The action (Sp(4)xU(4)):$M_{4,4}$(C)

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This Maple worksheet provides details concerning an example presented in section 4.6.4 of our paper *Spaces of Bounded Spherical Functions on Heisenberg Groups: Part I*. At issue is the (multiplicity free) action of the compact group Sp(4)xU(4) on the space V=$M_{4,4}$(C) of 4x4 complex matrices via **(k1, k2). Z = k1 Z k2^t**. We will show that this action is well-behaved as defined in Section 2 of the paper.

```
> restart: with(linalg):
```

Functions **dot** and **sym** below implement the inner product of 4x4 matrices and the symplectic product of vectors in C^4. In the inner product complex conjugation should be applied to the matrix entries of the second input. But for our purposes it will suffice to restrict attention to matrices all of whose entries are real.

```
> dot:=(a,b)->sum(sum(a['u','v']*b['u','v'],'u'=1..4),'v'=1..4);
```

$$dot := (a, b) \rightarrow \sum_{'v'=1}^{4} \left(\sum_{'u'=1}^{4} a_{'u', 'v'} b_{'u', 'v'} \right)$$

```
> sym:=(a,b)->sum(a['i']*b['i'+2]-a['i'+2]*b['i'],'i'=1..2);
```

$$sym := (a, b) \rightarrow \sum_{'i'=1}^{2} \left(a_{'i'} b_{'i'+2} - a_{'i'+2} b_{'i'} \right)$$

So for example......

```
> z:=matrix(4,4): dot(z,z); sym(col(z,1),col(z,2));
```

$$z_{1,1}^2 + z_{2,1}^2 + z_{3,1}^2 + z_{4,1}^2 + z_{1,2}^2 + z_{2,2}^2 + z_{3,2}^2 + z_{4,2}^2 + z_{1,3}^2 + z_{2,3}^2 + z_{3,3}^2 + z_{4,3}^2 + z_{1,4}^2 + z_{2,4}^2 + z_{3,4}^2 + z_{4,4}^2$$

$$z_{1,1} z_{3,2} - z_{3,1} z_{1,2} + z_{2,1} z_{4,2} - z_{4,1} z_{2,2}$$

This multiplicity free action has rank 6. Fundamental highest weights and highest weight vectors were given in a paper by Howe and Umeda. We implement these below as A1,...,A6 (highest weights) and h1,...h6 (highest weight vectors).

```
> h1:=z->z[1,1];h1(z);
```

$$h1 := z \rightarrow z_{1,1}$$

$$z_{1,1}$$

```
> A1:=[[1,0],[1,0,0,0] ];
```

$$A1 := [[1, 0], [1, 0, 0, 0]]$$

```
> h2:=z->z[1,1]*z[2,2]-z[1,2]*z[2,1];h2(z);
```

$$h2 := z \rightarrow z_{1,1} z_{2,2} - z_{1,2} z_{2,1}$$

$$z_{1,1} z_{2,2} - z_{1,2} z_{2,1}$$

```
> A2:=[[1,1],[1,1,0,0]] ;
```

$$A2 := [[1, 1], [1, 1, 0, 0]]$$

```
> h3:=z->det(delcols(delrows(z,3...3),4..4)); h3(z);
```

$$h3 := z \rightarrow linalg\text{:-}det(linalg\text{:-}delcols(linalg\text{:-}delrows(z, 3..3), 4..4))$$

$$z_{1,1} z_{2,2} z_{4,3} - z_{1,1} z_{2,3} z_{4,2} - z_{2,1} z_{1,2} z_{4,3} + z_{2,1} z_{1,3} z_{4,2} + z_{4,1} z_{1,2} z_{2,3} - z_{4,1} z_{1,3} z_{2,2}$$

```
> A3:=[[1,0],[1,1,1,0]];
```

$$A3 := [[1, 0], [1, 1, 1, 0]]$$

```
> h4:=z->z[1,1]*z[3,2]-z[1,2]*z[3,1]+z[2,1]*z[4,2]-z[2,2 ]*z[4,1];
  h4(z);
```

$$h4 := z \rightarrow z_{1,1} z_{3,2} - z_{1,2} z_{3,1} + z_{2,1} z_{4,2} - z_{2,2} z_{4,1}$$

$$z_{1,1} z_{3,2} - z_{3,1} z_{1,2} + z_{2,1} z_{4,2} - z_{4,1} z_{2,2}$$

```
> A4:=[[0,0],[1,1,0,0]];
```

$$A4 := [[0, 0], [1, 1, 0, 0]]$$

```
> m1:=z->matrix([[z[1,1],0,z[1,2],z[1,3]],
  [z[2,1],0,z[2,2],z[2,3]],
  [0,z[1,1],z[1,2], z[1,3]],
  [0,z[3,1],z[3,2],z[3,3]]]):
> m2:=z->matrix([[z[1,1],0,z[1,2],z[1,3]],
  [z[2,1],0,z[2,2],z[2,3]],
  [0,z[2,1],z[2,2], z[2,3]],
  [0,z[4,1],z[4,2],z[4,3]]]):
> h5:=z->det(m1(z))+det(m2(z));
  h5(z);
```

$$h5 := z \rightarrow linalg\text{:-}det(m1(z)) + linalg\text{:-}det(m2(z))$$

$$-z_{1,1}^2 z_{2,2} z_{3,3} + z_{1,1}^2 z_{2,3} z_{3,2} + z_{1,1} z_{3,1} z_{1,3} z_{2,2} - z_{1,1} z_{3,1} z_{1,2} z_{2,3} + z_{2,1} z_{1,1} z_{1,2} z_{3,3} - z_{2,1} z_{1,1} z_{1,3} z_{3,2} - z_{2,1} z_{1,1} z_{2,2} z_{4,3} + z_{2,1} z_{1,1} z_{2,3} z_{4,2} + z_{2,1}^2 z_{1,2} z_{4,3} - z_{2,1}^2 z_{1,3} z_{4,2} - z_{2,1} z_{4,1} z_{1,2} z_{2,3} + z_{2,1} z_{4,1} z_{1,3} z_{2,2}$$

```
> A5:=[[1,1],[2,1,1,0]];
```

$$A5 := [[1, 1], [2, 1, 1, 0]]$$

```
> h6:=z->det(z);h6(z);
```

$$h6 := z \rightarrow linalg\text{:-}det(z)$$

$$z_{1,1} z_{2,2} z_{3,3} z_{4,4} - z_{1,1} z_{2,2} z_{3,4} z_{4,3} - z_{1,1} z_{3,2} z_{2,3} z_{4,4} + z_{1,1} z_{3,2} z_{2,4} z_{4,3} + z_{1,1} z_{4,2} z_{2,3} z_{3,4} - z_{1,1} z_{4,2} z_{2,4} z_{3,3} - z_{2,1} z_{1,2} z_{3,3} z_{4,4} + z_{2,1} z_{1,2} z_{3,4} z_{4,3} + z_{2,1} z_{3,2} z_{1,3} z_{4,4} - z_{2,1} z_{3,2} z_{1,4} z_{4,3} - z_{2,1} z_{4,2} z_{1,3} z_{3,4} + z_{2,1} z_{4,2} z_{1,4} z_{3,3} + z_{3,1} z_{1,2} z_{2,3} z_{4,4} - z_{3,1} z_{1,2} z_{2,4} z_{4,3} - z_{3,1} z_{2,2} z_{1,3} z_{4,4} + z_{3,1} z_{2,2} z_{1,4} z_{4,3}$$

$$+ z_{3,1} z_{4,2} z_{1,3} z_{2,4} - z_{3,1} z_{4,2} z_{1,4} z_{2,3} - z_{4,1} z_{1,2} z_{2,3} z_{3,4} + z_{4,1} z_{1,2} z_{2,4} z_{3,3}$$

$$+ z_{4,1} z_{2,2} z_{1,3} z_{3,4} - z_{4,1} z_{2,2} z_{1,4} z_{3,3} - z_{4,1} z_{3,2} z_{1,3} z_{2,4} + z_{4,1} z_{3,2} z_{1,4} z_{2,3}$$

```
> A6:=[[0,0],[1,1,1,1]];
```

$$A6 := [[0, 0], [1, 1, 1, 1]]$$

Matrix X will be an arbitrary element of the Lie algebra sp(4,C)....

```
> X:=matrix(4,4): X[1,4]:= X[2,3]: X[3,3]:=-X[1,1]: X[4,3]:=-X[1,2]:
  X[3,4]:=-X[2,1]: X[4,4]:=- X[2,2]: X[4,1]:= X[3,2]: evalm(X) ;
```

$$\begin{bmatrix} X_{1,1} & X_{1,2} & X_{1,3} & X_{2,3} \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} \\ X_{3,1} & X_{3,2} & -X_{1,1} & -X_{2,1} \\ X_{3,2} & X_{4,2} & -X_{1,2} & -X_{2,2} \end{bmatrix}$$

Matrix Y will be an arbitrary element of gl(4,C)....

```
> Y:=matrix(4,4);
```

$$Y := array(1..4, 1..4, [\ ])$$

The moment map takes V to the dual of sp(4) x gl(4). We implement this below as function **mom**.

```
> mom:=z->  simplify(dot(evalm(X&*z),z) )   +simplify(dot(evalm(z&*
  transpose(Y)),z));
```

mom := *z* → *simplify* (*dot* (*evalm* (*X* &* *z*), *z*)) + *simplify* (*dot* (*evalm* (*z* &* *linalg*:-
transpose (*Y*)), *z*))

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

Using numerical methods, we found the points in V which map to diagonals under the moment map. We code the results of this investigation as procedure **spt** below. We will show that this produces a spherical point which maps to the weight aA1+bA2+cA3+dA4+eA5+fA6.

```
> spt:=proc(a,b,c,d,e,f)
  local z11,z12,z13,z14,z21,z22,z23,z24,z31,z32,z33,z34,z41,z42,z43,
  z44,sp;

  z11:=(a*(a+b+e)*(a+c+e)*(a+b+c+2*e)*(a+2*b+c+d+2*e)*(a+b+c+d+e)*(a+
  b+c+d+2*e+f))/
  ((a+c)*(a+e)*(a+b+c+e)*(a+b+d+e)*(a+2*b+c+2*e)*(a+b+c+d+2*e));
  z12:=(b*c*e*(a+b+c+d+e)*(a+2*b+c+d+2*e)*(b+c+e)*(b+c+d+e+f))/
  ((a+c)*(a+e)*(b+d)*(a+b+c+e)*(a+2*b+c+2*e)*(b+c+d+e));
  z13:=(c*d*(a+b+c+2*e)*(a+c+e)*(b+c+e)*(b+d+e)*(c+e+f))/
  ((a+c)*(b+d)*(c+e)*(a+b+c+e)*(a+2*b+c+2*e)*(a+b+d+e));
  z14:=(a*b*d*e*f*(a+b+e)*(b+d+e))/
  ((a+c)*(c+e)*(a+b+c+e)*(a+2*b+c+2*e)*(a+b+c+d+2*e)*(b+c+d+e));

  z21:=(c*e*(a+b+e)*(a+b+c+2*e)*(a+2*b+c+d+2*e)*(b+d+e)*(a+b+c+d+2*e+
  f))/
  ((a+c)*(a+e)*(b+e)*(a+2*b+c+2*e)*(a+b+c+d+2*e)*(a+b+d+e));
  z22:=(a*b*(a+2*b+c+d+2*e)*(a+c+e)*(b+c+e)*(b+d+e)*(b+c+d+e+f))/
  ((a+c)*(a+e)*(b+d)*(b+e)*(a+2*b+c+2*e)*(b+c+d+e));
  z23:=(a*d*e*(a+b+c+2*e)*(a+b+c+d+e)*(b+c+e)*(c+e+f))/
  ((a+c)*(b+d)*(b+e)*(c+e)*(a+2*b+c+2*e)*(a+b+d+e));
  z24:=(b*c*d*f*(a+b+e)*(a+b+c+d+e)*(a+c+e))/
```

```
((a+c)*(b+e)*(c+e)*(a+2*b+c+2*e)*(a+b+c+d+2*e)*(b+c+d+e));

z31:=(b*c*d*e*(b+c+e)*(b+d+e)*(a+b+c+d+2*e+f))/
((a+c)*(a+e)*(a+b+c+e)*(a+2*b+c+2*e)*(a+b+c+d+2*e)*(a+b+d+e));
z32:=(a*d*(a+b+e)*(a+b+c+2*e)*(a+c+e)*(b+d+e)*(b+c+d+e+f))/
((a+c)*(a+e)*(b+d)*(a+b+c+e)*(a+2*b+c+2*e)*(b+c+d+e));
z33:=(a*b*e*(a+b+c+d+e)*(a+2*b+c+d+2*e)*(a+b+e)*(c+e+f))/
((a+c)*(b+d)*(c+e)*(a+b+c+e)*(a+2*b+c+2*e)*(a+b+d+e));
z34:=(c*f*(a+b+c+2*e)*(a+b+c+d+e)*(a+2*b+c+d+2*e)*(a+c+e)*(b+c+e))/
((a+c)*(c+e)*(a+b+c+e)*(a+2*b+c+2*e)*(a+b+c+d+2*e)*(b+c+d+e));

z41:=(a*b*d*(a+b+c+d+e)*(a+c+e)*(b+c+e)*(a+b+c+d+2*e+f))/
((a+c)*(a+e)*(b+e)*(a+2*b+c+2*e)*(a+b+d+e)*(a+b+c+d+2*e));
z42:=(c*d*e*(a+b+c+2*e)*(a+b+c+d+e)*(a+b+e)*(b+c+d+e+f))/
((a+c)*(a+e)*(b+d)*(b+e)*(a+2*b+c+2*e)*(b+c+d+e));
z43:=(b*c*(a+2*b+c+d+2*e)*(a+b+e)*(a+c+e)*(b+d+e)*(c+e+f))/
((a+c)*(b+d)*(b+e)*(c+e)*(a+2*b+c+2*e)*(a+b+d+e));
z44:=(a*e*f*(a+b+c+2*e)*(a+2*b+c+d+2*e)*(b+d+e)*(c+b+e)*(c+d+e))/
((a+c)*(b+e)*(c+e)*(a+2*b+c+2*e)*(a+b+c+d+2*e)*(b+c+d+e)*(c+d+e));

sp:=matrix([[z11,z12,z13,z14],[z21,z22,z23,z24],[z31,z32,z33,z34],
[z41,z42,z43,z44]]);
sp:=map(x->sqrt(x),sp);
sp[1,2]:=-sp[1,2]; sp[1,3]:=-sp[1,3]; sp[1,4]:=-sp[1,4]; sp[2,1]:=-
sp[2,1]; sp[2,2]:=-sp[2,2]; sp[2,3]:=-sp[2,3];
sp[3,3]:=-sp[3,3]; sp[4,2]:=-sp[4,2];
evalm(sp);
end:
```

