2c+2f) c (b+d+f) b}{(a+b+2c+f)(b+f)(b+d)}}\right.$$

$$\Bigg],$$

$$\left[0, \sqrt{\frac{(a+b+2c+2f)(c+f+a+b) d (b+d+f)}{(a+b+2c+f)(a+2b+2c+2f)(b+d)}},\right.$$

$$\left.\sqrt{\frac{b f (a+2b+2c+d+2f)(c+f+a+b)}{(a+b+2c+f)(a+2b+2c+2f)(b+d)}}\right],$$

$$\left[-\sqrt{\frac{b d (c+f+b)}{(a+2b+2c+2f)(b+f)}}, 0, 0\right],$$

$$\left.\left[-\sqrt{\frac{c f (b+d+f)}{(a+b+2c+f)(b+f)}}, 0, 0\right]\right]$$

$$[f=0]$$

$$\left[\left[\sqrt{\frac{(a+b)(a+b+e+d)(a+2b+2c+d+e)(c+e+a+b)}{(a+b+d)(a+2b+2c+e)(e+a+b)}}, 0,\right.\right.$$

$$\left.\sqrt{\frac{e d (c+e+a+b)(b+2c+e)}{(a+2b+2c+e)(a+b+d)(e+a+b)}}\right],$$

$$\left[0, \sqrt{\frac{(a+2b+2c+d+e)\,(b+2c+e)\,(c+b)}{(a+2b+2c+e)\,(2c+b)}}, 0\right],$$

$$\left[-\sqrt{\frac{c\,(a+2b+2c+d+e)\,e\,d}{(2c+b)\,(e+a+b)\,(a+b+d)}}, 0,\right.$$

$$\left.\sqrt{\frac{(b+2c+e)\,(a+b+e+d)\,c\,(a+b)}{(2c+b)\,(e+a+b)\,(a+b+d)}}\right],$$

$$\left[0, \sqrt{\frac{(c+e+a+b)\,d\,(a+b)}{(a+2b+2c+e)\,(e+a+b)}}, 0\right],$$

$$\left[-\sqrt{\frac{d\,(b+2c+e)\,(c+b)\,(a+b+e+d)}{(a+2b+2c+e)\,(2c+b)\,(a+b+d)}}, 0,\right.$$

$$\left.\sqrt{\frac{e\,(a+b)\,(c+b)\,(a+2b+2c+d+e)}{(a+b+d)\,(2c+b)\,(a+2b+2c+e)}}\right],$$

$$\left.\left[0, -\sqrt{\frac{(a+b+e+d)\,c\,e}{(e+a+b)\,(2c+b)}}, 0\right]\right]$$

$$[a=0, b=0]$$

$$\begin{bmatrix} 0 & 0 & \sqrt{c+e+f} \\ \sqrt{\frac{(2c+d+e+2f)\,(c+f)}{2c+f}} & 0 & 0 \\ 0 & \sqrt{\frac{(2c+d+e+2f)\,c}{2c+f}} & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{\frac{(c+f)\,(e+d+f)}{2c+f}} & 0 \\ -\sqrt{\frac{(e+d+f)\,c}{2c+f}} & 0 & 0 \end{bmatrix}$$

$$[a=0, c=0]$$

$$\left[\left[0, -\sqrt{\frac{b\,(2b+d+e+2f)\,(b+e+d+f)}{(2b+e+2f)\,(b+d)}}, \sqrt{\frac{d\,(b+e+2f)\,(e+f)}{(2b+e+2f)\,(b+d)}}\right],\right.$$

$$\left[\sqrt{\frac{(2b+d+e+2f)\,(b+e+2f)}{2b+e+2f}}, 0, 0\right],$$

$$[0, 0, 0],$$

$$\left[\sqrt{\frac{b\,d}{2\,b+e+2\,f}}, 0, 0\right],$$

$$\left[0, \sqrt{\frac{d\,(b+e+2\,f)\,(b+e+d+f)}{(b+d)\,(2\,b+e+2\,f)}}, \sqrt{\frac{b\,(e+f)\,(2\,b+d+e+2\,f)}{(b+d)\,(2\,b+e+2\,f)}}\right],$$

$$\left[0, 0, 0\right]\Big]$$

$$[a=0, d=0]$$

$$\begin{bmatrix} 0 & -\sqrt{c+e+f+b} & 0 \\ \sqrt{\frac{(b+2\,c+e+2\,f)\,(c+f+b)}{2\,c+f+b}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{(b+2\,c+e+2\,f)\,c}{2\,c+f+b}} \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{(e+f)\,(c+f+b)}{2\,c+f+b}} \\ -\sqrt{\frac{c\,(e+f)}{2\,c+f+b}} & 0 & 0 \end{bmatrix}$$

$$[a=0, e=0]$$

$$\Big[\Big[0, -\sqrt{\frac{b\,(2\,b+2\,c+d+2\,f)\,(b+d+f)\,(c+f+b)}{(2\,b+2\,c+2\,f)\,(b+f)\,(b+d)}},$$

$$\sqrt{\frac{d\,(b+2\,c+2\,f)\,(c+f+b)\,f}{(2\,b+2\,c+2\,f)\,(b+d)\,(b+f)}}\Big],$$

$$\left[\sqrt{\frac{(2\,b+2\,c+d+2\,f)\,(b+2\,c+2\,f)\,(c+f+b)}{(2\,b+2\,c+2\,f)\,(2\,c+f+b)}}, 0, 0\right],$$

$$\left[0, \sqrt{\frac{(2\,b+2\,c+d+2\,f)\,c\,f\,d}{(2\,c+f+b)\,(b+f)\,(b+d)}}, \sqrt{\frac{(b+2\,c+2\,f)\,(b+d+f)\,c\,b}{(2\,c+f+b)\,(b+f)\,(b+d)}}\right],$$

$$\left[\sqrt{\frac{b\,d\,(c+f+b)}{(2\,b+2\,c+2\,f)\,(b+f)}}, 0, 0\right],$$

$$\Big[0, \sqrt{\frac{d\,(b+2\,c+2\,f)\,(c+f+b)\,(b+d+f)}{(b+d)\,(2\,b+2\,c+2\,f)\,(2\,c+f+b)}},$$

$$\sqrt{\frac{b\,f\,(c+f+b)\,(2\,b+2\,c+d+2\,f)}{(b+d)\,(2\,c+f+b)\,(2\,b+2\,c+2\,f)}}\Big],$$

$$\left[-\sqrt{\frac{(b+d+f)\,c\,f}{(2\,c+f+b)\,(b+f)}},0,0\right]\Bigg]$$

$$[a=0, f=0]$$

$$\Bigg[\Bigg[0,-\sqrt{\frac{b\,(2\,b+2\,c+d+e)\,(b+e+d)\,(c+e+b)}{(2\,b+2\,c+e)\,(e+b)\,(b+d)}},$$

$$\sqrt{\frac{d\,(b+2\,c+e)\,(c+e+b)\,e}{(2\,b+2\,c+e)\,(b+d)\,(e+b)}}\Bigg],$$

$$\left[\sqrt{\frac{(2\,b+2\,c+d+e)\,(b+2\,c+e)\,(c+b)}{(2\,b+2\,c+e)\,(2\,c+b)}},0,0\right],$$

$$\left[0,\sqrt{\frac{(2\,b+2\,c+d+e)\,c\,e\,d}{(2\,c+b)\,(e+b)\,(b+d)}},\sqrt{\frac{(b+2\,c+e)\,(b+e+d)\,c\,b}{(2\,c+b)\,(e+b)\,(b+d)}}\right],$$

$$\left[\sqrt{\frac{b\,d\,(c+e+b)}{(2\,b+2\,c+e)\,(e+b)}},0,0\right],$$

$$\left[0,\sqrt{\frac{d\,(b+2\,c+e)\,(c+b)\,(b+e+d)}{(b+d)\,(2\,b+2\,c+e)\,(2\,c+b)}},\sqrt{\frac{b\,e\,(c+b)\,(2\,b+2\,c+d+e)}{(b+d)\,(2\,c+b)\,(2\,b+2\,c+e)}}\right],$$

$$\left[-\sqrt{\frac{(b+e+d)\,c\,e}{(2\,c+b)\,(e+b)}},0,0\right]\Bigg]$$

$$[b=0, c=0]$$

$$\Bigg[\Bigg[\sqrt{\frac{a\,(a+e+d+f)\,(a+d+e+2\,f)}{(a+e)\,(a+d+f)}},0,\sqrt{\frac{e\,(d+f)\,(e+f)}{(a+e)\,(a+d+f)}}\Bigg],$$