Here now is our general spherical point:

```
> z:=spt(a,b,c,d,e,f);
```

$z :=$ [

[$((a\,(a+b+e)\,(a+c+e)\,(a+b+c+2\,e)\,(a+2\,b+c+d$

$+2\,e)\,(a+b+c+d+e)\,(a+b+c+d+2\,e+f))\,/\,((a+c)\,(a+e)\,(a+b+c$

$+e)\,(a+b+d+e)\,(a+2\,b+c+2\,e)\,(a+b+c+d+2\,e)))^{1/2}$,

$-$

$((b\,c\,e\,(a+b+c+d+e)\,(a+2\,b+c+d+2\,e)\,(b+c+e)\,(b+c$

$+d+e+f))\,/\,((a+c)\,(a+e)\,(b+d)\,(a+b+c+e)\,(a+2\,b+c+2\,e)\,(b+c$

$+ d + e)))^{1/2}$,

$-\sqrt{\frac{c\, d\, (a + b + c + 2\, e)\, (a + c + e)\, (b + c + e)\, (b + d + e)\, (c + e + f)}{(a + c)\, (b + d)\, (c + e)\, (a + b + c + e)\, (a + 2\, b + c + 2\, e)\, (a + b + d + e)}}$,

$-((a\, b\, d\, e\, f\, (a + b + e)\, (b + d + e))/((a + c)\, (c + e)\, (a + b + c$

$+ e)\, (a + 2\, b + c + 2\, e)\, (a + b + c + d + 2\, e)\, (b + c + d + e)))^{1/2}\Big]$,

$\Big[$

$-((c\, e\, (a + b + e)\, (a + b + c + 2\, e)\, (a + 2\, b + c + d + 2\, e)\, (b + d$

$+ e)\, (a + b + c + d + 2\, e + f))/((a + c)\, (a + e)\, (b + e)\, (a + 2\, b + c + 2\, e)\, (a$

$+ b + c + d + 2\, e)\, (a + b + d + e)))^{1/2}$,

$-((a\, b\, (a + 2\, b + c + d + 2\, e)\, (a + c + e)\, (b + c + e)\, (b + d + e)\, (b$

$+ c + d + e + f))/((a + c)\, (a + e)\, (b + d)\, (b + e)\, (a + 2\, b + c + 2\, e)\, (b + c + d$

$+ e)))^{1/2}$,

$-\sqrt{\frac{a\, d\, e\, (a + b + c + 2\, e)\, (a + b + c + d + e)\, (b + c + e)\, (c + e + f)}{(a + c)\, (b + d)\, (b + e)\, (c + e)\, (a + 2\, b + c + 2\, e)\, (a + b + d + e)}}$,

$((b\, c\, d\, f\, (a + b + e)\, (a + b + c + d + e)\, (a + c + e))/((a + c)\, (b$

$+ e)\, (c + e)\, (a + 2\, b + c + 2\, e)\, (a + b + c + d + 2\, e)\, (b + c + d + e)))^{1/2}\Big]$,

$\Big[((b\, c\, d\, e\, (b + c + e)\, (b + d + e)\, (a + b + c + d + 2\, e + f))/((a$

$+ c)\, (a + e)\, (a + b + c + e)\, (a + b + d + e)\, (a + 2\, b + c + 2\, e)\, (a + b + c + d$

$$+2e)))^{1/2},$$

$$\sqrt{\frac{a\,d\,(a+b+e)\,(a+b+c+2e)\,(a+c+e)\,(b+d+e)\,(b+c+d+e+f)}{(a+c)\,(a+e)\,(b+d)\,(a+b+c+e)\,(a+2b+c+2e)\,(b+c+d+e)}},$$

$$-\sqrt{\frac{a\,b\,e\,(a+b+c+d+e)\,(a+2b+c+d+2e)\,(a+b+e)\,(c+e+f)}{(a+c)\,(b+d)\,(c+e)\,(a+b+c+e)\,(a+2b+c+2e)\,(a+b+d+e)}},$$

$$((c\,f\,(a+b+c+2e)\,(a+b+c+d+e)\,(a+2b+c+d+2e)\,(a$$

$$+c+e)\,(b+c+e))/((a+c)\,(c+e)\,(a+b+c+e)\,(a+2b+c+2e)\,(a+b$$

$$+c+d+2e)\,(b+c+d+e)))^{1/2}\Big],$$

$$\Big[$$

$$((a\,b\,d\,(a+b+c+d+e)\,(a+c+e)\,(b+c+e)\,(a+b+c+d+2e$$

$$+f))/((a+c)\,(a+e)\,(b+e)\,(a+2b+c+2e)\,(a+b+c+d+2e)\,(a+b$$

$$+d+e)))^{1/2},$$

$$-\sqrt{\frac{c\,d\,e\,(a+b+c+2e)\,(a+b+c+d+e)\,(a+b+e)\,(b+c+d+e+f)}{(a+c)\,(a+e)\,(b+d)\,(b+e)\,(a+2b+c+2e)\,(b+c+d+e)}},$$

$$\sqrt{\frac{b\,c\,(a+2b+c+d+2e)\,(a+b+e)\,(a+c+e)\,(b+d+e)\,(c+e+f)}{(a+c)\,(b+d)\,(b+e)\,(c+e)\,(a+2b+c+2e)\,(a+b+d+e)}},$$

$$((a\,e\,f\,(a+b+c+2e)\,(a+2b+c+d+2e)\,(b+d+e)\,(b+c$$

$$+e))/((a+c)\,(b+e)\,(c+e)\,(a+2b+c+2e)\,(a+b+c+d+2e)\,(b+c$$

$$+d+e)))^{1/2}\Big]\Big]$$

To help with simplification, we replace each square root in the matrix above with a single variable, t1.... t26.

```
> getallterms:=proc(sp)
      local ntermsarray, dtermsarray, Trms, i, j;
      ntermsarray:=map(x->{op(convert(numer(x^2), list))}, sp);
```

```
    dtermsarray:=map(x->{op(convert(denom(x^2), list))}, sp);
    Trms:={};
    for i from 1 to 4 do
        for j from 1 to 4 do
            Trms:=Trms union ntermsarray[i,j] union dtermsarray[i,
j]
    od od;
    convert(Trms, list)
end:
> Terms:=getallterms(z); nops(%);
```

$Terms := [a, b, c, d, e, f, a+c, a+e, b+d, b+e, c+e, a+b+e, a+c+e, b+c+e, b+d+e, c+e+f, a+b+c+e, a+b+c+2e, a+b+d+e, a+2b+c+2e, b+c+d+e, a+b+c+d+e, a+b+c+d+2e, a+2b+c+d+2e, b+c+d+e+f, a+b+c+d+2e+f]$

$$26$$

```
> rewrite := proc(q)
    local num, den, numr, denr, j:
    global Terms;
    num:={op(convert(numer(q), list))};
    den:={op(convert(denom(q), list))};
    numr:=1; denr:=1;
    for j from 1 to nops(Terms) do
        if Terms[j] in num then numr:=numr*t[j] fi;
        if Terms[j] in den then denr:=denr*t[j] fi
    od;
    numr/denr
end:
```

The matrix zn is the spherical point expressed in terms of the t's.

```
> zn:=zip((x,y)->sign(x)*(rewrite(y^2)), z, z);
```

$$zn := \left[\left[\frac{t_1 t_{12} t_{13} t_{18} t_{22} t_{24} t_{26}}{t_7 t_8 t_{17} t_{19} t_{20} t_{23}}, -\frac{t_2 t_3 t_5 t_{14} t_{22} t_{24} t_{25}}{t_7 t_8 t_9 t_{17} t_{20} t_{21}}, -\frac{t_3 t_4 t_{13} t_{14} t_{15} t_{16} t_{18}}{t_7 t_9 t_{11} t_{17} t_{19} t_{20}}, -\frac{t_1 t_2 t_4 t_5 t_6 t_{12} t_{15}}{t_7 t_{11} t_{17} t_{20} t_{21} t_{23}}\right],\right.$$

$$\left[-\frac{t_3 t_5 t_{12} t_{15} t_{18} t_{24} t_{26}}{t_7 t_8 t_{10} t_{19} t_{20} t_{23}}, -\frac{t_1 t_2 t_{13} t_{14} t_{15} t_{24} t_{25}}{t_7 t_8 t_9 t_{10} t_{20} t_{21}}, -\frac{t_1 t_4 t_5 t_{14} t_{16} t_{18} t_{22}}{t_7 t_9 t_{10} t_{11} t_{19} t_{20}}, \frac{t_2 t_3 t_4 t_6 t_{12} t_{13} t_{22}}{t_7 t_{10} t_{11} t_{20} t_{21} t_{23}}\right],$$

$$\left[\frac{t_2 t_3 t_4 t_5 t_{14} t_{15} t_{26}}{t_7 t_8 t_{17} t_{19} t_{20} t_{23}}, \frac{t_1 t_4 t_{12} t_{13} t_{15} t_{18} t_{25}}{t_7 t_8 t_9 t_{17} t_{20} t_{21}}, -\frac{t_1 t_2 t_5 t_{12} t_{16} t_{22} t_{24}}{t_7 t_9 t_{11} t_{17} t_{19} t_{20}}, \frac{t_3 t_6 t_{13} t_{14} t_{18} t_{22} t_{24}}{t_7 t_{11} t_{17} t_{20} t_{21} t_{23}}\right],$$

$$\left[\frac{t_1 t_2 t_4 t_{13} t_{14} t_{22} t_{26}}{t_7 t_8 t_{10} t_{19} t_{20} t_{23}}, -\frac{t_3 t_4 t_5 t_{12} t_{18} t_{22} t_{25}}{t_7 t_8 t_9 t_{10} t_{20} t_{21}}, \frac{t_2 t_3 t_{12} t_{13} t_{15} t_{16} t_{24}}{t_7 t_9 t_{10} t_{11} t_{19} t_{20}}, \frac{t_1 t_5 t_6 t_{14} t_{15} t_{18} t_{24}}{t_7 t_{10} t_{11} t_{20} t_{21} t_{23}} \right. \Big]\Big]$$

To convert back from "t-variables" to parameters (a,b,...) use this substitution......

```
> subt:={seq(t[j]=sqrt(Terms[j]), j=1..nops(Terms))};
```

$$subt := \{t_1 = \sqrt{a}, t_2 = \sqrt{b}, t_3 = \sqrt{c}, t_4 = \sqrt{d}, t_5 = \sqrt{e}, t_6 = \sqrt{f}, t_7 = \sqrt{a+c}, t_8 = \sqrt{a+e}, t_9 = \sqrt{b+d}, t_{10} = \sqrt{b+e}, t_{11} = \sqrt{c+e}, t_{12} = \sqrt{a+b+e}, t_{13} = \sqrt{a+c+e}, t_{14} = \sqrt{b+c+e}, t_{15} = \sqrt{b+d+e}, t_{16} = \sqrt{c+e+f}, t_{17} = \sqrt{a+b+c+e}, t_{18} = \sqrt{a+b+c+2e}, t_{19} = \sqrt{a+b+d+e}, t_{20} = \sqrt{a+2b+c+2e}, t_{21} = \sqrt{b+c+d+e}, t_{22} = \sqrt{a+b+c+d+e}, t_{23} = \sqrt{a+b+c+d+2e}, t_{24} = \sqrt{a+2b+c+d+2e}, t_{25} = \sqrt{b+c+d+e+f}, t_{26} = \sqrt{a+b+c+d+2e+f}\}$$

Now we can apply the moment map to our general spherical point to show that we indeed get aA1+ bA2+cA3+dA4+eA5+fA6=[[a+b+c+e,b+e],[a+b+c+d+2*e+f,b+c+d+e+f,c+e+f,f]]. Here recall that our weights are:

```
> A1;A2;A3;A4;A5;A6;
```

$$[[1, 0], [1, 0, 0, 0]]$$
$$[[1, 1], [1, 1, 0, 0]]$$
$$[[1, 0], [1, 1, 1, 0]]$$
$$[[0, 0], [1, 1, 0, 0]]$$
$$[[1, 1], [2, 1, 1, 0]]$$
$$[[0, 0], [1, 1, 1, 1]]$$

We apply the moment map to zn, convert back to parameters (a,...,f) , simplify and collect terms...

```
> mom(zn):
  subs(subt,%):
  simplify(%):
  collect(%,[X[1,1],X[2,2],Y[1,1],Y[2,2],Y[3,3],Y[4,4]]);
```

$$(a+b+c+e)\,X_{1,1} + (b+e)\,X_{2,2} + (a+b+c+d+2e+f)\,Y_{1,1} + (b+c+d+e+f)\,Y_{2,2} + (c+e+f)\,Y_{3,3} + f\,Y_{4,4}$$

This completes the justification that z:=spt(a,b,c,d,e,f) is indeed a (generalized) spherical point for the weight aA1+bA2+cA3+dA4+eA5+fA6.

%%

Next we evaluate the highest weight vectors h1,...,h6 at our general spherical point z. For this we work in terms of the t's and simplify.

```
> w:=matrix(4,4):
  h1(w);
```

$$w_{1,1}$$

```
> h1(zn);
  h1n:=subs(subt,%);
```

$$\frac{t_1\, t_{12}\, t_{13}\, t_{18}\, t_{22}\, t_{24}\, t_{26}}{t_7\, t_8\, t_{17}\, t_{19}\, t_{20}\, t_{23}}$$

$$h1n := \left(\sqrt{a}\,\sqrt{a+b+e}\,\sqrt{a+c+e}\,\sqrt{a+b+c+2\,e}\,\sqrt{a+b+c+d+e}\,\sqrt{a+2\,b+c+d+2\,e}\,\sqrt{a+b+c+d+2\,e+f}\right) \Big/ \left(\sqrt{a+c}\,\sqrt{a+e}\,\sqrt{a+b+c+e}\,\sqrt{a+b+d+e}\,\sqrt{a+2\,b+c+2\,e}\,\sqrt{a+b+c+d+2\,e}\right)$$

```
> h2(w);
```

$$w_{1,1}\, w_{2,2} - w_{1,2}\, w_{2,1}$$

```
> factor(h2(zn));subs(subt,%);h2n:=factor(%);
```

$$-\frac{t_{12}\, t_{18}\, t_{22}\, t_{24}^2\, t_{26}\, t_2\, t_{14}\, t_{15}\, t_{25}\left(t_1^2\, t_{13}^2 + t_3^2\, t_5^2\right)}{t_7^2\, t_8^2\, t_{17}\, t_{19}\, t_{20}^2\, t_{23}\, t_9\, t_{10}\, t_{21}}$$

$$-\left(\sqrt{a+b+e}\,\sqrt{a+b+c+2\,e}\,\sqrt{a+b+c+d+e}\,(a+2\,b+c+d+2\,e)\,\sqrt{a+b+c+d+2\,e+f}\,\sqrt{b}\,\sqrt{b+c+e}\,\sqrt{b+d+e}\,\sqrt{b+c+d+e+f}\,(a\,(a+c+e)+c\,e)\right) \Big/ \left((a+c)\,(a+e)\,\sqrt{a+b+c+e}\,\sqrt{a+b+d+e}\,(a+2\,b+c+2\,e)\,\sqrt{a+b+c+d+2\,e}\,\sqrt{b+d}\,\sqrt{b+e}\,\sqrt{b+c+d+e}\right)$$

$$h2n := -\left(\sqrt{b+c+d+e+f}\,\sqrt{b+d+e}\,\sqrt{b+c+e}\,\sqrt{b}\,\sqrt{a+b+c+d+2\,e+f}\,(a+2\,b+c+d+2\,e)\,\sqrt{a+b+c+d+e}\,\sqrt{a+b+c+2\,e}\,\sqrt{a+b+e}\right) \Big/ \left(\sqrt{b+c+d+e}\,\sqrt{b+e}\,\sqrt{b+d}\,\sqrt{a+b+c+d+2\,e}\,(a+2\,b+c+2\,e)\,\sqrt{a+b+d+e}\,\sqrt{a+b+c+e}\right)$$

```
> h3(w);
```

$$w_{1,1}\, w_{2,2}\, w_{4,3} - w_{1,1}\, w_{2,3}\, w_{4,2} - w_{2,1}\, w_{1,2}\, w_{4,3} + w_{2,1}\, w_{1,3}\, w_{4,2} + w_{4,1}\, w_{1,2}\, w_{2,3} - w_{4,1}\, w_{1,3}\, w_{2,2}$$

```
> factor(h3(zn));subs(subt,%):h3n:=factor(%);
```

$$\frac{1}{t_7^3\, t_8^2\, t_{17}\, t_{19}^2\, t_{20}^3\, t_{23}\, t_9^2\, t_{10}^2\, t_{21}\, t_{11}}\left(t_{13}\, t_{18}\, t_{22}\, t_{24}\, t_{26}\, t_{14}\, t_{25}\, t_3\, t_{16}\left(-t_1^2\, t_{12}^2\, t_2^2\, t_{13}^2\, t_{15}^2\, t_{24}^2 - t_1^2\, t_{12}^2\, t_4^2\right.\right.$$

$t_5^2\, t_{18}^2\, t_{22}^2 - t_3^2\, t_5^2\, t_{12}^2\, t_{15}^2\, t_2^2\, t_{24}^2 - t_3^2\, t_5^2\, t_{12}^2\, t_{15}^2\, t_4^2\, t_{18}^2 + t_1^2\, t_2^2\, t_4^2\, t_{14}^2\, t_5^2\, t_{22}^2 - t_1^2\, t_2^2\, t_4^2\, t_{14}^2\, t_{13}^2\, t_{15}^2))$

$$h3n := -\left(\sqrt{a+c+e}\,\sqrt{a+b+c+2e}\,\sqrt{a+b+c+d+e}\,\sqrt{a+2b+c+d+2e}\,\sqrt{a+b+c+d+2e+f}\,\sqrt{b+c+e}\,\sqrt{b+c+d+e+f}\,\sqrt{c}\,\sqrt{c+e+f}\right) \Big/ \left(\sqrt{a+c}\,\sqrt{a+2b+c+2e}\,\sqrt{c+e}\,\sqrt{b+c+d+e}\,\sqrt{a+b+c+d+2e}\,\sqrt{a+b+c+e}\right)$$

```
> h4(w);
```

$$w_{1,1}\, w_{3,2} - w_{1,2}\, w_{3,1} + w_{2,1}\, w_{4,2} - w_{2,2}\, w_{4,1}$$

```
> factor(h4(zn));subs(subt,%):h4n:=factor(%);
```

$$\frac{t_{22}\, t_{24}\, t_{26}\, t_4\, t_{15}\, t_{25} \left(t_1^2\, t_{12}^2\, t_{13}^2\, t_{18}^2\, t_{10}^2 + t_2^2\, t_3^2\, t_5^2\, t_{14}^2\, t_{10}^2 + t_3^2\, t_5^2\, t_{12}^2\, t_{18}^2\, t_{17}^2 + t_1^2\, t_2^2\, t_{13}^2\, t_{14}^2\, t_{17}^2\right)}{t_7^2\, t_8^2\, t_{17}^2\, t_{19}\, t_{20}^2\, t_{23}\, t_9\, t_{21}\, t_{10}^2}$$

$$h4n := \left(\sqrt{b+c+d+e+f}\,\sqrt{b+d+e}\,\sqrt{d}\,\sqrt{a+b+c+d+2e+f}\,\sqrt{a+2b+c+d+2e}\,\sqrt{a+b+c+d+e}\right) \Big/ \left(\sqrt{b+c+d+e}\,\sqrt{b+d}\,\sqrt{a+b+c+d+2e}\,\sqrt{a+b+d+e}\right)$$

```
> h5(w);
```

$$-w_{1,1}^2\, w_{2,2}\, w_{3,3} + w_{1,1}^2\, w_{2,3}\, w_{3,2} + w_{1,1}\, w_{3,1}\, w_{1,3}\, w_{2,2} - w_{1,1}\, w_{3,1}\, w_{1,2}\, w_{2,3} + w_{2,1}\, w_{1,1}\, w_{1,2}\, w_{3,3} - w_{2,1}\, w_{1,1}\, w_{1,3}\, w_{3,2} - w_{2,1}\, w_{1,1}\, w_{2,2}\, w_{4,3} + w_{2,1}\, w_{1,1}\, w_{2,3}\, w_{4,2} + w_{2,1}^2\, w_{1,2}\, w_{4,3} - w_{2,1}^2\, w_{1,3}\, w_{4,2} - w_{2,1}\, w_{4,1}\, w_{1,2}\, w_{2,3} + w_{2,1}\, w_{4,1}\, w_{1,3}\, w_{2,2}$$

```
> factor(h5(zn));h5n:=factor(subs(subt,%));
```

$$-\frac{1}{t_7^4\, t_8^3\, t_{17}^3\, t_{19}^3\, t_{20}^4\, t_{23}^2\, t_9^2\, t_{21}\, t_{10}^3\, t_{11}} \Big(t_{12}\, t_{13}\, t_{18}^2\, t_{22}\, t_{24}^2\, t_{26}^2\, t_{14}\, t_{15}\, t_{25}\, t_5\, t_{16} \big(t_1^4\, t_{10}^2\, t_{12}^2\, t_{13}^2\, t_{22}^2\, t_2^2\, t_{24}^2 + t_1^4\, t_{10}^2\, t_{12}^2\, t_{13}^2\, t_{22}^2\, t_4^2\, t_{18}^2 + t_1^2\, t_{10}^2\, t_2^2\, t_3^2\, t_4^2\, t_{14}^2\, t_5^2\, t_{22}^2 - t_1^2\, t_{10}^2\, t_2^2\, t_3^2\, t_4^2\, t_{14}^2\, t_{13}^2\, t_{15}^2 + t_1^2\, t_{10}^2\, t_3^2\, t_{12}^2\, t_2^2\, t_5^2\, t_{22}^2\, t_{24}^2 + t_1^2\, t_{10}^2\, t_3^2\, t_{12}^2\, t_4^2\, t_{13}^2\, t_{15}^2\, t_{18}^2 + t_3^2\, t_{17}^2\, t_1^2\, t_{12}^2\, t_2^2\, t_{13}^2\, t_{15}^2\, t_{24}^2 + t_3^2\, t_{17}^2\, t_1^2\, t_{12}^2\, t_4^2\, t_5^2\, t_{18}^2\, t_{22}^2 + t_3^4\, t_{17}^2\, t_5^2\, t_{12}^2\, t_{15}^2\, t_2^2\, t_{24}^2 + t_3^4\, t_{17}^2\, t_5^2\, t_{12}^2\, t_{15}^2\, t_4^2\, t_{18}^2 - t_3^2\, t_{17}^2\, t_1^2\, t_2^2\, t_4^2\, t_{14}^2\, t_5^2\, t_{22}^2 + t_3^2\, t_{17}^2\, t_1^2\, t_2^2\, t_4^2\, t_{14}^2\, t_{13}^2\, t_{15}^2 \big)\Big)$$

$$h5n := -\Big(\sqrt{c+e+f}\,\sqrt{e}\,\sqrt{b+c+d+e+f}\,\sqrt{b+d+e}\,\sqrt{b+c+e}\,(a+b+c+d+2e+f)\,(a+2b+c+d+2e)\,\sqrt{a+b+c+d+e}\,(a+b+c$$

$$+ 2 e) \sqrt{a + c + e} \sqrt{a + b + e} \Big) \Big/$$
$$\Big(\sqrt{b + e} \sqrt{a + e} \sqrt{a + b + d + e} \sqrt{a + b + c + e} \sqrt{c + e} \sqrt{b + c + d + e} \; (a + b$$
$$+ c + d + 2 e) \; (a + 2 b + c + 2 e) \Big)$$

```
> h6(w);
```