$$\left[\sqrt{\frac{e\,(a+d+e+2\,f)\,(d+f)}{(a+e)\,(a+d+f)}},0,-\sqrt{\frac{a\,(e+f)\,(a+e+d+f)}{(a+e)\,(a+d+f)}}\right],$$

$$[0,0,0],$$

$$\left[0,\sqrt{\frac{a\,(d+f)}{a+e}},0\right],$$

$$\left[0,\sqrt{\frac{e\,(a+e+d+f)}{a+e}},0\right],$$

$$[0,0,0]\Bigg]$$

$$[b=0, d=0]$$

$$\Bigg[\Bigg[\sqrt{\frac{a\,(a+e+f)\,(a+2\,c+e+2\,f)\,(c+e+f+a)}{(a+e)\,(a+2\,c+e+f)\,(a+f)}},0,$$

$$\sqrt{\frac{e f (c+e+f+a)\,(2c+e+f)}{(a+e)\,(a+2c+e+f)\,(a+f)}}\Bigg],$$

$$\Bigg[\sqrt{\frac{e\,(a+2c+e+2f)\,f\,(c+f)}{(a+e)\,(a+f)\,(2c+f)}},0,-\sqrt{\frac{a\,(2c+e+f)\,(a+e+f)\,(c+f)}{(a+e)\,(a+f)\,(2c+f)}}\Bigg],$$

$$\Bigg[0,\sqrt{\frac{(a+2c+e+2f)\,(2c+e+f)\,c}{(a+2c+e+f)\,(2c+f)}},0\Bigg],$$

$$\Bigg[0,\sqrt{\frac{(c+e+f+a)\,a f}{(a+2c+e+f)\,(a+e)}},0\Bigg],$$

$$\Bigg[0,\sqrt{\frac{e\,(c+f)\,(a+e+f)}{(a+e)\,(2c+f)}},0\Bigg],$$

$$\Bigg[-\sqrt{\frac{(2c+e+f)\,(a+e+f)\,c f}{(a+2c+e+f)\,(2c+f)\,(a+f)}},0,$$

$$\sqrt{\frac{a\,(a+2c+e+2f)\,c e}{(a+2c+e+f)\,(2c+f)\,(a+f)}}\Bigg]\Bigg]$$

$$[b=0,e=0]$$

$$\begin{bmatrix} \sqrt{\frac{(a+2c+d+2f)\,(c+f+a)}{a+2c+f}} & 0 & 0 \\ 0 & 0 & -\sqrt{c+f} \\ 0 & \sqrt{\frac{(a+2c+d+2f)\,c}{a+2c+f}} & 0 \\ 0 & \sqrt{\frac{(c+f+a)\,(d+f)}{a+2c+f}} & 0 \\ 0 & 0 & 0 \\ -\sqrt{\frac{c\,(d+f)}{a+2c+f}} & 0 & 0 \end{bmatrix}$$

$$[b=0,f=0]$$

$$\Bigg[\Bigg[\sqrt{\frac{a\,(a+e+d)\,(a+2c+d+e)\,(c+e+a)}{(a+e)\,(a+2c+e)\,(a+d)}},0,$$

$$\sqrt{\frac{e d\,(c+e+a)\,(2c+e)}{(a+e)\,(a+2c+e)\,(a+d)}}\Bigg],$$

$$\Bigg[\frac{1}{2}\sqrt{2}\sqrt{\frac{e\,(a+2c+d+e)\,d}{(a+e)\,(a+d)}},0,-\frac{1}{2}\sqrt{2}\sqrt{\frac{a\,(2c+e)\,(a+e+d)}{(a+e)\,(a+d)}}\Bigg],$$

$$\left[0, \frac{1}{2}\sqrt{2}\sqrt{\frac{(a+2c+d+e)(2c+e)}{a+2c+e}}, 0\right],$$

$$\left[0, \sqrt{\frac{(c+e+a)\,a\,d}{(a+2c+e)(a+e)}}, 0\right],$$

$$\left[0, \frac{1}{2}\sqrt{2}\sqrt{\frac{e(a+e+d)}{a+e}}, 0\right],$$

$$\left[-\frac{1}{2}\sqrt{2}\sqrt{\frac{(2c+e)(a+e+d)\,d}{(a+2c+e)(a+d)}}, 0, \frac{1}{2}\sqrt{2}\sqrt{\frac{a(a+2c+d+e)\,e}{(a+2c+e)(a+d)}}\right]\right]$$

$$[c=0, d=0]$$

$$\left[\left[\sqrt{\frac{a(a+e+f)(a+b+e+2f)}{(a+e)(a+f)}}, -\sqrt{\frac{ef(e+f+b)}{(a+e)(a+f)}}, 0\right],\right.$$

$$\left[\sqrt{\frac{ef(a+b+e+2f)}{(a+e)(a+f)}}, \sqrt{\frac{a(e+f+b)(a+e+f)}{(a+e)(a+f)}}, 0\right],$$

$$[0, 0, 0],$$

$$\left[0, 0, \sqrt{\frac{af}{a+e}}\right],$$

$$\left[0, 0, \sqrt{\frac{e(a+e+f)}{a+e}}\right],$$

$$\left.[0, 0, 0]\right]$$

$$[c=0, e=0]$$

$$\left[\left[\sqrt{\frac{(a+b+2f)(a+2b+d+2f)}{a+2b+2f}}, 0, 0\right],\right.$$

$$\left[0, \sqrt{\frac{(a+2b+d+2f)\,b\,(b+d+f)}{(a+2b+2f)(b+d)}}, -\sqrt{\frac{df(a+b+2f)}{(a+2b+2f)(b+d)}}\right],$$

$$[0, 0, 0],$$

$$\left[0, \sqrt{\frac{(a+b+2f)\,d\,(b+d+f)}{(a+2b+2f)(b+d)}}, \sqrt{\frac{bf(a+2b+d+2f)}{(a+2b+2f)(b+d)}}\right],$$

$$\left[-\sqrt{\frac{b\,d}{a+2b+2f}}, 0, 0\right],$$

$$\left[0, 0, 0\right]\Big]$$

$$[c = 0, f = 0]$$

$$\left[\left[\sqrt{\frac{(a+b)\,(a+b+e+d)\,(a+2\,b+d+e)}{(a+b+d)\,(a+2\,b+e)}}, 0, \sqrt{\frac{e\,d\,(e+b)}{(a+2\,b+e)\,(a+b+d)}}\right],\right.$$

$$\left[0, \sqrt{\frac{(a+2\,b+d+e)\,(e+b)}{a+2\,b+e}}, 0\right],$$

$$\left[0, 0, 0\right],$$

$$\left[0, \sqrt{\frac{d\,(a+b)}{a+2\,b+e}}, 0\right],$$

$$\left[-\sqrt{\frac{d\,(e+b)\,(a+b+e+d)}{(a+2\,b+e)\,(a+b+d)}}, 0, \sqrt{\frac{e\,(a+b)\,(a+2\,b+d+e)}{(a+b+d)\,(a+2\,b+e)}}\right],$$

$$\left.\left[0, 0, 0\right]\right]$$

$$[d = 0, e = 0]$$

$$\begin{bmatrix} \sqrt{\frac{(a+b+2\,c+2\,f)\,(c+f+a+b)}{a+b+2\,c+f}} & 0 & 0 \\ 0 & \sqrt{c+f+b} & 0 \\ 0 & 0 & \sqrt{\frac{(a+b+2\,c+2\,f)\,c}{a+b+2\,c+f}} \\ 0 & 0 & \sqrt{\frac{f\,(c+f+a+b)}{a+b+2\,c+f}} \\ 0 & 0 & 0 \\ -\sqrt{\frac{c\,f}{a+b+2\,c+f}} & 0 & 0 \end{bmatrix}$$

$$[d = 0, f = 0]$$

$$\begin{bmatrix} \sqrt{c+e+a+b} & 0 & 0 \\ 0 & \sqrt{\frac{(b+2c+e)(c+b)}{2c+b}} & 0 \\ 0 & 0 & \sqrt{\frac{(b+2c+e)c}{2c+b}} \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{e(c+b)}{2c+b}} \\ 0 & -\sqrt{\frac{ce}{2c+b}} & 0 \end{bmatrix}$$