$$w_{1, 1} w_{2, 2} w_{3, 3} w_{4, 4} - w_{1, 1} w_{2, 2} w_{3, 4} w_{4, 3} - w_{1, 1} w_{3, 2} w_{2, 3} w_{4, 4} + w_{1, 1} w_{3, 2} w_{2, 4} w_{4, 3}$$
$$+ w_{1, 1} w_{4, 2} w_{2, 3} w_{3, 4} - w_{1, 1} w_{4, 2} w_{2, 4} w_{3, 3} - w_{2, 1} w_{1, 2} w_{3, 3} w_{4, 4}$$
$$+ w_{2, 1} w_{1, 2} w_{3, 4} w_{4, 3} + w_{2, 1} w_{3, 2} w_{1, 3} w_{4, 4} - w_{2, 1} w_{3, 2} w_{1, 4} w_{4, 3}$$
$$- w_{2, 1} w_{4, 2} w_{1, 3} w_{3, 4} + w_{2, 1} w_{4, 2} w_{1, 4} w_{3, 3} + w_{3, 1} w_{1, 2} w_{2, 3} w_{4, 4}$$
$$- w_{3, 1} w_{1, 2} w_{2, 4} w_{4, 3} - w_{3, 1} w_{2, 2} w_{1, 3} w_{4, 4} + w_{3, 1} w_{2, 2} w_{1, 4} w_{4, 3}$$
$$+ w_{3, 1} w_{4, 2} w_{1, 3} w_{2, 4} - w_{3, 1} w_{4, 2} w_{1, 4} w_{2, 3} - w_{4, 1} w_{1, 2} w_{2, 3} w_{3, 4}$$
$$+ w_{4, 1} w_{1, 2} w_{2, 4} w_{3, 3} + w_{4, 1} w_{2, 2} w_{1, 3} w_{3, 4} - w_{4, 1} w_{2, 2} w_{1, 4} w_{3, 3}$$
$$- w_{4, 1} w_{3, 2} w_{1, 3} w_{2, 4} + w_{4, 1} w_{3, 2} w_{1, 4} w_{2, 3}$$

```
> factor(h6(zn));h6n:=factor(subs(subt,%));
```

$$\frac{1}{t_7^4 t_8^2 t_{17}^2 t_{19}^2 t_{20}^4 t_{23}^2 t_9^2 t_{10}^2 t_{21}^2 t_{11}^2} \Big(t_{26} t_{25} t_{16} t_6 \Big(t_1^4 t_{12}^2 t_{13}^2 t_{18}^2 t_{22}^2 t_{24}^4 t_2^2 t_{14}^2 t_{15}^2 t_5^2 + t_1^2 t_{12}^2 t_{13}^4 t_{18}^2 t_{22}^2$$
$$t_{24}^4 t_2^2 t_{14}^2 t_{15}^2 t_3^2 + t_1^4 t_{12}^2 t_{13}^2 t_{18}^4 t_{22}^2 t_{24}^2 t_4^2 t_{15}^2 t_5^2 t_{14}^2 + t_1^2 t_{12}^4 t_{13}^4 t_{18}^2 t_{22}^2 t_{24}^2 t_4^2 t_{15}^2 t_2^2 t_3^2 + t_1^2 t_{12}^2$$
$$t_{13}^2 t_{18}^4 t_{22}^4 t_{24}^2 t_3^2 t_4^2 t_5^2 t_{14}^2 - t_1^2 t_{12}^4 t_{13}^2 t_{18}^2 t_{22}^4 t_{24}^2 t_3^2 t_4^2 t_5^2 t_2^2 + t_3^2 t_5^4 t_{12}^2 t_{15}^2 t_{18}^2 t_{24}^4 t_2^2 t_{14}^2 t_{22}^2 t_1^2 +$$
$$t_3^4 t_5^2 t_{12}^2 t_{15}^2 t_{18}^2 t_{24}^4 t_2^2 t_{14}^2 t_{22}^2 t_{13}^2 + t_3^2 t_5^2 t_{12}^2 t_{15}^4 t_{18}^4 t_{24}^2 t_1^2 t_4^2 t_{13}^2 t_{14}^2 - t_3^2 t_5^2 t_{12}^4 t_{15}^4 t_{18}^2 t_{24}^2 t_1^2 t_4^2$$
$$t_{13}^2 t_2^2 + t_3^4 t_5^2 t_{12}^2 t_{15}^2 t_{18}^4 t_{24}^2 t_4^2 t_{22}^2 t_{13}^2 t_{14}^2 + t_3^2 t_5^4 t_{12}^4 t_{15}^2 t_{18}^2 t_{24}^2 t_4^2 t_{22}^2 t_1^2 t_2^2 + t_2^2 t_3^2 t_4^2 t_5^4 t_{14}^4 t_{15}^2$$
$$t_{22}^2 t_{24}^2 t_1^2 t_{18}^2 + t_2^4 t_3^4 t_4^2 t_5^2 t_{14}^2 t_{15}^2 t_{22}^2 t_{24}^2 t_{12}^2 t_{13}^2 - t_2^2 t_3^2 t_4^2 t_5^2 t_{14}^4 t_{15}^4 t_1^2 t_{13}^2 t_{24}^2 t_{18}^2 + t_2^4 t_3^2 t_4^2 t_5^2$$
$$t_{14}^2 t_{15}^4 t_1^2 t_{12}^2 t_{13}^2 t_{24}^2 + t_2^2 t_3^2 t_4^4 t_5^4 t_{14}^2 t_{15}^2 t_1^2 t_{12}^2 t_{18}^2 t_{22}^2 + t_2^2 t_3^4 t_4^4 t_5^2 t_{14}^2 t_{15}^2 t_{12}^2 t_{18}^2 t_{22}^2 t_{13}^2 - t_1^2 t_2^2$$
$$t_4^2 t_{13}^2 t_{14}^4 t_{22}^4 t_3^2 t_5^2 t_{24}^2 t_{18}^2 + t_1^2 t_2^4 t_4^2 t_{13}^2 t_{14}^2 t_{22}^4 t_3^2 t_{12}^2 t_5^2 t_{24}^2 + t_1^2 t_2^2 t_4^2 t_{13}^4 t_{14}^4 t_{22}^2 t_{15}^2 t_{24}^2 t_3^2 t_{18}^2 +$$
$$t_1^4 t_2^4 t_4^2 t_{13}^2 t_{14}^2 t_{22}^2 t_{15}^2 t_{24}^2 t_5^2 t_{12}^2 + t_1^4 t_2^2 t_4^4 t_{13}^2 t_{14}^2 t_{22}^2 t_{12}^2 t_{15}^2 t_{18}^2 t_5^2 + t_1^2 t_2^2 t_4^4 t_{13}^4 t_{14}^2 t_{22}^2 t_3^2 t_{12}^2 t_{15}^2$$
$$t_{18}^2 \Big) \Big)$$

$$h6n := \sqrt{f} \sqrt{c + e + f} \sqrt{b + c + d + e + f} \sqrt{a + b + c + d + 2 e + f}$$

These are the formulas given in section 4.10 of our paper. They show that for positive real parameters (a, b,c,d,e,f) each fundamental highest weight vector $h_j(z)$ takes a non-zero value at z=spt(a,b,c,d,e,f). It follows that z lies in the open Borel orbit in V and Corollary 3.4 in our paper implies that (for positive integer values of the parameters a,...,f) the spherical point z is well-adapted to the highest weight vector

$h_1^{\,a}\, h_2^{\,b}\, h_3^{\,c}\, h_4^{\,d}\, h_5^{\,e}\, h_6^{\,f}$. But we can also demonstrate this via direct computation as follows...

The well-adapted property uses directional derivatives of the HWV's evaluated at the spherical points. We simplify using the t-substitution. The i,j'th entry of dkn is the i,j'th derivative of h_k evaluated at the general spherical point (expressed in terms of the t's).

```
> d1n:=matrix(4,4,(i,j)->coeff(coeff(h1(evalm(zn+t*w)),t),w[i,j])):
> matrix(4,4,(i,j)->coeff(coeff(h2(evalm(zn+t*w)),t),w[i,j])):
  d2n:=map(simplify,subs(subt,%)):
> s:='s':matrix(4,4,(i,j)->coeff(coeff(h3(evalm(zn+s*w)),s),w[i,j])):
  d3n:=map(simplify,subs(subt,%)):
> matrix(4,4,(i,j)->coeff(coeff(h4(evalm(zn+s*w)),s),w[i,j])):
  d4n:=map(simplify,subs(subt,%)):
> matrix(4,4,(i,j)->coeff(coeff(h5(evalm(zn+s*w)),s),w[i,j])):
  d5n:=map(simplify,subs(subt,%)):
> matrix(4,4,(i,j)->coeff(coeff(h6(evalm(zn+s*w)),s),w[i,j])):
  d6n:=map(simplify,subs(subt,%)):
```

The well-adapted condition says that the sum of the following six terms is the spherical point z.

```
> term1:=map(simplify,evalm(a/h1n*d1n)):
> term2:=map(simplify,evalm(b/h2n*d2n)):
> term3:=map(simplify,evalm(c/h3n*d3n)):
> term4:=map(simplify,evalm(d/h4n*d4n)):
> term5:=map(simplify,evalm(e/h5n*d5n)):
> term6:=map(simplify,evalm(f/h6n*d6n)):
> ans:=map(factor, evalm(term1+term2+term3+term4+term5+term6));
```

$ans :=\Big[$

$\Big[\big(\sqrt{a}\,\sqrt{a+b+e}\,\sqrt{a+c+e}\,\sqrt{a+b+c+2\,e}\,\sqrt{a+b+c+d+e}$

$\sqrt{a+2\,b+c+d+2\,e}\,\sqrt{a+b+c+d+2\,e+f}\big)\Big/$

$\big(\sqrt{a+c}\,\sqrt{a+e}\,\sqrt{a+b+c+e}\,\sqrt{a+b+d+e}\,\sqrt{a+2\,b+c+2\,e}$

$\sqrt{a+b+c+d+2\,e}\big),$

$-\big(\sqrt{b}\,\sqrt{c}\,\sqrt{e}\,\sqrt{b+c+e}\,\sqrt{a+b+c+d+e}\,\sqrt{a+2\,b+c+d+2\,e}$

$\sqrt{b+c+d+e+f}\big)\Big/$

$\big(\sqrt{a+c}\,\sqrt{a+e}\,\sqrt{b+d}\,\sqrt{a+b+c+e}\,\sqrt{a+2\,b+c+2\,e}\,\sqrt{b+c+d+e}\big),$

$$-\left(\sqrt{d}\,\sqrt{c}\,\sqrt{b+c+e}\,\sqrt{a+c+e}\,\sqrt{a+b+c+2e}\,\sqrt{c+e+f}\,\sqrt{b+d+e}\right)$$

$$\Big/\left(\sqrt{a+c}\,\sqrt{a+2b+c+2e}\,\sqrt{a+b+d+e}\,\sqrt{b+d}\,\sqrt{a+b+c+e}\,\sqrt{c+e}\right),$$

$$-\left(\sqrt{f}\,\sqrt{a+b+e}\,\sqrt{e}\,\sqrt{b+d+e}\,\sqrt{d}\,\sqrt{b}\,\sqrt{a}\right)\Big/$$
$$\left(\sqrt{a+c}\,\sqrt{a+b+c+e}\,\sqrt{a+2b+c+2e}\,\sqrt{a+b+c+d+2e}\,\sqrt{b+c+d+e}\right.$$
$$\left.\sqrt{c+e}\right)\Big],$$

$$\Big[$$

$$-\left(\sqrt{c}\,\sqrt{e}\,\sqrt{a+b+e}\,\sqrt{b+d+e}\,\sqrt{a+b+c+2e}\,\sqrt{a+2b+c+d+2e}\right.$$

$$\left.\sqrt{a+b+c+d+2e+f}\right)\Big/$$

$$\left(\sqrt{a+c}\,\sqrt{a+e}\,\sqrt{b+e}\,\sqrt{a+b+d+e}\,\sqrt{a+2b+c+2e}\right.$$

$$\left.\sqrt{a+b+c+d+2e}\right),$$

$$-\left(\sqrt{a}\,\sqrt{b}\,\sqrt{a+c+e}\,\sqrt{b+c+e}\,\sqrt{b+d+e}\,\sqrt{a+2b+c+d+2e}\right.$$

$$\left.\sqrt{b+c+d+e+f}\right)\Big/$$

$$\left(\sqrt{a+c}\,\sqrt{a+e}\,\sqrt{b+d}\,\sqrt{b+e}\,\sqrt{a+2b+c+2e}\,\sqrt{b+c+d+e}\right),$$

$$-\frac{\sqrt{a}\,\sqrt{d}\,\sqrt{e}\,\sqrt{c+e+f}\,\sqrt{b+c+e}\,\sqrt{a+b+c+2e}\,\sqrt{a+b+c+d+e}}{\sqrt{a+c}\,\sqrt{b+d}\,\sqrt{b+e}\,\sqrt{a+b+d+e}\,\sqrt{a+2b+c+2e}\,\sqrt{c+e}},$$
$$\frac{\sqrt{f}\,\sqrt{a+b+e}\,\sqrt{c}\,\sqrt{a+b+c+d+e}\,\sqrt{a+c+e}\,\sqrt{d}\,\sqrt{b}}{\sqrt{a+c}\,\sqrt{b+e}\,\sqrt{c+e}\,\sqrt{a+2b+c+2e}\,\sqrt{b+c+d+e}\,\sqrt{a+b+c+d+2e}}$$

$$\Big],$$

$$\left[\left(\sqrt{b}\,\sqrt{c}\,\sqrt{d}\,\sqrt{e}\,\sqrt{b+c+e}\,\sqrt{b+d+e}\,\sqrt{a+b+c+d+2e+f}\right)\Big/\right.$$

$$\left(\sqrt{a+c}\,\sqrt{a+e}\,\sqrt{a+b+c+e}\,\sqrt{a+b+d+e}\,\sqrt{a+2b+c+2e}\right.$$