$$[e=0, f=0]$$

$$\begin{bmatrix} \sqrt{\frac{(a+2b+2c+d)(c+a+b)}{a+2b+2c}} & 0 & 0 \\ 0 & \sqrt{\frac{(a+2b+2c+d)(c+b)}{a+2b+2c}} & 0 \\ 0 & 0 & \sqrt{c} \\ 0 & \sqrt{\frac{(c+a+b)d}{a+2b+2c}} & 0 \\ -\sqrt{\frac{d(c+b)}{a+2b+2c}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[a=0, b=0, c=0]$$

$$\begin{bmatrix} 0 & 0 & \sqrt{e+f} \\ \sqrt{d+e+2f} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{e+d+f} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[a=0, b=0, d=0]$$

$$\left[\begin{array}{ccc} 0 & 0 & \sqrt{c+e+f} \\ \sqrt{\frac{(2c+e+2f)(c+f)}{2c+f}} & 0 & 0 \\ 0 & \sqrt{\frac{(2c+e+2f)c}{2c+f}} & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{\frac{(c+f)(e+f)}{2c+f}} & 0 \\ -\sqrt{\frac{(e+f)c}{2c+f}} & 0 & 0 \end{array}\right]$$

$$[a=0, b=0, e=0]$$

$$\left[\begin{array}{ccc} 0 & 0 & \sqrt{c+f} \\ \sqrt{\frac{(2c+d+2f)(c+f)}{2c+f}} & 0 & 0 \\ 0 & \sqrt{\frac{(2c+d+2f)c}{2c+f}} & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{\frac{(c+f)(d+f)}{2c+f}} & 0 \\ -\sqrt{\frac{(d+f)c}{2c+f}} & 0 & 0 \end{array}\right]$$

$$[a=0, b=0, f=0]$$

$$\left[\begin{array}{ccc} 0 & 0 & \sqrt{c+e} \\ \sqrt{c+\frac{1}{2}d+\frac{1}{2}e} & 0 & 0 \\ 0 & \sqrt{c+\frac{1}{2}d+\frac{1}{2}e} & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}e+\frac{1}{2}d} & 0 \\ -\sqrt{\frac{1}{2}e+\frac{1}{2}d} & 0 & 0 \end{array}\right]$$

$$[a = 0, c = 0, d = 0]$$

$$\begin{bmatrix} 0 & -\sqrt{e+f+b} & 0 \\ \sqrt{b+e+2f} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{e+f} \\ 0 & 0 & 0 \end{bmatrix}$$

$$[a = 0, c = 0, e = 0]$$

$$\left[\left[0, -\sqrt{\frac{b\,(2\,b+d+2\,f)\,(b+d+f)}{(2\,b+2\,f)\,(b+d)}}, \sqrt{\frac{d\,(b+2\,f)\,f}{(2\,b+2\,f)\,(b+d)}}\right],\right.$$

$$\left[\sqrt{\frac{(2\,b+d+2\,f)\,(b+2\,f)}{2\,b+2\,f}}, 0, 0\right],$$

$$[0, 0, 0],$$

$$\left[\sqrt{\frac{b\,d}{2\,b+2\,f}}, 0, 0\right],$$

$$\left[0, \sqrt{\frac{d\,(b+2\,f)\,(b+d+f)}{(b+d)\,(2\,b+2\,f)}}, \sqrt{\frac{b\,f\,(2\,b+d+2\,f)}{(b+d)\,(2\,b+2\,f)}}\right],$$

$$\left.[0, 0, 0]\right]$$

$$[a = 0, c = 0, f = 0]$$

$$\left[\left[0, -\sqrt{\frac{b\,(2\,b+d+e)\,(b+e+d)}{(2\,b+e)\,(b+d)}}, \sqrt{\frac{d\,(e+b)\,e}{(2\,b+e)\,(b+d)}}\right],\right.$$

$$\left[\sqrt{\frac{(2\,b+d+e)\,(e+b)}{2\,b+e}}, 0, 0\right],$$

$$[0, 0, 0],$$

$$\left[\sqrt{\frac{b\,d}{2\,b+e}}, 0, 0\right],$$

$$\left[0, \sqrt{\frac{d\,(e+b)\,(b+e+d)}{(b+d)\,(2\,b+e)}}, \sqrt{\frac{b\,e\,(2\,b+d+e)}{(b+d)\,(2\,b+e)}}\right],$$

$$\left[0, 0, 0\right]\Big]\Big]$$

$$[a = 0, d = 0, e = 0]$$

$$\begin{bmatrix} 0 & -\sqrt{c+f+b} & 0 \\ \sqrt{\frac{(b+2c+2f)(c+f+b)}{2c+f+b}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{(b+2c+2f)c}{2c+f+b}} \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{f(c+f+b)}{2c+f+b}} \\ -\sqrt{\frac{cf}{2c+f+b}} & 0 & 0 \end{bmatrix}$$

$$[a = 0, d = 0, f = 0]$$

$$\begin{bmatrix} 0 & -\sqrt{c+e+b} & 0 \\ \sqrt{\frac{(b+2c+e)(c+b)}{2c+b}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{(b+2c+e)c}{2c+b}} \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{e(c+b)}{2c+b}} \\ -\sqrt{\frac{ce}{2c+b}} & 0 & 0 \end{bmatrix}$$

$$[a = 0, e = 0, f = 0]$$

$$\begin{bmatrix} 0 & -\sqrt{\frac{(2b+2c+d)(c+b)}{2b+2c}} & 0 \\ \sqrt{\frac{(2b+2c+d)(c+b)}{2b+2c}} & 0 & 0 \\ 0 & 0 & \sqrt{c} \\ \sqrt{\frac{d(c+b)}{2b+2c}} & 0 & 0 \\ 0 & \sqrt{\frac{d(c+b)}{2b+2c}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[b=0, c=0, d=0]$$

$$\begin{bmatrix} \sqrt{\frac{a(a+e+f)(a+e+2f)}{(a+e)(a+f)}} & 0 & \sqrt{\frac{ef(e+f)}{(a+e)(a+f)}} \\ \sqrt{\frac{e(a+e+2f)f}{(a+e)(a+f)}} & 0 & -\sqrt{\frac{a(e+f)(a+e+f)}{(a+e)(a+f)}} \\ 0 & 0 & 0 \\ 0 & \sqrt{\frac{af}{a+e}} & 0 \\ 0 & \sqrt{\frac{e(a+e+f)}{a+e}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[b=0, c=0, e=0]$$

$$\begin{bmatrix} \sqrt{a+d+2f} & 0 & 0 \\ 0 & 0 & -\sqrt{f} \\ 0 & 0 & 0 \\ 0 & \sqrt{d+f} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[b=0, c=0, f=0]$$

$$\begin{bmatrix} \sqrt{\frac{a\,(a+e+d)^2}{(a+e)\,(a+d)}} & 0 & \sqrt{\frac{e^2\,d}{(a+e)\,(a+d)}} \\ \sqrt{\frac{e\,(a+e+d)\,d}{(a+e)\,(a+d)}} & 0 & -\sqrt{\frac{a\,e\,(a+e+d)}{(a+e)\,(a+d)}} \\ 0 & 0 & 0 \\ 0 & \sqrt{\frac{a\,d}{a+e}} & 0 \\ 0 & \sqrt{\frac{e\,(a+e+d)}{a+e}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[b=0, d=0, e=0]$$

$$\begin{bmatrix} \sqrt{\frac{(a+2\,c+2\,f)\,(c+f+a)}{a+2\,c+f}} & 0 & 0 \\ 0 & 0 & -\sqrt{c+f} \\ 0 & \sqrt{\frac{(a+2\,c+2\,f)\,c}{a+2\,c+f}} & 0 \\ 0 & \sqrt{\frac{(c+f+a)\,f}{a+2\,c+f}} & 0 \\ 0 & 0 & 0 \\ -\sqrt{\frac{c\,f}{a+2\,c+f}} & 0 & 0 \end{bmatrix}$$

$$[b=0, d=0, f=0]$$

$$\begin{bmatrix} \sqrt{c+e+a} & 0 & 0 \\ 0 & 0 & -\sqrt{c+\frac{1}{2}\,e} \\ 0 & \sqrt{c+\frac{1}{2}\,e} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2}\,\sqrt{e} & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{2}\,\sqrt{e} \end{bmatrix}$$