$$\left.\sqrt{a+b+c+d+2e}\right),$$

$$\left(\sqrt{a}\,\sqrt{d}\,\sqrt{a+b+e}\,\sqrt{a+c+e}\,\sqrt{b+d+e}\,\sqrt{a+b+c+2e}\right.$$

$$\left.\sqrt{b+c+d+e+f}\right)\Big/$$

$$\left(\sqrt{a+c}\,\sqrt{a+e}\,\sqrt{b+d}\,\sqrt{a+b+c+e}\,\sqrt{a+2b+c+2e}\,\sqrt{b+c+d+e}\right),$$

$$-\left(\sqrt{e}\,\sqrt{a+2b+c+d+2e}\,\sqrt{a+b+c+d+e}\,\sqrt{a+b+e}\,\sqrt{b}\,\sqrt{a}\right.$$

$$\left.\sqrt{c+e+f}\right)\Big/$$

$$\left(\sqrt{a+c}\,\sqrt{b+d}\,\sqrt{c+e}\,\sqrt{a+b+c+e}\,\sqrt{a+b+d+e}\,\sqrt{a+2b+c+2e}\right),$$

$$\left(\sqrt{f}\,\sqrt{a+c+e}\,\sqrt{a+b+c+2e}\,\sqrt{a+b+c+d+e}\,\sqrt{a+2b+c+d+2e}\right.$$

$$\left.\sqrt{b+c+e}\,\sqrt{c}\right)\Big/$$

$$\left(\sqrt{a+c}\,\sqrt{a+2b+c+2e}\,\sqrt{c+e}\,\sqrt{b+c+d+e}\,\sqrt{a+b+c+d+2e}\right.$$

$$\left.\left.\sqrt{a+b+c+e}\right)\right],$$

$$\left[\left(\sqrt{a}\,\sqrt{b}\,\sqrt{d}\,\sqrt{a+c+e}\,\sqrt{b+c+e}\,\sqrt{a+b+c+d+e}\right.\right.$$

$$\left.\sqrt{a+b+c+d+2e+f}\right)\Big/$$

$$\left(\sqrt{a+c}\,\sqrt{a+e}\,\sqrt{b+e}\,\sqrt{a+b+d+e}\,\sqrt{a+2b+c+2e}\right.$$

$$\left.\sqrt{a+b+c+d+2e}\right),$$

$$-\left(\sqrt{c}\,\sqrt{d}\,\sqrt{e}\,\sqrt{a+b+e}\,\sqrt{a+b+c+2e}\,\sqrt{a+b+c+d+e}\right.$$

$$\left.\sqrt{b+c+d+e+f}\right) \Big/$$

$$\left(\sqrt{a+c}\,\sqrt{a+e}\,\sqrt{b+d}\,\sqrt{b+e}\,\sqrt{a+2b+c+2e}\,\sqrt{b+c+d+e}\right),$$

$$\frac{\sqrt{c}\,\sqrt{a+2b+c+d+2e}\,\sqrt{b+d+e}\,\sqrt{b}\,\sqrt{a+b+e}\,\sqrt{c+e+f}\,\sqrt{a+c+e}}{\sqrt{a+2b+c+2e}\,\sqrt{b+e}\,\sqrt{b+d}\,\sqrt{a+b+d+e}\,\sqrt{a+c}\,\sqrt{c+e}},$$

$$\frac{\sqrt{f}\,\sqrt{e}\,\sqrt{b+d+e}\,\sqrt{b+c+e}\,\sqrt{a+2b+c+d+2e}\,\sqrt{a+b+c+2e}\,\sqrt{a}}{\sqrt{a+c}\,\sqrt{b+e}\,\sqrt{c+e}\,\sqrt{a+2b+c+2e}\,\sqrt{b+c+d+e}\,\sqrt{a+b+c+d+2e}}$$

]]

The following code checks that this **ans** does equal the spherical point **z** by comparing the squares and signs of each entry in **ans** and **z**.

```
> for i from 1 to 4 do;
      for j from 1 to 4 do;
          simplify(ans[i,j]^2/z[i,j]^2);
          print(%);
      end do;
  end do;
```

1

1

1

1

1

1

1

1

1

1

1

1

1

1

1

$$1$$

```
> evalm(map(sign,z)-map(sign,ans));
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

%%

Our results hold for all positive reals parameters a...f. To complete the verification that our action is well-behaved we must also consider non-generic spherical points for which one or more coefficient a...f is zero. Here we apply Lemma 3.5 from our paper. First we check condition (3) in the Lemma: limits of (generalized) spherical points exist if we take some variables to zero. Since the moment map is continuous, these limits are also (generalized) spherical points....

First, we generate all possible ways the variables can go to zero. We eliminate the empty set. There are 63 non-empty subsets of {a=0, b=0, c=0, d=0, e=0, f=0}.

```
> with(combinat):
> zs:=[a=0,b=0,c=0,d=0,e=0,f=0];
  ch:=[seq(choose(%)[i],i=2..64)]:
  :nops(%);
```

$$zs := [a = 0, b = 0, c = 0, d = 0, e = 0, f = 0]$$

$$63$$

The following code lists each possible setting of parameters a...f to zero and prints the limiting (generalized) spherical point. This verifies condition (3) in Lemma 3.5. In particular no singularities arise as we perfom these limits via setting parameters to zero in succession.