$$[b=0, e=0, f=0]$$

$$\left[\begin{array}{ccc} \sqrt{\frac{(a+2c+d)(c+a)}{a+2c}} & 0 & 0 \\ 0 & 0 & -\sqrt{c} \\ 0 & \sqrt{\frac{(a+2c+d)c}{a+2c}} & 0 \\ 0 & \sqrt{\frac{(c+a)d}{a+2c}} & 0 \\ 0 & 0 & 0 \\ -\sqrt{\frac{cd}{a+2c}} & 0 & 0 \end{array}\right]$$

$$[c=0, d=0, e=0]$$

$$\left[\begin{array}{ccc} \sqrt{a+b+2f} & 0 & 0 \\ 0 & \sqrt{b+f} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{f} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$[c=0, d=0, f=0]$$

$$\left[\begin{array}{ccc} \sqrt{e+a+b} & 0 & 0 \\ 0 & \sqrt{e+b} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{e} \\ 0 & 0 & 0 \end{array}\right]$$

$$[c=0, e=0, f=0]$$

$$\begin{bmatrix} \sqrt{\frac{(a+b)\,(a+2\,b+d)}{a+2\,b}} & 0 & 0 \\ 0 & \sqrt{\frac{(a+2\,b+d)\,b}{a+2\,b}} & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{\frac{(a+b)\,d}{a+2\,b}} & 0 \\ -\sqrt{\frac{b\,d}{a+2\,b}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[d=0,\ e=0,\ f=0]$$

$$\begin{bmatrix} \sqrt{c+a+b} & 0 & 0 \\ 0 & \sqrt{c+b} & 0 \\ 0 & 0 & \sqrt{c} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[a=0,\ b=0,\ c=0,\ d=0]$$

$$\begin{bmatrix} 0 & 0 & \sqrt{e+f} \\ \sqrt{e+2\,f} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{e+f} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[a=0,\ b=0,\ c=0,\ e=0]$$

$$\begin{bmatrix} 0 & 0 & \sqrt{f} \\ \sqrt{d+2\,f} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{d+f} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[a = 0, b = 0, c = 0, f = 0]$$

$$\begin{bmatrix} 0 & 0 & \sqrt{e} \\ \sqrt{e+d} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{e+d} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[a = 0, b = 0, d = 0, e = 0]$$

$$\begin{bmatrix} 0 & 0 & \sqrt{c+f} \\ \sqrt{\frac{(2\,c+2\,f)\,(c+f)}{2\,c+f}} & 0 & 0 \\ 0 & \sqrt{\frac{(2\,c+2\,f)\,c}{2\,c+f}} & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{\frac{(c+f)\,f}{2\,c+f}} & 0 \\ -\sqrt{\frac{f\,c}{2\,c+f}} & 0 & 0 \end{bmatrix}$$

$$[a = 0, b = 0, d = 0, f = 0]$$

$$\begin{bmatrix} 0 & 0 & \sqrt{c+e} \\ \sqrt{c+\frac{1}{2}\,e} & 0 & 0 \\ 0 & \sqrt{c+\frac{1}{2}\,e} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2}\,\sqrt{2}\,\sqrt{e} & 0 \\ -\frac{1}{2}\,\sqrt{2}\,\sqrt{e} & 0 & 0 \end{bmatrix}$$

$$[a = 0, b = 0, e = 0, f = 0]$$

$$\begin{bmatrix} 0 & 0 & \sqrt{c} \\ \sqrt{c + \frac{1}{2}\,d} & 0 & 0 \\ 0 & \sqrt{c + \frac{1}{2}\,d} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2}\,\sqrt{d} & 0 \\ -\frac{1}{2}\sqrt{2}\,\sqrt{d} & 0 & 0 \end{bmatrix}$$

$$[a = 0, c = 0, d = 0, e = 0]$$

$$\begin{bmatrix} 0 & -\sqrt{b + f} & 0 \\ \sqrt{b + 2\,f} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{f} \\ 0 & 0 & 0 \end{bmatrix}$$

$$[a = 0, c = 0, d = 0, f = 0]$$

$$\begin{bmatrix} 0 & -\sqrt{e + b} & 0 \\ \sqrt{e + b} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{e} \\ 0 & 0 & 0 \end{bmatrix}$$

$$[a = 0, c = 0, e = 0, f = 0]$$

$$\begin{bmatrix} 0 & -\sqrt{b+\frac{1}{2}d} & 0 \\ \sqrt{b+\frac{1}{2}d} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2}\sqrt{2}\sqrt{d} & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2}\sqrt{d} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[a=0, d=0, e=0, f=0]$$

$$\begin{bmatrix} 0 & -\sqrt{c+b} & 0 \\ \sqrt{c+b} & 0 & 0 \\ 0 & 0 & \sqrt{c} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[b=0, c=0, d=0, e=0]$$

$$\begin{bmatrix} \sqrt{a+2f} & 0 & 0 \\ 0 & 0 & -\sqrt{f} \\ 0 & 0 & 0 \\ 0 & \sqrt{f} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[b=0, c=0, d=0, f=0]$$

$$\begin{bmatrix} \sqrt{a+e} & 0 & 0 \\ 0 & 0 & -\sqrt{e} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{e} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[b = 0, c = 0, e = 0, f = 0]$$

$$\begin{bmatrix} \sqrt{a+d} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{d} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[b = 0, d = 0, e = 0, f = 0]$$

$$\begin{bmatrix} \sqrt{c+a} & 0 & 0 \\ 0 & 0 & -\sqrt{c} \\ 0 & \sqrt{c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[c = 0, d = 0, e = 0, f = 0]$$

$$\begin{bmatrix} \sqrt{a+b} & 0 & 0 \\ 0 & \sqrt{b} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[a = 0, b = 0, c = 0, d = 0, e = 0]$$

$$\begin{bmatrix} 0 & 0 & \sqrt{f} \\ \sqrt{2}\,\sqrt{f} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{f} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[a = 0, b = 0, c = 0, d = 0, f = 0]$$

$$\begin{bmatrix} 0 & 0 & \sqrt{e} \\ \sqrt{e} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{e} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$[a = 0, b = 0, c = 0, e = 0, f = 0]$

$$\begin{bmatrix} 0 & 0 & 0 \\ \sqrt{d} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sqrt{d} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$[a = 0, b = 0, d = 0, e = 0, f = 0]$

$$\begin{bmatrix} 0 & 0 & \sqrt{c} \\ \sqrt{c} & 0 & 0 \\ 0 & \sqrt{c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$[a = 0, c = 0, d = 0, e = 0, f = 0]$

$$\begin{bmatrix} 0 & -\sqrt{b} & 0 \\ \sqrt{b} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$[b = 0, c = 0, d = 0, e = 0, f = 0]$

$$\begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[a = 0, b = 0, c = 0, d = 0, e = 0, f = 0]$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

%%

To complete the verification that our action is well-behaved it remains to show that condition (4) in Lemma 3.5 applies. For each subset of the parameters (a,...,f) we obtained above a spherical point for the weight with those parameters set equal to zero. We must also check that the fundamental highest weight vectors associated with the complementary parameters are non-zero at this spherical point. For example at the spherical point Z for a weight of the sort aA1+0A2+cA3+0A4+0A5+fA6 (i.e. with b=d=e=0) we require that each of $h_1(Z)$, $h_3(Z)$ and $h_6(Z)$ be non-zero.

The code below generates the following output for each of the 63 non-empty subsets of {a=0, b=0, c=0, d=0, e=0, f=0}:

- A listing of the subset. These parameters are set to zero in succession to obtain a non-generic (generalized) spherical point as in the previous output.
- A list of the fundamental highest weight vectors ($h_1 ... h_6$) associated with the complementary parameters.
- A list of values for these h_j's at the spherical point.

The output shows that in all cases each fundamental highest weight vector for a complementary parameter takes a non-zero value at the limiting spherical point in question. Thus condition (4) from Lemma 3.5 does hold here, completing our analysis for this example.

```
> [1,2,3,4,5,6]: ch:=[seq(choose(%)[i],i=2..64)]:
  h:=[h1,h2,h3,h4,h5,h6]: hn:=[h1n,h2n,h3n,h4n,h5n,h6n]:
  zs:=[a=0,b=0,c=0,d=0,e=0,f=0]:
> for i from 1 to nops(ch) do:
      s:=map(x->zs[x],ch[i]):
      print(s);
      zc:=[op({1,2,3,4,5,6}minus{op(ch[i])})]:
      hs:=map(x->hn[x],zc):
      for j from 1 to nops(s) do:
```

```
        hs:=subs(s[j],hs):
      end do:
      print(map(x->h[x],zc));
      print(hs);
  end do:
```

$$[a=0]$$

$$[h2, h3, h4, h5, h6]$$

$$\left[\left(\sqrt{c+f+b}\,\sqrt{b}\,(2b+2c+d+e+2f)\,\sqrt{c+e+f+b}\,\sqrt{b+e+d+f}\,\sqrt{b+2c+e+2f}\right)\Big/\left(\sqrt{2c+f+b}\,\sqrt{b+d}\,(2b+2c+e+2f)\,\sqrt{e+f+b}\right),\right.$$

$$\frac{\sqrt{c}\,\sqrt{c+f+b}\,(2b+2c+d+e+2f)\,\sqrt{c+e+f+b}\,(b+2c+e+2f)}{(2c+f+b)\,(2b+2c+e+2f)},$$

$$\frac{\sqrt{d}\,\sqrt{2b+2c+d+e+2f}\,\sqrt{b+e+d+f}}{\sqrt{b+d}},$$

$$\frac{1}{\sqrt{e+f+b}\,\sqrt{2b+2c+e+2f}}\left(\sqrt{2b+2c+d+e+2f}\,\sqrt{c+e+f+b}\,\sqrt{b+e+d+f}\,\sqrt{b+2c+e+2f}\,\sqrt{e+f}\right), -\left(\sqrt{c+f+b}\,(2b+2c+d+e+2f)\,\sqrt{c+e+f+b}\,\sqrt{b+e+d+f}\,(b+2c+e+2f)\,\sqrt{e+f}\right)\Big/$$

$$\left.\left(\sqrt{2c+f+b}\,\sqrt{e+f+b}\,(2b+2c+e+2f)\right)\right]$$

$$[b=0]$$

$$[h1, h3, h4, h5, h6]$$

$$\left[\frac{\sqrt{a}\,\sqrt{a+e+d+f}\,\sqrt{c+e+f+a}\,\sqrt{a+2c+d+e+2f}}{\sqrt{a+e}\,\sqrt{a+d+f}\,\sqrt{a+2c+e+f}},\right.$$

$$\frac{\sqrt{c}\,(2c+e+f)\,\sqrt{c+f}\,(a+2c+d+e+2f)\,\sqrt{c+e+f+a}}{(2c+f)\,(a+2c+e+f)},$$

$$\frac{\sqrt{d+f}\,\sqrt{a+2c+d+e+2f}\,\sqrt{a+e+d+f}}{\sqrt{a+d+f}},$$

$$\frac{\sqrt{e}\,\sqrt{2c+e+f}\,\sqrt{a+2c+d+e+2f}\,\sqrt{c+e+f+a}\,\sqrt{a+e+d+f}}{\sqrt{a+e}\,\sqrt{a+2c+e+f}},$$

$$-\frac{1}{\sqrt{2c+f}\,\sqrt{a+2c+e+f}\,\sqrt{a+d+f}}\Big(\sqrt{d+f}\,\sqrt{2c+e+f}\,\sqrt{c+f}\,(a+2c+d+e+2f)\,\sqrt{c+e+f+a}\,\sqrt{a+e+d+f}\Big)\Bigg]$$

$$[a=0, b=0]$$

$$[h3, h4, h5, h6]$$

$$\Bigg[\frac{\sqrt{c}\,\sqrt{c+f}\,(2c+d+e+2f)\,\sqrt{c+e+f}}{2c+f}, \sqrt{2c+d+e+2f}\,\sqrt{e+d+f},$$

$$\sqrt{2c+d+e+2f}\,\sqrt{c+e+f}\,\sqrt{e+d+f},$$

$$-\frac{\sqrt{c+f}\,(2c+d+e+2f)\,\sqrt{c+e+f}\,\sqrt{e+d+f}}{\sqrt{2c+f}}\Bigg]$$

$$[c=0]$$

$$[h1, h2, h4, h5, h6]$$

$$\Bigg[\Big(\sqrt{a}\,\sqrt{a+b+f}\,\sqrt{a+e+f}\,\sqrt{a+b+e+2f}\,\sqrt{a+b+e+d+f}\,\sqrt{a+2b+d+e+2f}\Big)\Big/$$

$$\Big(\sqrt{e+f+a+b}\,\sqrt{a+e}\,\sqrt{a+f}\,\sqrt{a+b+d+f}\,\sqrt{a+2b+e+2f}\Big),$$

$$\Big(\sqrt{e+f+b}\,\sqrt{b+d+f}\,\sqrt{b}\,(a+2b+d+e+2f)\,\sqrt{a+b+e+d+f}\,\sqrt{a+b+e+2f}\,\sqrt{a+b+f}\Big)\Big/$$

$$\Big(\sqrt{b+f}\,\sqrt{e+f+a+b}\,\sqrt{b+d}\,(a+2b+e+2f)\,\sqrt{a+b+d+f}\Big),$$

$$\frac{\sqrt{b+d+f}\,\sqrt{d}\,\sqrt{a+2b+d+e+2f}\,\sqrt{a+b+e+d+f}}{\sqrt{b+d}\,\sqrt{a+b+d+f}},$$

$$\frac{1}{\sqrt{a+e}\,\sqrt{e+f+a+b}\,\sqrt{a+2b+e+2f}}\Big(\sqrt{e}\,\sqrt{e+f+b}\,\sqrt{a+2b+d+e+2f}$$

$\sqrt{a+b+e+d+f}\,\sqrt{a+b+e+2f}\,\sqrt{a+e+f}\Big),$

$-\Big(\sqrt{b+d+f}\,\sqrt{e+f+b}\,\sqrt{f}\,(a+2b+d+e+2f)\,\sqrt{a+b+e+d+f}\,(a+b+e+2f)\,\sqrt{a+e+f}\,\sqrt{a+b+f}\Big)\Big/\Big(\sqrt{b+f}\,\sqrt{a+f}\,\sqrt{e+f+a+b}\,\sqrt{a+b+d+f}\,(a+2b+e+2f)\Big)\Big]$

$[a=0, c=0]$

$[h2, h4, h5, h6]$

$$\left[\frac{\sqrt{b}\,(2b+d+e+2f)\,\sqrt{b+e+d+f}\,\sqrt{b+e+2f}}{\sqrt{b+d}\,(2b+e+2f)}, \frac{\sqrt{d}\,\sqrt{2b+d+e+2f}\,\sqrt{b+e+d+f}}{\sqrt{b+d}}, \frac{\sqrt{2b+d+e+2f}\,\sqrt{b+e+d+f}\,\sqrt{b+e+2f}\,\sqrt{e+f}}{\sqrt{2b+e+2f}}, -\frac{(2b+d+e+2f)\,\sqrt{b+e+d+f}\,(b+e+2f)\,\sqrt{e+f}}{2b+e+2f}\right]$$

$[b=0, c=0]$

$[h1, h4, h5, h6]$

$$\left[\frac{\sqrt{a}\,\sqrt{a+e+d+f}\,\sqrt{a+d+e+2f}}{\sqrt{a+e}\,\sqrt{a+d+f}}, \frac{\sqrt{d+f}\,\sqrt{a+d+e+2f}\,\sqrt{a+e+d+f}}{\sqrt{a+d+f}}, \frac{\sqrt{e}\,\sqrt{e+f}\,\sqrt{a+d+e+2f}\,\sqrt{a+e+d+f}}{\sqrt{a+e}}, -\frac{\sqrt{d+f}\,\sqrt{e+f}\,(a+d+e+2f)\,\sqrt{a+e+d+f}}{\sqrt{a+d+f}}\right]$$

$[a=0, b=0, c=0]$

$[h4, h5, h6]$

$$\left[\sqrt{d+e+2f}\,\sqrt{e+d+f}, \sqrt{d+e+2f}\,\sqrt{e+f}\,\sqrt{e+d+f}, -(d+e+2f)\,\sqrt{e+f}\,\sqrt{e+d+f}\right]$$