```
> for i from 1 to nops(ch) do:
      print(ch[i]);
      zf:=evalm(z):
      for j from 1 to nops(ch[i]) do:
          zf:=subs(ch[i][j],evalm(zf)):
      end do:
      print(zf);
  end do:
```

$$[a = 0]$$

$$\left[\left[0, -\sqrt{\frac{b\,(2\,b+d+2\,e+c)\,(b+c+d+e+f)}{(b+d)\,(2\,b+c+2\,e)}}, -\sqrt{\frac{d\,(b+c+2\,e)\,(c+e+f)}{(b+d)\,(2\,b+c+2\,e)}},\right.\right.$$

$$\left.0\right],$$

$$\left[-\sqrt{\frac{(b+c+2\,e)\,(2\,b+d+2\,e+c)\,(f+c+b+d+2\,e)}{(2\,b+c+2\,e)\,(2\,e+b+c+d)}}, 0, 0,\right.$$

$$\sqrt{\frac{b\,d\,f}{(2\,b+c+2\,e)\,(2\,e+b+c+d)}}\Bigg],$$

$$\Bigg[\sqrt{\frac{b\,d\,(f+c+b+d+2\,e)}{(2\,b+c+2\,e)\,(2\,e+b+c+d)}},0,0,\sqrt{\frac{f\,(b+c+2\,e)\,(2\,b+d+2\,e+c)}{(2\,b+c+2\,e)\,(2\,e+b+c+d)}}$$

$$\Bigg],$$

$$\Bigg[0,-\sqrt{\frac{d\,(b+c+2\,e)\,(b+c+d+e+f)}{(b+d)\,(2\,b+c+2\,e)}},$$

$$\sqrt{\frac{b\,(2\,b+d+2\,e+c)\,(c+e+f)}{(b+d)\,(2\,b+c+2\,e)}},0\Bigg]\Bigg]$$

$$[b=0]$$

$$\Bigg[\Bigg[\sqrt{\frac{a\,(d+e+a+c)\,(d+2\,e+f+a+c)}{(a+c)\,(a+d+e)}},0,-\sqrt{\frac{c\,(d+e)\,(c+e+f)}{(a+c)\,(a+d+e)}},0\Bigg],$$

$$\Bigg[-\sqrt{\frac{c\,(d+e)\,(d+2\,e+f+a+c)}{(a+c)\,(a+d+e)}},0,-\sqrt{\frac{a\,(d+e+a+c)\,(c+e+f)}{(a+c)\,(a+d+e)}},0\Bigg],$$

$$\Bigg[0,\sqrt{\frac{a\,(d+e)\,(d+e+f+c)}{(a+c)\,(d+c+e)}},0,\sqrt{\frac{c\,f\,(d+e+a+c)}{(a+c)\,(d+c+e)}}\Bigg],$$

$$\Bigg[0,-\sqrt{\frac{c\,(d+e+a+c)\,(d+e+f+c)}{(a+c)\,(d+c+e)}},0,\sqrt{\frac{a\,f\,(d+e)}{(a+c)\,(d+c+e)}}\Bigg]\Bigg]$$

$$[c=0]$$

$$\Bigg[\Bigg[\sqrt{\frac{(a+b+2\,e)\,(a+2\,b+d+2\,e)\,(a+b+d+2\,e+f)}{(a+2\,b+2\,e)\,(a+b+d+2\,e)}},0,0,$$

$$-\sqrt{\frac{b\,d\,f}{(a+2\,b+2\,e)\,(a+b+d+2\,e)}}\Bigg],$$

$$\Bigg[0,-\sqrt{\frac{b\,(a+2\,b+d+2\,e)\,(b+d+e+f)}{(b+d)\,(a+2\,b+2\,e)}},-\sqrt{\frac{d\,(a+b+2\,e)\,(e+f)}{(b+d)\,(a+2\,b+2\,e)}},0\Bigg],$$

$$\Bigg[0,\sqrt{\frac{d\,(a+b+2\,e)\,(b+d+e+f)}{(b+d)\,(a+2\,b+2\,e)}},-\sqrt{\frac{b\,(a+2\,b+d+2\,e)\,(e+f)}{(b+d)\,(a+2\,b+2\,e)}},0\Bigg],$$

$$\Bigg[\sqrt{\frac{b\,d\,(a+b+d+2\,e+f)}{(a+2\,b+2\,e)\,(a+b+d+2\,e)}},0,0,\sqrt{\frac{f\,(a+b+2\,e)\,(a+2\,b+d+2\,e)}{(a+2\,b+2\,e)\,(a+b+d+2\,e)}}$$

$$\Bigg]\Bigg]$$

$$[d=0]$$

$$\left[\left[\sqrt{\frac{a\,(a+c+e)\,(a+b+c+2\,e+f)}{(a+c)\,(a+e)}}, -\sqrt{\frac{c\,e\,(b+c+e+f)}{(a+c)\,(a+e)}}, 0, 0\right],\right.$$

$$\left[-\sqrt{\frac{c\,e\,(a+b+c+2\,e+f)}{(a+c)\,(a+e)}}, -\sqrt{\frac{a\,(a+c+e)\,(b+c+e+f)}{(a+c)\,(a+e)}}, 0, 0\right],$$

$$\left[0, 0, -\sqrt{\frac{a\,e\,(c+e+f)}{(a+c)\,(c+e)}}, \sqrt{\frac{c\,f\,(a+c+e)}{(a+c)\,(c+e)}}\right],$$

$$\left.\left[0, 0, \sqrt{\frac{c\,(a+c+e)\,(c+e+f)}{(a+c)\,(c+e)}}, \sqrt{\frac{a\,e\,f}{(a+c)\,(c+e)}}\right]\right]$$

$$[e=0]$$

$$\left[\left[\sqrt{\frac{(a+b)\,(a+2\,b+c+d)\,(a+b+c+d+f)}{(a+b+d)\,(a+2\,b+c)}}, 0,\right.\right.$$

$$\left.-\sqrt{\frac{d\,(b+c)\,(c+f)}{(a+2\,b+c)\,(a+b+d)}}, 0\right],$$

$$\left[0, -\sqrt{\frac{(a+2\,b+c+d)\,(b+c)\,(b+c+d+f)}{(a+2\,b+c)\,(b+c+d)}}, 0,\right.$$

$$\left.\sqrt{\frac{d\,f\,(a+b)}{(a+2\,b+c)\,(b+c+d)}}\right],$$

$$\left[0, \sqrt{\frac{d\,(a+b)\,(b+c+d+f)}{(a+2\,b+c)\,(b+c+d)}}, 0, \sqrt{\frac{f\,(a+2\,b+c+d)\,(b+c)}{(a+2\,b+c)\,(b+c+d)}}\right],$$

$$\left.\left[\sqrt{\frac{d\,(b+c)\,(a+b+c+d+f)}{(a+2\,b+c)\,(a+b+d)}}, 0, \sqrt{\frac{(a+2\,b+c+d)\,(a+b)\,(c+f)}{(a+2\,b+c)\,(a+b+d)}}, 0\right]\right]$$

$$[f=0]$$

$$\left[\left[((a\,(a+b+e)\,(a+c+e)\,(a+b+c+2\,e)\,(a+2\,b+c+d+2\,e)\,(a+b+c+d\right.\right.$$

$$+e))\Big/((a+c)\,(a+e)\,(a+b+c+e)\,(a+b+d+e)\,(a+2\,b+c+2\,e)))^{1/2},$$

$$-\sqrt{\frac{b\,c\,e\,(a+b+c+d+e)\,(a+2\,b+c+d+2\,e)\,(b+c+e)}{(a+c)\,(a+e)\,(b+d)\,(a+b+c+e)\,(a+2\,b+c+2\,e)}},$$

$$\left.-\sqrt{\frac{c\,d\,(a+b+c+2\,e)\,(a+c+e)\,(b+c+e)\,(b+d+e)}{(a+c)\,(b+d)\,(a+b+c+e)\,(a+2\,b+c+2\,e)\,(a+b+d+e)}}, 0\right],$$

$$\left[-\sqrt{\frac{c\,e\,(a+b+e)\,(a+b+c+2\,e)\,(a+2\,b+c+d+2\,e)\,(b+d+e)}{(a+c)\,(a+e)\,(b+e)\,(a+2\,b+c+2\,e)\,(a+b+d+e)}},\right.$$

$$-\sqrt{\frac{a\,b\,(a+2\,b+c+d+2\,e)\,(a+c+e)\,(b+c+e)\,(b+d+e)}{(a+c)\,(a+e)\,(b+d)\,(b+e)\,(a+2\,b+c+2\,e)}},$$

$$-\sqrt{\frac{a\,d\,e\,(a+b+c+2\,e)\,(a+b+c+d+e)\,(b+c+e)}{(a+c)\,(b+d)\,(b+e)\,(a+2\,b+c+2\,e)\,(a+b+d+e)}},0\Bigg],$$

$$\Bigg[\sqrt{\frac{b\,c\,d\,e\,(b+c+e)\,(b+d+e)}{(a+c)\,(a+e)\,(a+b+c+e)\,(a+b+d+e)\,(a+2\,b+c+2\,e)}},$$

$$\sqrt{\frac{a\,d\,(a+b+e)\,(a+b+c+2\,e)\,(a+c+e)\,(b+d+e)}{(a+c)\,(a+e)\,(b+d)\,(a+b+c+e)\,(a+2\,b+c+2\,e)}},$$

$$-\sqrt{\frac{a\,b\,e\,(a+b+c+d+e)\,(a+2\,b+c+d+2\,e)\,(a+b+e)}{(a+c)\,(b+d)\,(a+b+c+e)\,(a+2\,b+c+2\,e)\,(a+b+d+e)}},0\Bigg],$$

$$\Bigg[\sqrt{\frac{a\,b\,d\,(a+b+c+d+e)\,(a+c+e)\,(b+c+e)}{(a+c)\,(a+e)\,(b+e)\,(a+2\,b+c+2\,e)\,(a+b+d+e)}},$$

$$-\sqrt{\frac{c\,d\,e\,(a+b+c+2\,e)\,(a+b+c+d+e)\,(a+b+e)}{(a+c)\,(a+e)\,(b+d)\,(b+e)\,(a+2\,b+c+2\,e)}},$$

$$\sqrt{\frac{b\,c\,(a+2\,b+c+d+2\,e)\,(a+b+e)\,(a+c+e)\,(b+d+e)}{(a+c)\,(b+d)\,(b+e)\,(a+2\,b+c+2\,e)\,(a+b+d+e)}},0\Bigg]\Bigg]$$

$$[a=0, b=0]$$

$$\begin{bmatrix} 0 & 0 & -\sqrt{c+e+f} & 0 \\ -\sqrt{f+c+d+2\,e} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \\ 0 & -\sqrt{d+e+f+c} & 0 & 0 \end{bmatrix}$$

$$[a=0, c=0]$$

$$\Bigg[\Bigg[0, -\sqrt{\frac{b\,(2\,b+d+2\,e)\,(b+d+e+f)}{(b+d)\,(2\,b+2\,e)}}, -\sqrt{\frac{d\,(b+2\,e)\,(e+f)}{(b+d)\,(2\,b+2\,e)}},0\Bigg],$$

$$\Bigg[-\sqrt{\frac{(b+2\,e)\,(2\,b+d+2\,e)\,(f+b+d+2\,e)}{(2\,b+2\,e)\,(b+d+2\,e)}},0,0,$$

$$\sqrt{\frac{b\,d\,f}{(2\,b+2\,e)\,(b+d+2\,e)}}\Bigg],$$

$$\Bigg[\sqrt{\frac{b\,d\,(f+b+d+2\,e)}{(2\,b+2\,e)\,(b+d+2\,e)}},0,0,\sqrt{\frac{f\,(b+2\,e)\,(2\,b+d+2\,e)}{(2\,b+2\,e)\,(b+d+2\,e)}}\Bigg],$$

$$\Bigg[0, -\sqrt{\frac{d\,(b+2\,e)\,(b+d+e+f)}{(b+d)\,(2\,b+2\,e)}},\sqrt{\frac{b\,(2\,b+d+2\,e)\,(e+f)}{(b+d)\,(2\,b+2\,e)}},0\Bigg]\Bigg]$$

$$[a=0, d=0]$$

$$\begin{bmatrix} 0 & -\sqrt{b+c+e+f} & 0 & 0 \\ -\sqrt{f+c+b+2e} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \\ 0 & 0 & \sqrt{c+e+f} & 0 \end{bmatrix}$$

$$[a=0, e=0]$$

$$\left[\left[0, -\sqrt{\frac{b(2b+d+c)(b+c+d+f)}{(b+d)(2b+c)}}, -\sqrt{\frac{d(b+c)(c+f)}{(b+d)(2b+c)}}, 0\right],\right.$$
$$\left[-\sqrt{\frac{(b+c)(2b+d+c)(b+c+d+f)}{(2b+c)(b+c+d)}}, 0, 0, \sqrt{\frac{b\,d\,f}{(2b+c)(b+c+d)}}\right],$$
$$\left[\sqrt{\frac{b\,d(b+c+d+f)}{(2b+c)(b+c+d)}}, 0, 0, \sqrt{\frac{f(b+c)(2b+d+c)}{(2b+c)(b+c+d)}}\right],$$
$$\left.\left[0, -\sqrt{\frac{d(b+c)(b+c+d+f)}{(b+d)(2b+c)}}, \sqrt{\frac{b(2b+d+c)(c+f)}{(b+d)(2b+c)}}, 0\right]\right]$$

$$[a=0, f=0]$$

$$\left[\left[0, -\sqrt{\frac{b(2b+d+2e+c)(b+c+d+e)}{(b+d)(2b+c+2e)}}, -\sqrt{\frac{d(b+c+2e)(c+e)}{(b+d)(2b+c+2e)}}, 0\right],\right.$$
$$\left[-\sqrt{\frac{(b+c+2e)(2b+d+2e+c)}{2b+c+2e}}, 0, 0, 0\right],$$
$$\left[\sqrt{\frac{b\,d}{2b+c+2e}}, 0, 0, 0\right],$$
$$\left.\left[0, -\sqrt{\frac{d(b+c+2e)(b+c+d+e)}{(b+d)(2b+c+2e)}}, \sqrt{\frac{b(2b+d+2e+c)(c+e)}{(b+d)(2b+c+2e)}}, 0\right]\right]$$

$$[b=0, c=0]$$

$$\begin{bmatrix} \sqrt{d+2e+f+a} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{e+f} & 0 \\ 0 & \sqrt{d+e+f} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \end{bmatrix}$$

$$[b=0, d=0]$$

$$\left[\left[\sqrt{\frac{a(a+c+e)(2e+f+a+c)}{(a+c)(a+e)}}, 0, -\sqrt{\frac{c\,e(c+e+f)}{(a+c)(a+e)}}, 0\right],\right.$$

$$\left[-\sqrt{\frac{c\,e\,(2\,e+f+a+c)}{(a+c)\,(a+e)}},0,-\sqrt{\frac{a\,(a+c+e)\,(c+e+f)}{(a+c)\,(a+e)}},0\right],$$

$$\left[0,\sqrt{\frac{a\,e\,(c+e+f)}{(a+c)\,(c+e)}},0,\sqrt{\frac{c\,f\,(a+c+e)}{(a+c)\,(c+e)}}\right],$$

$$\left[0,-\sqrt{\frac{c\,(a+c+e)\,(c+e+f)}{(a+c)\,(c+e)}},0,\sqrt{\frac{a\,e\,f}{(a+c)\,(c+e)}}\right]\right]$$

$$[b=0,e=0]$$

$$\left[\left[\sqrt{\frac{a\,(d+a+c)\,(d+f+a+c)}{(a+c)\,(a+d)}},0,-\sqrt{\frac{c\,d\,(c+f)}{(a+c)\,(a+d)}},0\right],\right.$$

$$\left[-\sqrt{\frac{c\,d\,(d+f+a+c)}{(a+c)\,(a+d)}},0,-\sqrt{\frac{a\,(d+a+c)\,(c+f)}{(a+c)\,(a+d)}},0\right],$$

$$\left[0,\sqrt{\frac{a\,d\,(d+f+c)}{(a+c)\,(d+c)}},0,\sqrt{\frac{c\,f\,(d+a+c)}{(a+c)\,(d+c)}}\right],$$

$$\left.\left[0,-\sqrt{\frac{c\,(d+a+c)\,(d+f+c)}{(a+c)\,(d+c)}},0,\sqrt{\frac{a\,f\,d}{(a+c)\,(d+c)}}\right]\right]$$

$$[b=0,f=0]$$

$$\left[\left[\sqrt{\frac{a\,(d+e+a+c)\,(a+c+d+2\,e)}{(a+c)\,(a+d+e)}},0,-\sqrt{\frac{c\,(d+e)\,(c+e)}{(a+c)\,(a+d+e)}},0\right],\right.$$

$$\left[-\sqrt{\frac{c\,(d+e)\,(a+c+d+2\,e)}{(a+c)\,(a+d+e)}},0,-\sqrt{\frac{a\,(d+e+a+c)\,(c+e)}{(a+c)\,(a+d+e)}},0\right],$$

$$\left[0,\sqrt{\frac{a\,(d+e)}{a+c}},0,0\right],$$

$$\left.\left[0,-\sqrt{\frac{c\,(d+e+a+c)}{a+c}},0,0\right]\right]$$

$$[c=0,d=0]$$

$$\begin{bmatrix}\sqrt{a+b+2\,e+f} & 0 & 0 & 0\\ 0 & -\sqrt{b+e+f} & 0 & 0\\ 0 & 0 & -\sqrt{e+f} & 0\\ 0 & 0 & 0 & \sqrt{f}\end{bmatrix}$$

$$[c=0,e=0]$$

$$\left[\left[\sqrt{\frac{(a+b)\,(a+2\,b+d)\,(a+b+d+f)}{(a+2\,b)\,(a+b+d)}},0,0,-\sqrt{\frac{b\,d\,f}{(a+2\,b)\,(a+b+d)}}\right],\right.$$

$$\left[0, -\sqrt{\frac{b\,(a+2\,b+d)\,(b+d+f)}{(b+d)\,(a+2\,b)}}, -\sqrt{\frac{d\,(a+b)\,f}{(b+d)\,(a+2\,b)}}, 0\right],$$

$$\left[0, \sqrt{\frac{d\,(a+b)\,(b+d+f)}{(b+d)\,(a+2\,b)}}, -\sqrt{\frac{b\,(a+2\,b+d)\,f}{(b+d)\,(a+2\,b)}}, 0\right],$$

$$\left[\sqrt{\frac{b\,d\,(a+b+d+f)}{(a+2\,b)\,(a+b+d)}}, 0, 0, \sqrt{\frac{f\,(a+b)\,(a+2\,b+d)}{(a+2\,b)\,(a+b+d)}}\right]\Bigg]$$

$$[c=0, f=0]$$

$$\left[\left[\sqrt{\frac{(a+b+2\,e)\,(a+2\,b+d+2\,e)}{a+2\,b+2\,e}}, 0, 0, 0\right],\right.$$

$$\left[0, -\sqrt{\frac{b\,(a+2\,b+d+2\,e)\,(b+d+e)}{(b+d)\,(a+2\,b+2\,e)}}, -\sqrt{\frac{d\,(a+b+2\,e)\,e}{(b+d)\,(a+2\,b+2\,e)}}, 0\right],$$

$$\left[0, \sqrt{\frac{d\,(a+b+2\,e)\,(b+d+e)}{(b+d)\,(a+2\,b+2\,e)}}, -\sqrt{\frac{b\,(a+2\,b+d+2\,e)\,e}{(b+d)\,(a+2\,b+2\,e)}}, 0\right],$$

$$\left.\left[\sqrt{\frac{b\,d}{a+2\,b+2\,e}}, 0, 0, 0\right]\right]$$

$$[d=0, e=0]$$

$$\begin{bmatrix} \sqrt{a+b+c+f} & 0 & 0 & 0 \\ 0 & -\sqrt{b+c+f} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \\ 0 & 0 & \sqrt{c+f} & 0 \end{bmatrix}$$

$$[d=0, f=0]$$

$$\left[\left[\sqrt{\frac{a\,(a+c+e)\,(a+b+c+2\,e)}{(a+c)\,(a+e)}}, -\sqrt{\frac{c\,e\,(b+c+e)}{(a+c)\,(a+e)}}, 0, 0\right],\right.$$

$$\left[-\sqrt{\frac{c\,e\,(a+b+c+2\,e)}{(a+c)\,(a+e)}}, -\sqrt{\frac{a\,(a+c+e)\,(b+c+e)}{(a+c)\,(a+e)}}, 0, 0\right],$$

$$\left[0, 0, -\sqrt{\frac{a\,e}{a+c}}, 0\right],$$

$$\left.\left[0, 0, \sqrt{\frac{c\,(a+c+e)}{a+c}}, 0\right]\right]$$

$$[e=0, f=0]$$

$$\left[\left[\sqrt{\frac{(a+b)\,(a+2\,b+c+d)\,(a+b+c+d)}{(a+b+d)\,(a+2\,b+c)}}, 0, -\sqrt{\frac{d\,(b+c)\,c}{(a+2\,b+c)\,(a+b+d)}}, 0\right.\right.$$

$$\Bigg],$$

$$\left[0, -\sqrt{\frac{(a+2b+c+d)\,(b+c)}{a+2b+c}}, 0, 0\right],$$

$$\left[0, \sqrt{\frac{d\,(a+b)}{a+2b+c}}, 0, 0\right],$$

$$\left[\sqrt{\frac{d\,(b+c)\,(a+b+c+d)}{(a+2b+c)\,(a+b+d)}}, 0, \sqrt{\frac{(a+2b+c+d)\,(a+b)\,c}{(a+2b+c)\,(a+b+d)}}, 0\right]\Bigg]$$

$$[a=0, b=0, c=0]$$

$$\begin{bmatrix} 0 & 0 & -\sqrt{e+f} & 0 \\ -\sqrt{f+d+2e} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \\ 0 & -\sqrt{d+e+f} & 0 & 0 \end{bmatrix}$$

$$[a=0, b=0, d=0]$$

$$\begin{bmatrix} 0 & 0 & -\sqrt{c+e+f} & 0 \\ -\sqrt{f+c+2e} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \\ 0 & -\sqrt{c+e+f} & 0 & 0 \end{bmatrix}$$

$$[a=0, b=0, e=0]$$

$$\begin{bmatrix} 0 & 0 & -\sqrt{c+f} & 0 \\ -\sqrt{d+f+c} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \\ 0 & -\sqrt{d+f+c} & 0 & 0 \end{bmatrix}$$

$$[a=0, b=0, f=0]$$

$$\begin{bmatrix} 0 & 0 & -\sqrt{c+e} & 0 \\ -\sqrt{d+2e+c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{d+c+e} & 0 & 0 \end{bmatrix}$$

$$[a=0, c=0, d=0]$$

$$\begin{bmatrix} 0 & -\sqrt{b+e+f} & 0 & 0 \\ -\sqrt{b+2e+f} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \\ 0 & 0 & \sqrt{e+f} & 0 \end{bmatrix}$$

$$[a=0, c=0, e=0]$$

$$\left[\left[0, -\frac{1}{2}\sqrt{2}\sqrt{\frac{(2b+d)(b+d+f)}{b+d}}, -\frac{1}{2}\sqrt{2}\sqrt{\frac{df}{b+d}}, 0\right],\right.$$
$$\left[-\frac{1}{2}\sqrt{2}\sqrt{\frac{(2b+d)(b+d+f)}{b+d}}, 0, 0, \frac{1}{2}\sqrt{2}\sqrt{\frac{df}{b+d}}\right],$$
$$\left[\frac{1}{2}\sqrt{2}\sqrt{\frac{d(b+d+f)}{b+d}}, 0, 0, \frac{1}{2}\sqrt{2}\sqrt{\frac{(2b+d)f}{b+d}}\right],$$
$$\left.\left[0, -\frac{1}{2}\sqrt{2}\sqrt{\frac{d(b+d+f)}{b+d}}, \frac{1}{2}\sqrt{2}\sqrt{\frac{(2b+d)f}{b+d}}, 0\right]\right]$$

$$[a=0, c=0, f=0]$$

$$\left[\left[0, -\sqrt{\frac{b(2b+d+2e)(b+d+e)}{(b+d)(2b+2e)}}, -\sqrt{\frac{d(b+2e)e}{(b+d)(2b+2e)}}, 0\right],\right.$$
$$\left[-\sqrt{\frac{(b+2e)(2b+d+2e)}{2b+2e}}, 0, 0, 0\right],$$
$$\left[\sqrt{\frac{bd}{2b+2e}}, 0, 0, 0\right],$$
$$\left.\left[0, -\sqrt{\frac{d(b+2e)(b+d+e)}{(b+d)(2b+2e)}}, \sqrt{\frac{b(2b+d+2e)e}{(b+d)(2b+2e)}}, 0\right]\right]$$

$$[a=0, d=0, e=0]$$

$$\begin{bmatrix} 0 & -\sqrt{b+c+f} & 0 & 0 \\ -\sqrt{b+c+f} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \\ 0 & 0 & \sqrt{c+f} & 0 \end{bmatrix}$$

$$[a=0, d=0, f=0]$$

$$\begin{bmatrix} 0 & -\sqrt{b+c+e} & 0 & 0 \\ -\sqrt{b+c+2e} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{c+e} & 0 \end{bmatrix}$$

$$[a=0, e=0, f=0]$$

$$\left[\left[0, -\sqrt{\frac{b(2b+d+c)(b+c+d)}{(b+d)(2b+c)}}, -\sqrt{\frac{d(b+c)c}{(b+d)(2b+c)}}, 0\right], \left[-\sqrt{\frac{(b+c)(2b+d+c)}{2b+c}}, 0, 0, 0\right], \left[\sqrt{\frac{bd}{2b+c}}, 0, 0, 0\right], \left[0, -\sqrt{\frac{d(b+c)(b+c+d)}{(b+d)(2b+c)}}, \sqrt{\frac{b(2b+d+c)c}{(b+d)(2b+c)}}, 0\right]\right]$$

$$[b=0, c=0, d=0]$$

$$\begin{bmatrix} \sqrt{f+a+2e} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{e+f} & 0 \\ 0 & \sqrt{e+f} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \end{bmatrix}$$

$$[b=0, c=0, e=0]$$

$$\begin{bmatrix} \sqrt{d+f+a} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{f} & 0 \\ 0 & \sqrt{d+f} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \end{bmatrix}$$

$$[b=0, c=0, f=0]$$

$$\begin{bmatrix} \sqrt{d+2e+a} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{e} & 0 \\ 0 & \sqrt{d+e} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[b=0, d=0, e=0]$$

$$\begin{bmatrix} \sqrt{f+a+c} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{c+f} & 0 \\ 0 & 0 & 0 & \sqrt{f} \\ 0 & -\sqrt{c+f} & 0 & 0 \end{bmatrix}$$

$[b = 0, d = 0, f = 0]$

$$\begin{bmatrix} \sqrt{\frac{a\,(a+c+e)\,(a+2\,e+c)}{(a+c)\,(a+e)}} & 0 & -\sqrt{\frac{c\,e\,(c+e)}{(a+c)\,(a+e)}} & 0 \\ -\sqrt{\frac{c\,e\,(a+2\,e+c)}{(a+c)\,(a+e)}} & 0 & -\sqrt{\frac{a\,(a+c+e)\,(c+e)}{(a+c)\,(a+e)}} & 0 \\ 0 & \sqrt{\frac{a\,e}{a+c}} & 0 & 0 \\ 0 & -\sqrt{\frac{c\,(a+c+e)}{a+c}} & 0 & 0 \end{bmatrix}$$

$[b = 0, e = 0, f = 0]$

$$\begin{bmatrix} \sqrt{\frac{a\,(d+a+c)^2}{(a+c)\,(a+d)}} & 0 & -\sqrt{\frac{c^2\,d}{(a+c)\,(a+d)}} & 0 \\ -\sqrt{\frac{c\,d\,(d+a+c)}{(a+c)\,(a+d)}} & 0 & -\sqrt{\frac{a\,(d+a+c)\,c}{(a+c)\,(a+d)}} & 0 \\ 0 & \sqrt{\frac{a\,d}{a+c}} & 0 & 0 \\ 0 & -\sqrt{\frac{c\,(d+a+c)}{a+c}} & 0 & 0 \end{bmatrix}$$

$[c = 0, d = 0, e = 0]$

$$\begin{bmatrix} \sqrt{a+b+f} & 0 & 0 & 0 \\ 0 & -\sqrt{b+f} & 0 & 0 \\ 0 & 0 & -\sqrt{f} & 0 \\ 0 & 0 & 0 & \sqrt{f} \end{bmatrix}$$

$[c = 0, d = 0, f = 0]$

$$\begin{bmatrix} \sqrt{a+b+2e} & 0 & 0 & 0 \\ 0 & -\sqrt{b+e} & 0 & 0 \\ 0 & 0 & -\sqrt{e} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[c=0, e=0, f=0]$$

$$\begin{bmatrix} \sqrt{\frac{(a+b)\,(a+2b+d)}{a+2b}} & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{b\,(a+2b+d)}{a+2b}} & 0 & 0 \\ 0 & \sqrt{\frac{d\,(a+b)}{a+2b}} & 0 & 0 \\ \sqrt{\frac{b\,d}{a+2b}} & 0 & 0 & 0 \end{bmatrix}$$

$$[d=0, e=0, f=0]$$

$$\begin{bmatrix} \sqrt{a+b+c} & 0 & 0 & 0 \\ 0 & -\sqrt{b+c} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{c} & 0 \end{bmatrix}$$

$$[a=0, b=0, c=0, d=0]$$

$$\begin{bmatrix} 0 & 0 & -\sqrt{e+f} & 0 \\ -\sqrt{2e+f} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \\ 0 & -\sqrt{e+f} & 0 & 0 \end{bmatrix}$$

$$[a=0, b=0, c=0, e=0]$$

$$\begin{bmatrix} 0 & 0 & -\sqrt{f} & 0 \\ -\sqrt{d+f} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \\ 0 & -\sqrt{d+f} & 0 & 0 \end{bmatrix}$$

$$[a=0, b=0, c=0, f=0]$$

$$\begin{bmatrix} 0 & 0 & -\sqrt{e} & 0 \\ -\sqrt{d+2e} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{d+e} & 0 & 0 \end{bmatrix}$$

$$[a = 0, b = 0, d = 0, e = 0]$$

$$\begin{bmatrix} 0 & 0 & -\sqrt{c+f} & 0 \\ -\sqrt{c+f} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \\ 0 & -\sqrt{c+f} & 0 & 0 \end{bmatrix}$$

$$[a = 0, b = 0, d = 0, f = 0]$$

$$\begin{bmatrix} 0 & 0 & -\sqrt{c+e} & 0 \\ -\sqrt{c+2e} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{c+e} & 0 & 0 \end{bmatrix}$$

$$[a = 0, b = 0, e = 0, f = 0]$$

$$\begin{bmatrix} 0 & 0 & -\sqrt{c} & 0 \\ -\sqrt{d+c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{d+c} & 0 & 0 \end{bmatrix}$$

$$[a = 0, c = 0, d = 0, e = 0]$$

$$\begin{bmatrix} 0 & -\sqrt{b+f} & 0 & 0 \\ -\sqrt{b+f} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \\ 0 & 0 & \sqrt{f} & 0 \end{bmatrix}$$

$$[a = 0, c = 0, d = 0, f = 0]$$

$$\begin{bmatrix} 0 & -\sqrt{b+e} & 0 & 0 \\ -\sqrt{b+2e} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{e} & 0 \end{bmatrix}$$

$[a = 0, c = 0, e = 0, f = 0]$

$$\begin{bmatrix} 0 & -\frac{1}{2}\sqrt{2}\sqrt{2b+d} & 0 & 0 \\ -\frac{1}{2}\sqrt{2}\sqrt{2b+d} & 0 & 0 & 0 \\ \frac{1}{2}\sqrt{2}\sqrt{d} & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\sqrt{2}\sqrt{d} & 0 & 0 \end{bmatrix}$$

$[a = 0, d = 0, e = 0, f = 0]$

$$\begin{bmatrix} 0 & -\sqrt{b+c} & 0 & 0 \\ -\sqrt{b+c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{c} & 0 \end{bmatrix}$$

$[b = 0, c = 0, d = 0, e = 0]$

$$\begin{bmatrix} \sqrt{f+a} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{f} & 0 \\ 0 & \sqrt{f} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \end{bmatrix}$$

$[b = 0, c = 0, d = 0, f = 0]$

$$\begin{bmatrix} \sqrt{a+2e} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{e} & 0 \\ 0 & \sqrt{e} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$[b = 0, c = 0, e = 0, f = 0]$

$$\begin{bmatrix} \sqrt{a+d} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sqrt{d} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[b = 0, d = 0, e = 0, f = 0]$$

$$\begin{bmatrix} \sqrt{a+c} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{c} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{c} & 0 & 0 \end{bmatrix}$$

$$[c = 0, d = 0, e = 0, f = 0]$$

$$\begin{bmatrix} \sqrt{a+b} & 0 & 0 & 0 \\ 0 & -\sqrt{b} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[a = 0, b = 0, c = 0, d = 0, e = 0]$$

$$\begin{bmatrix} 0 & 0 & -\sqrt{f} & 0 \\ -\sqrt{f} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{f} \\ 0 & -\sqrt{f} & 0 & 0 \end{bmatrix}$$

$$[a = 0, b = 0, c = 0, d = 0, f = 0]$$

$$\begin{bmatrix} 0 & 0 & -\sqrt{e} & 0 \\ -\sqrt{2}\,\sqrt{e} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{e} & 0 & 0 \end{bmatrix}$$

$$[a = 0, b = 0, c = 0, e = 0, f = 0]$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -\sqrt{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{d} & 0 & 0 \end{bmatrix}$$

$$[a = 0, b = 0, d = 0, e = 0, f = 0]$$

$$\begin{bmatrix} 0 & 0 & -\sqrt{c} & 0 \\ -\sqrt{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{c} & 0 & 0 \end{bmatrix}$$

$$[a = 0, c = 0, d = 0, e = 0, f = 0]$$

$$\begin{bmatrix} 0 & -\sqrt{b} & 0 & 0 \\ -\sqrt{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[b = 0, c = 0, d = 0, e = 0, f = 0]$$

$$\begin{bmatrix} \sqrt{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[a = 0, b = 0, c = 0, d = 0, e = 0, f = 0]$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

%%

To complete the verification that our action is well-behaved it remains to show that condition (4) in Lemma 3.5 applies. For each subset of the parameters (a,...,f) we obtained above a spherical point for the weight with those parameters set equal to zero. We must also check that the fundamental highest weight vectors associated with the complementary parameters are non-zero at this spherical point. For example at the spherical point Z for a weight of the sort aA1+0A2+cA3+0A4+0A5+fA6 (i.e. with b=d=e=0) we require that each of $h_1(Z)$, $h_3(Z)$ and $h_6(Z)$ be non-zero.

The code below generates the following output for each of the 63 non-empty subsets of {a=0, b=0, c=0, d=0, e=0, f=0}:

- A listing of the subset. These parameters are set to zero in succession to obtain a non-generic (generalized) spherical point as in the previous output.
- A list of the fundamental highest weight vectors ($h_1 \ldots h_6$) associated with the complementary parameters.
- A list of values for these h_j's at the spherical point.

The output shows that in all cases each fundamental highest weight vector for a complementary parameter takes a non-zero value at the limiting spherical point in question. Thus condition (4) from Lemma 3.5 does hold here, completing our analysis for this example.