$[d=0]$

$[h1, h2, h3, h5, h6]$

$$\left[\frac{\sqrt{a}\,\sqrt{a+e+f}\,\sqrt{a+b+2c+e+2f}\,\sqrt{c+e+f+a+b}}{\sqrt{a+e}\,\sqrt{a+f}\,\sqrt{a+b+2c+e+f}}, \right.$$

$$\frac{\sqrt{b+2c+e+f}\,\sqrt{c+f+b}\,\sqrt{c+e+f+a+b}\,\sqrt{a+b+2c+e+2f}}{\sqrt{2c+f+b}\,\sqrt{a+b+2c+e+f}},$$

$$\frac{\sqrt{c}\,(b+2c+e+f)\,\sqrt{c+f+b}\,\sqrt{c+e+f+a+b}\,(a+b+2c+e+2f)}{(2c+f+b)\,(a+b+2c+e+f)},$$

$$\frac{\sqrt{e}\,\sqrt{b+2c+e+f}\,\sqrt{c+e+f+a+b}\,\sqrt{a+b+2c+e+2f}\,\sqrt{a+e+f}}{\sqrt{a+e}\,\sqrt{a+b+2c+e+f}},$$

$$-\frac{1}{\sqrt{2c+f+b}\,\sqrt{a+f}\,\sqrt{a+b+2c+e+f}}\Big(\sqrt{b+2c+e+f}\,\sqrt{c+f+b}\,\sqrt{f}$$

$$\left.\sqrt{c+e+f+a+b}\,(a+b+2c+e+2f)\,\sqrt{a+e+f}\Big)\right]$$

$$[a=0, d=0]$$

$$[h2, h3, h5, h6]$$

$$\left[\frac{\sqrt{c+f+b}\,\sqrt{c+e+f+b}\,\sqrt{b+2c+e+2f}}{\sqrt{2c+f+b}}, \right.$$

$$\frac{\sqrt{c}\,\sqrt{c+f+b}\,\sqrt{c+e+f+b}\,(b+2c+e+2f)}{2c+f+b},$$

$$\sqrt{c+e+f+b}\,\sqrt{b+2c+e+2f}\,\sqrt{e+f},$$

$$\left. -\frac{\sqrt{c+f+b}\,\sqrt{c+e+f+b}\,(b+2c+e+2f)\,\sqrt{e+f}}{\sqrt{2c+f+b}} \right]$$

$$[b=0, d=0]$$

$$[h1, h3, h5, h6]$$

$$\left[\frac{\sqrt{a}\,\sqrt{a+e+f}\,\sqrt{c+e+f+a}\,\sqrt{a+2c+e+2f}}{\sqrt{a+e}\,\sqrt{a+f}\,\sqrt{a+2c+e+f}}, \right.$$

$$\frac{\sqrt{c}\,(2c+e+f)\,\sqrt{c+f}\,(a+2c+e+2f)\,\sqrt{c+e+f+a}}{(2c+f)\,(a+2c+e+f)},$$

$$\frac{\sqrt{e}\,\sqrt{2c+e+f}\,\sqrt{a+2c+e+2f}\,\sqrt{c+e+f+a}\,\sqrt{a+e+f}}{\sqrt{a+e}\,\sqrt{a+2c+e+f}},$$

$$-\frac{\sqrt{f}\sqrt{2c+e+f}\sqrt{c+f}\,(a+2c+e+2f)\sqrt{c+e+f+a}\sqrt{a+e+f}}{\sqrt{2c+f}\sqrt{a+2c+e+f}\sqrt{a+f}}\Bigg]$$

$$[a=0, b=0, d=0]$$

$$[h3, h5, h6]$$

$$\Bigg[\frac{\sqrt{c}\sqrt{c+f}\,(2c+e+2f)\sqrt{c+e+f}}{2c+f}, \sqrt{2c+e+2f}\sqrt{c+e+f}\sqrt{e+f},$$

$$-\frac{\sqrt{c+f}\,(2c+e+2f)\sqrt{c+e+f}\sqrt{e+f}}{\sqrt{2c+f}}\Bigg]$$

$$[c=0, d=0]$$

$$[h1, h2, h5, h6]$$

$$\Bigg[\frac{\sqrt{a}\sqrt{a+e+f}\sqrt{a+b+e+2f}}{\sqrt{a+e}\sqrt{a+f}}, \sqrt{e+f+b}\sqrt{a+b+e+2f},$$

$$\frac{\sqrt{e}\sqrt{e+f+b}\sqrt{a+b+e+2f}\sqrt{a+e+f}}{\sqrt{a+e}},$$

$$-\frac{\sqrt{e+f+b}\sqrt{f}\,(a+b+e+2f)\sqrt{a+e+f}}{\sqrt{a+f}}\Bigg]$$

$$[a=0, c=0, d=0]$$

$$[h2, h5, h6]$$

$$\Big[\sqrt{e+f+b}\sqrt{b+e+2f}, \sqrt{e+f+b}\sqrt{b+e+2f}\sqrt{e+f}, -\sqrt{e+f+b}\,(b+e$$
$$+2f)\sqrt{e+f}\Big]$$

$$[b=0, c=0, d=0]$$

$$[h1, h5, h6]$$

$$\Bigg[\frac{\sqrt{a}\sqrt{a+e+f}\sqrt{a+e+2f}}{\sqrt{a+e}\sqrt{a+f}}, \frac{\sqrt{e}\sqrt{e+f}\sqrt{a+e+2f}\sqrt{a+e+f}}{\sqrt{a+e}},$$

$$-\frac{\sqrt{f}\sqrt{e+f}\,(a+e+2f)\sqrt{a+e+f}}{\sqrt{a+f}}\Bigg]$$

$$[a=0, b=0, c=0, d=0]$$

$$[h5, h6]$$

$$\Big[\sqrt{e+2f}\,(e+f), -(e+2f)\,(e+f)\Big]$$

$$[e=0]$$

$$[h1, h2, h3, h4, h6]$$

$$\left[\frac{\sqrt{a+b+2c+2f}\,\sqrt{c+f+a+b}\,\sqrt{a+2b+2c+d+2f}}{\sqrt{a+b+2c+f}\,\sqrt{a+2b+2c+2f}}, \right.$$

$$\left(\sqrt{c+f+b}\,\sqrt{b+d+f}\,\sqrt{b}\,(a+2b+2c+d\right.$$

$$\left.+2f)\,\sqrt{c+f+a+b}\,\sqrt{a+b+2c+2f}\right) \Big/ \left(\sqrt{b+f}\,\sqrt{b+d}\,(a+2b+2c\right.$$

$$\left.+2f)\,\sqrt{a+b+2c+f}\right),$$

$$\frac{\sqrt{c}\,\sqrt{c+f+b}\,(a+2b+2c+d+2f)\,\sqrt{c+f+a+b}\,(a+b+2c+2f)}{(a+b+2c+f)\,(a+2b+2c+2f)},$$

$$\frac{\sqrt{b+d+f}\,\sqrt{d}\,\sqrt{a+2b+2c+d+2f}}{\sqrt{b+d}}, -\left(\sqrt{b+d+f}\,\sqrt{c+f+b}\,\sqrt{f}\,(a\right.$$

$$\left.+2b+2c+d+2f)\,\sqrt{c+f+a+b}\,(a+b+2c+2f)\right) \Big/$$

$$\left.\left(\sqrt{b+f}\,\sqrt{a+b+2c+f}\,(a+2b+2c+2f)\right)\right]$$

$$[a=0, e=0]$$

$$[h2, h3, h4, h6]$$

$$\left[\frac{(c+f+b)\,\sqrt{b}\,(2b+2c+d+2f)\,\sqrt{b+d+f}\,\sqrt{b+2c+2f}}{\sqrt{2c+f+b}\,\sqrt{b+d}\,(2b+2c+2f)\,\sqrt{b+f}}, \right.$$

$$\frac{\sqrt{c}\,(c+f+b)\,(2b+2c+d+2f)\,(b+2c+2f)}{(2c+f+b)\,(2b+2c+2f)},$$

$$\frac{\sqrt{d}\,\sqrt{2b+2c+d+2f}\,\sqrt{b+d+f}}{\sqrt{b+d}},$$

$$\left. -\frac{(c+f+b)\,(2b+2c+d+2f)\,\sqrt{b+d+f}\,(b+2c+2f)\,\sqrt{f}}{\sqrt{2c+f+b}\,\sqrt{b+f}\,(2b+2c+2f)} \right]$$

$$[b=0, e=0]$$

$$[h1, h3, h4, h6]$$

$$\left[\frac{\sqrt{c+f+a}\,\sqrt{a+2c+d+2f}}{\sqrt{a+2c+f}}, \frac{\sqrt{c}\,\sqrt{c+f}\,(a+2c+d+2f)\,\sqrt{c+f+a}}{a+2c+f}, \right.$$

$$\left. \sqrt{d+f}\,\sqrt{a+2c+d+2f}, -\frac{\sqrt{d+f}\,\sqrt{c+f}\,(a+2c+d+2f)\,\sqrt{c+f+a}}{\sqrt{a+2c+f}} \right]$$

$$[a=0, b=0, e=0]$$

$$[h3, h4, h6]$$

$$\left[\frac{\sqrt{c}\,(c+f)\,(2c+d+2f)}{2c+f}, \sqrt{2c+d+2f}\,\sqrt{d+f}, -\frac{(c+f)\,(2c+d+2f)\,\sqrt{d+f}}{\sqrt{2c+f}}\right]$$

$$[c=0, e=0]$$

$$[h1, h2, h4, h6]$$

$$\left[\frac{\sqrt{a+b+2f}\,\sqrt{a+2b+d+2f}}{\sqrt{a+2b+2f}}, \frac{\sqrt{b+d+f}\,\sqrt{b}\,(a+2b+d+2f)\,\sqrt{a+b+2f}}{\sqrt{b+d}\,(a+2b+2f)}, \frac{\sqrt{b+d+f}\,\sqrt{d}\,\sqrt{a+2b+d+2f}}{\sqrt{b+d}}, -\frac{\sqrt{b+d+f}\,\sqrt{f}\,(a+2b+d+2f)\,(a+b+2f)}{a+2b+2f}\right]$$

$$[a=0, c=0, e=0]$$

$$[h2, h4, h6]$$

$$\left[\frac{\sqrt{b}\,(2b+d+2f)\,\sqrt{b+d+f}\,\sqrt{b+2f}}{\sqrt{b+d}\,(2b+2f)}, \frac{\sqrt{d}\,\sqrt{2b+d+2f}\,\sqrt{b+d+f}}{\sqrt{b+d}}, -\frac{(2b+d+2f)\,\sqrt{b+d+f}\,(b+2f)\,\sqrt{f}}{2b+2f}\right]$$

$$[b=0, c=0, e=0]$$

$$[h1, h4, h6]$$

$$\left[\sqrt{a+d+2f}, \sqrt{d+f}\,\sqrt{a+d+2f}, -\sqrt{d+f}\,\sqrt{f}\,(a+d+2f)\right]$$

$$[a=0, b=0, c=0, e=0]$$

$$[h4, h6]$$

$$\left[\sqrt{d+2f}\,\sqrt{d+f}, -(d+2f)\,\sqrt{f}\,\sqrt{d+f}\right]$$

$$[d=0, e=0]$$

$$[h1, h2, h3, h6]$$

$$\left[\frac{\sqrt{a+b+2c+2f}\,\sqrt{c+f+a+b}}{\sqrt{a+b+2c+f}}, \frac{\sqrt{c+f+b}\,\sqrt{c+f+a+b}\,\sqrt{a+b+2c+2f}}{\sqrt{a+b+2c+f}}, \frac{\sqrt{c}\,\sqrt{c+f+b}\,\sqrt{c+f+a+b}\,(a+b+2c+2f)}{a+b+2c+f}, -\frac{\sqrt{c+f+b}\,\sqrt{f}\,\sqrt{c+f+a+b}\,(a+b+2c+2f)}{\sqrt{a+b+2c+f}}\right]$$

$$[a=0, d=0, e=0]$$

$$[h2, h3, h6]$$

$$\left[\frac{(c+f+b)\sqrt{b+2c+2f}}{\sqrt{2c+f+b}}, \frac{\sqrt{c}\,(c+f+b)\,(b+2c+2f)}{2c+f+b}, -\frac{(c+f+b)\,(b+2c+2f)\sqrt{f}}{\sqrt{2c+f+b}}\right]$$

$$[b=0, d=0, e=0]$$

$$[h1, h3, h6]$$

$$\left[\frac{\sqrt{c+f+a}\,\sqrt{a+2c+2f}}{\sqrt{a+2c+f}}, \frac{\sqrt{c}\,\sqrt{c+f}\,(a+2c+2f)\sqrt{c+f+a}}{a+2c+f}, -\frac{\sqrt{f}\,\sqrt{c+f}\,(a+2c+2f)\sqrt{c+f+a}}{\sqrt{a+2c+f}}\right]$$

$$[a=0, b=0, d=0, e=0]$$

$$[h3, h6]$$

$$\left[\frac{\sqrt{c}\,(c+f)\,(2c+2f)}{2c+f}, -\frac{(c+f)\,(2c+2f)\sqrt{f}}{\sqrt{2c+f}}\right]$$

$$[c=0, d=0, e=0]$$

$$[h1, h2, h6]$$

$$\left[\sqrt{a+b+2f}, \sqrt{b+f}\,\sqrt{a+b+2f}, -\sqrt{b+f}\,\sqrt{f}\,(a+b+2f)\right]$$

$$[a=0, c=0, d=0, e=0]$$

$$[h2, h6]$$

$$\left[\sqrt{b+f}\,\sqrt{b+2f}, -\sqrt{b+f}\,(b+2f)\sqrt{f}\right]$$

$$[b=0, c=0, d=0, e=0]$$

$$[h1, h6]$$

$$\left[\sqrt{a+2f}, -f(a+2f)\right]$$

$$[a = 0, b = 0, c = 0, d = 0, e = 0]$$

$$[h6]$$

$$[-2 f^2]$$

$$[f = 0]$$

$$[h1, h2, h3, h4, h5]$$

$$\left[\frac{\sqrt{a+b}\,\sqrt{a+b+e+d}\,\sqrt{c+e+a+b}\,\sqrt{a+2b+2c+d+e}}{\sqrt{a+b+d}\,\sqrt{e+a+b}\,\sqrt{a+2b+2c+e}}, \right.$$

$$\left(\sqrt{b+2c+e}\,\sqrt{c+b}\,(a+2b+2c+d\right.$$

$$\left.+e)\,\sqrt{c+e+a+b}\,\sqrt{a+b+e+d}\,\sqrt{a+b}\right) \Big/ \left(\sqrt{2c+b}\,(a+2b+2c\right.$$

$$\left.+e)\,\sqrt{e+a+b}\,\sqrt{a+b+d}\right),$$

$$\frac{\sqrt{c}\,(b+2c+e)\,\sqrt{c+b}\,(a+2b+2c+d+e)\,\sqrt{c+e+a+b}}{(2c+b)\,(a+2b+2c+e)},$$

$$\frac{\sqrt{d}\,\sqrt{a+2b+2c+d+e}\,\sqrt{a+b+e+d}}{\sqrt{a+b+d}},$$

$$\left. \frac{\sqrt{e}\,\sqrt{b+2c+e}\,\sqrt{a+2b+2c+d+e}\,\sqrt{c+e+a+b}\,\sqrt{a+b+e+d}}{\sqrt{e+a+b}\,\sqrt{a+2b+2c+e}} \right]$$

$$[a = 0, f = 0]$$

$$[h2, h3, h4, h5]$$

$$\left[\frac{\sqrt{c+b}\,\sqrt{b}\,(2b+2c+d+e)\,\sqrt{c+e+b}\,\sqrt{b+e+d}\,\sqrt{b+2c+e}}{\sqrt{2c+b}\,\sqrt{b+d}\,(2b+2c+e)\,\sqrt{e+b}}, \right.$$

$$\frac{\sqrt{c}\,\sqrt{c+b}\,(2b+2c+d+e)\,\sqrt{c+e+b}\,(b+2c+e)}{(2c+b)\,(2b+2c+e)},$$

$$\frac{\sqrt{d}\,\sqrt{2b+2c+d+e}\,\sqrt{b+e+d}}{\sqrt{b+d}},$$

$$\left. \frac{\sqrt{2b+2c+d+e}\,\sqrt{c+e+b}\,\sqrt{b+e+d}\,\sqrt{b+2c+e}\,\sqrt{e}}{\sqrt{e+b}\,\sqrt{2b+2c+e}} \right]$$

$$[b = 0, f = 0]$$

$$[h1, h3, h4, h5]$$

$$\left[\frac{\sqrt{a}\,\sqrt{a+e+d}\,\sqrt{c+e+a}\,\sqrt{a+2c+d+e}}{\sqrt{a+e}\,\sqrt{a+d}\,\sqrt{a+2c+e}},\right.$$

$$\frac{1}{2}\,\frac{(2c+e)\,(a+2c+d+e)\,\sqrt{c+e+a}}{a+2c+e},\ \frac{\sqrt{d}\,\sqrt{a+2c+d+e}\,\sqrt{a+e+d}}{\sqrt{a+d}},$$

$$\left.\frac{\sqrt{e}\,\sqrt{2c+e}\,\sqrt{a+2c+d+e}\,\sqrt{c+e+a}\,\sqrt{a+e+d}}{\sqrt{a+e}\,\sqrt{a+2c+e}}\right]$$

$$[a=0,\ b=0,\ f=0]$$

$$[h3,\ h4,\ h5]$$

$$\left[\frac{1}{2}\,(2c+d+e)\,\sqrt{c+e},\ \sqrt{2c+d+e}\,\sqrt{e+d},\ \sqrt{2c+d+e}\,\sqrt{c+e}\,\sqrt{e+d}\right]$$

$$[c=0,\ f=0]$$

$$[h1,\ h2,\ h4,\ h5]$$

$$\left[\frac{\sqrt{a+b}\,\sqrt{a+b+e+d}\,\sqrt{a+2b+d+e}}{\sqrt{a+b+d}\,\sqrt{a+2b+e}},\right.$$

$$\frac{\sqrt{e+b}\,(a+2b+d+e)\,\sqrt{a+b+e+d}\,\sqrt{a+b}}{(a+2b+e)\,\sqrt{a+b+d}},$$

$$\frac{\sqrt{d}\,\sqrt{a+2b+d+e}\,\sqrt{a+b+e+d}}{\sqrt{a+b+d}},$$

$$\left.\frac{\sqrt{e}\,\sqrt{e+b}\,\sqrt{a+2b+d+e}\,\sqrt{a+b+e+d}}{\sqrt{a+2b+e}}\right]$$

$$[a=0,\ c=0,\ f=0]$$

$$[h2,\ h4,\ h5]$$

$$\left[\frac{\sqrt{b}\,(2b+d+e)\,\sqrt{b+e+d}\,\sqrt{e+b}}{\sqrt{b+d}\,(2b+e)},\ \frac{\sqrt{d}\,\sqrt{2b+d+e}\,\sqrt{b+e+d}}{\sqrt{b+d}},\right.$$

$$\left.\frac{\sqrt{2b+d+e}\,\sqrt{b+e+d}\,\sqrt{e+b}\,\sqrt{e}}{\sqrt{2b+e}}\right]$$

$$[b=0,\ c=0,\ f=0]$$

$$[h1,\ h4,\ h5]$$

$$\left[\frac{\sqrt{a}\,(a+e+d)}{\sqrt{a+e}\,\sqrt{a+d}},\ \frac{\sqrt{d}\,(a+e+d)}{\sqrt{a+d}},\ \frac{e\,(a+e+d)}{\sqrt{a+e}}\right]$$

$$[a=0,\ b=0,\ c=0,\ f=0]$$

$$[h4, h5]$$

$$\left[e + d, (e + d)\sqrt{e}\right]$$

$$[d = 0, f = 0]$$

$$[h1, h2, h3, h5]$$

$$\left[\sqrt{c + e + a + b}, \frac{\sqrt{b + 2c + e}\sqrt{c + b}\sqrt{c + e + a + b}}{\sqrt{2c + b}}, \frac{\sqrt{c}(b + 2c + e)\sqrt{c + b}\sqrt{c + e + a + b}}{2c + b}, \sqrt{e}\sqrt{b + 2c + e}\sqrt{c + e + a + b}\right]$$

$$[a = 0, d = 0, f = 0]$$

$$[h2, h3, h5]$$

$$\left[\frac{\sqrt{c + b}\sqrt{c + e + b}\sqrt{b + 2c + e}}{\sqrt{2c + b}}, \frac{\sqrt{c}\sqrt{c + b}\sqrt{c + e + b}(b + 2c + e)}{2c + b}, \sqrt{c + e + b}\sqrt{b + 2c + e}\sqrt{e}\right]$$

$$[b = 0, d = 0, f = 0]$$

$$[h1, h3, h5]$$

$$\left[\sqrt{c + e + a}, \frac{1}{2}(2c + e)\sqrt{c + e + a}, \sqrt{e}\sqrt{2c + e}\sqrt{c + e + a}\right]$$

$$[a = 0, b = 0, d = 0, f = 0]$$

$$[h3, h5]$$

$$\left[\frac{1}{2}(2c + e)\sqrt{c + e}, \sqrt{2c + e}\sqrt{c + e}\sqrt{e}\right]$$

$$[c = 0, d = 0, f = 0]$$

$$[h1, h2, h5]$$

$$\left[\sqrt{e + a + b}, \sqrt{e + b}\sqrt{e + a + b}, \sqrt{e}\sqrt{e + b}\sqrt{e + a + b}\right]$$

$$[a = 0, c = 0, d = 0, f = 0]$$

$$[h2, h5]$$

$$\left[e + b, (e + b)\sqrt{e}\right]$$

$$[b = 0, c = 0, d = 0, f = 0]$$

$$[h1, h5]$$

$$\left[\sqrt{a + e}, \sqrt{a + e}\, e\right]$$

$$[a = 0, b = 0, c = 0, d = 0, f = 0]$$

$$[h5]$$

$$[e^{3/2}]$$

$$[e = 0, f = 0]$$

$$[h1, h2, h3, h4]$$

$$\left[\frac{\sqrt{c+a+b}\,\sqrt{a+2b+2c+d}}{\sqrt{a+2b+2c}}, \frac{\sqrt{c+b}\,(a+2b+2c+d)\sqrt{c+a+b}}{a+2b+2c}, \frac{\sqrt{c}\,\sqrt{c+b}\,(a+2b+2c+d)\sqrt{c+a+b}}{a+2b+2c}, \sqrt{d}\,\sqrt{a+2b+2c+d}\right]$$

$$[a = 0, e = 0, f = 0]$$

$$[h2, h3, h4]$$

$$\left[\frac{(2b+2c+d)(c+b)}{2b+2c}, \frac{\sqrt{c}\,(c+b)(2b+2c+d)}{2b+2c}, \sqrt{d}\,\sqrt{2b+2c+d}\right]$$

$$[b = 0, e = 0, f = 0]$$

$$[h1, h3, h4]$$

$$\left[\frac{\sqrt{c+a}\,\sqrt{a+2c+d}}{\sqrt{a+2c}}, \frac{c\,(a+2c+d)\sqrt{c+a}}{a+2c}, \sqrt{d}\,\sqrt{a+2c+d}\right]$$

$$[a = 0, b = 0, e = 0, f = 0]$$

$$[h3, h4]$$

$$\left[\frac{1}{2}\sqrt{c}\,(2c+d), \sqrt{2c+d}\,\sqrt{d}\right]$$

$$[c = 0, e = 0, f = 0]$$

$$[h1, h2, h4]$$

$$\left[\frac{\sqrt{a+b}\,\sqrt{a+2b+d}}{\sqrt{a+2b}}, \frac{\sqrt{b}\,(a+2b+d)\sqrt{a+b}}{a+2b}, \sqrt{d}\,\sqrt{a+2b+d}\right]$$

$$[a = 0, c = 0, e = 0, f = 0]$$

$$[h2, h4]$$

$$\left[b + \frac{1}{2}d, \sqrt{d}\,\sqrt{2b+d}\right]$$

$$[b = 0, c = 0, e = 0, f = 0]$$

$$[h1, h4]$$

$$\left[\sqrt{a+d}, \sqrt{d}\,\sqrt{a+d}\right]$$

$$[a = 0, b = 0, c = 0, e = 0, f = 0]$$

$$[h4]$$

$$[d]$$

$$[d=0, e=0, f=0]$$

$$[h1, h2, h3]$$

$$\left[\sqrt{c+a+b}, \sqrt{c+b}\,\sqrt{c+a+b}, \sqrt{c}\,\sqrt{c+b}\,\sqrt{c+a+b}\right]$$

$$[a=0, d=0, e=0, f=0]$$

$$[h2, h3]$$

$$\left[c+b, \sqrt{c}\,(c+b)\right]$$

$$[b=0, d=0, e=0, f=0]$$

$$[h1, h3]$$

$$\left[\sqrt{c+a}, c\sqrt{c+a}\right]$$

$$[a=0, b=0, d=0, e=0, f=0]$$

$$[h3]$$

$$\left[c^{3/2}\right]$$

$$[c=0, d=0, e=0, f=0]$$

$$[h1, h2]$$

$$\left[\sqrt{a+b}, \sqrt{b}\,\sqrt{a+b}\right]$$

$$[a=0, c=0, d=0, e=0, f=0]$$

$$[h2]$$

$$[b]$$

$$[b=0, c=0, d=0, e=0, f=0]$$

$$[h1]$$

$$\left[\sqrt{a}\right]$$

$$[a=0, b=0, c=0, d=0, e=0, f=0]$$

$$[\ ]$$

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