```
> [1,2,3,4,5,6]: ch:=[seq(choose(%)[i],i=2..64)]:
> h:=[h1,h2,h3,h4,h5,h6]: hn:=[h1n,h2n,h3n,h4n,h5n,h6n]:
  zs:=[a=0,b=0,c=0,d=0,e=0,f=0]:
> for i from 1 to nops(ch) do:
      s:=map(x->zs[x],ch[i]):
      print(s);
      zc:=[op({1,2,3,4,5,6}minus{op(ch[i])})]:
      hs:=map(x->hn[x],zc):
      for j from 1 to nops(s) do:
          hs:=subs(s[j],hs):
      end do:
      print(map(x->h[x],zc));
      print(hs);
  end do:
```

$$[a=0]$$

$$[h2, h3, h4, h5, h6]$$

$$\left[-\frac{\sqrt{b+c+d+e+f}\,\sqrt{b}\,\sqrt{f+c+b+d+2e}\,(2b+d+2e+c)\,\sqrt{b+c+2e}}{\sqrt{b+d}\,\sqrt{2e+b+c+d}\,(2b+c+2e)},\right.$$

$$-\frac{1}{\sqrt{2b+c+2e}\,\sqrt{2e+b+c+d}}\left(\sqrt{b+c+2e}\,\sqrt{2b+d+2e+c}\right.$$

$$\left.\sqrt{f+c+b+d+2e}\,\sqrt{b+c+d+e+f}\,\sqrt{c+e+f}\right),$$

$$\frac{\sqrt{b+c+d+e+f}\,\sqrt{d}\,\sqrt{f+c+b+d+2e}\,\sqrt{2b+d+2e+c}}{\sqrt{b+d}\,\sqrt{2e+b+c+d}},$$

$$-\frac{1}{(2e+b+c+d)\,(2b+c+2e)}\left(\sqrt{c+e+f}\,\sqrt{b+c+d+e+f}\,(f+c+b\right.$$

$$\left.+d+2e)\,(2b+d+2e+c)\,(b+c+2e)\right),$$

$$\left.\sqrt{f}\,\sqrt{c+e+f}\,\sqrt{b+c+d+e+f}\,\sqrt{f+c+b+d+2e}\right]$$

$$[b=0]$$

$$[h1, h3, h4, h5, h6]$$

$$\left[\frac{\sqrt{a}\,\sqrt{d+e+a+c}\,\sqrt{d+2e+f+a+c}}{\sqrt{a+c}\,\sqrt{a+d+e}},\right.$$

$$-\frac{\sqrt{d+e+a+c}\,\sqrt{d+2e+f+a+c}\,\sqrt{d+e+f+c}\,\sqrt{c}\,\sqrt{c+e+f}}{\sqrt{a+c}\,\sqrt{d+c+e}},$$

$$\frac{\sqrt{d+e+f+c}\,\sqrt{d+e}\,\sqrt{d+2e+f+a+c}\,\sqrt{d+e+a+c}}{\sqrt{d+c+e}\,\sqrt{a+d+e}},$$

$$-\frac{\sqrt{c+e+f}\,\sqrt{d+e+f+c}\,\sqrt{d+e}\,(d+2e+f+a+c)\,\sqrt{d+e+a+c}}{\sqrt{a+d+e}\,\sqrt{d+c+e}},$$

$$\left.\sqrt{f}\,\sqrt{c+e+f}\,\sqrt{d+e+f+c}\,\sqrt{d+2e+f+a+c}\right]$$

$$[a=0, b=0]$$

$$[h3, h4, h5, h6]$$

$$\left[-\sqrt{f+c+d+2e}\,\sqrt{d+e+f+c}\,\sqrt{c+e+f}, \sqrt{d+e+f+c}\,\sqrt{f+c+d+2e},\right.$$

$$-\sqrt{c+e+f}\,\sqrt{d+e+f+c}\,(f+c+d+2e),$$

$$\left.\sqrt{f}\,\sqrt{c+e+f}\,\sqrt{d+e+f+c}\,\sqrt{f+c+d+2e}\right]$$

$$[c=0]$$

$$[h1, h2, h4, h5, h6]$$

$$\left[\frac{\sqrt{a+b+2e}\,\sqrt{a+2b+d+2e}\,\sqrt{a+b+d+2e+f}}{\sqrt{a+2b+2e}\,\sqrt{a+b+d+2e}},\right.$$

$$-\frac{\sqrt{b+d+e+f}\,\sqrt{b}\,\sqrt{a+b+d+2e+f}\,(a+2b+d+2e)\,\sqrt{a+b+2e}}{\sqrt{b+d}\,\sqrt{a+b+d+2e}\,(a+2b+2e)},$$

$$\frac{\sqrt{b+d+e+f}\,\sqrt{d}\,\sqrt{a+b+d+2e+f}\,\sqrt{a+2b+d+2e}}{\sqrt{b+d}\,\sqrt{a+b+d+2e}},$$

$$-\frac{\sqrt{e+f}\,\sqrt{b+d+e+f}\,(a+b+d+2e+f)\,(a+2b+d+2e)\,(a+b+2e)}{(a+b+d+2e)\,(a+2b+2e)},$$

$$\left.\sqrt{f}\,\sqrt{e+f}\,\sqrt{b+d+e+f}\,\sqrt{a+b+d+2e+f}\right]$$

$$[a=0, c=0]$$

$$[h2, h4, h5, h6]$$

$$\left[-\frac{\sqrt{b+d+e+f}\,\sqrt{b}\,\sqrt{f+b+d+2e}\,(2b+d+2e)\sqrt{b+2e}}{\sqrt{b+d}\,\sqrt{b+d+2e}\,(2b+2e)},\right.$$
$$\frac{\sqrt{b+d+e+f}\,\sqrt{d}\,\sqrt{f+b+d+2e}\,\sqrt{2b+d+2e}}{\sqrt{b+d}\,\sqrt{b+d+2e}},$$
$$-\frac{\sqrt{e+f}\,\sqrt{b+d+e+f}\,(f+b+d+2e)\,(2b+d+2e)\,(b+2e)}{(b+d+2e)\,(2b+2e)},$$
$$\left.\sqrt{f}\,\sqrt{e+f}\,\sqrt{b+d+e+f}\,\sqrt{f+b+d+2e}\right]$$

$$[b=0, c=0]$$

$$[h1, h4, h5, h6]$$

$$\left[\sqrt{d+2e+f+a}, \sqrt{d+e+f}\,\sqrt{d+2e+f+a}, -\sqrt{e+f}\,\sqrt{d+e+f}\,(d+2e+f+a), \sqrt{f}\,\sqrt{e+f}\,\sqrt{d+e+f}\,\sqrt{d+2e+f+a}\right]$$

$$[a=0, b=0, c=0]$$

$$[h4, h5, h6]$$

$$\left[\sqrt{d+e+f}\,\sqrt{f+d+2e}, -\sqrt{e+f}\,\sqrt{d+e+f}\,(f+d+2e), \sqrt{f}\,\sqrt{e+f}\,\sqrt{d+e+f}\,\sqrt{f+d+2e}\right]$$

$$[d=0]$$

$$[h1, h2, h3, h5, h6]$$

$$\left[\frac{\sqrt{a}\,\sqrt{a+c+e}\,\sqrt{a+b+c+2e+f}}{\sqrt{a+c}\,\sqrt{a+e}}, -\sqrt{b+c+e+f}\,\sqrt{a+b+c+2e+f},\right.$$
$$-\frac{\sqrt{a+c+e}\,\sqrt{a+b+c+2e+f}\,\sqrt{b+c+e+f}\,\sqrt{c}\,\sqrt{c+e+f}}{\sqrt{a+c}\,\sqrt{c+e}},$$
$$-\frac{\sqrt{c+e+f}\,\sqrt{e}\,\sqrt{b+c+e+f}\,(a+b+c+2e+f)\sqrt{a+c+e}}{\sqrt{a+e}\,\sqrt{c+e}},$$
$$\left.\sqrt{f}\,\sqrt{c+e+f}\,\sqrt{b+c+e+f}\,\sqrt{a+b+c+2e+f}\right]$$

$$[a=0, d=0]$$

$$[h2, h3, h5, h6]$$

$$\left[-\sqrt{b+c+e+f}\,\sqrt{f+c+b+2e}, -\sqrt{f+c+b+2e}\,\sqrt{b+c+e+f}\,\sqrt{c+e+f},\right.$$

$$-\sqrt{c+e+f}\,\sqrt{b+c+e+f}\,(f+c+b+2e),$$
$$\sqrt{f}\,\sqrt{c+e+f}\,\sqrt{b+c+e+f}\,\sqrt{f+c+b+2e}\Big]$$

$$[b=0, d=0]$$

$$[h1, h3, h5, h6]$$

$$\left[\frac{\sqrt{a}\,\sqrt{a+c+e}\,\sqrt{2e+f+a+c}}{\sqrt{a+c}\,\sqrt{a+e}}, -\frac{\sqrt{a+c+e}\,\sqrt{2e+f+a+c}\,(c+e+f)\,\sqrt{c}}{\sqrt{a+c}\,\sqrt{c+e}},\right.$$
$$\left.-\frac{(c+e+f)\,\sqrt{e}\,(2e+f+a+c)\,\sqrt{a+c+e}}{\sqrt{a+e}\,\sqrt{c+e}}, \sqrt{f}\,(c+e+f)\,\sqrt{2e+f+a+c}\right]$$

$$[a=0, b=0, d=0]$$

$$[h3, h5, h6]$$

$$\left[-\sqrt{f+c+2e}\,(c+e+f), -(c+e+f)\,(f+c+2e), \sqrt{f}\,(c+e+f)\,\sqrt{f+c+2e}\right]$$

$$[c=0, d=0]$$

$$[h1, h2, h5, h6]$$

$$\Big[\sqrt{a+b+2e+f}, -\sqrt{b+e+f}\,\sqrt{a+b+2e+f}, -\sqrt{e+f}\,\sqrt{b+e+f}\,(a+b+2e$$
$$+f), \sqrt{f}\,\sqrt{e+f}\,\sqrt{b+e+f}\,\sqrt{a+b+2e+f}\Big]$$

$$[a=0, c=0, d=0]$$

$$[h2, h5, h6]$$

$$\Big[-\sqrt{b+e+f}\,\sqrt{b+2e+f}, -\sqrt{e+f}\,\sqrt{b+e+f}\,(b+2e+f),$$
$$\sqrt{f}\,\sqrt{e+f}\,\sqrt{b+e+f}\,\sqrt{b+2e+f}\Big]$$

$$[b=0, c=0, d=0]$$

$$[h1, h5, h6]$$

$$\left[\sqrt{f+a+2e}, -(e+f)\,(f+a+2e), \sqrt{f}\,(e+f)\,\sqrt{f+a+2e}\right]$$

$$[a=0, b=0, c=0, d=0]$$

$$[h5, h6]$$

$$\left[-(e+f)\,(2e+f), \sqrt{f}\,(e+f)\,\sqrt{2e+f}\right]$$

$$[e=0]$$

$$[h1, h2, h3, h4, h6]$$

$$\left[\frac{\sqrt{a+b}\,\sqrt{a+2b+c+d}\,\sqrt{a+b+c+d+f}}{\sqrt{a+b+d}\,\sqrt{a+2b+c}},\right.$$

$$-\frac{\sqrt{b+c+d+f}\,\sqrt{b+c}\,\sqrt{a+b+c+d+f}\,(a+2b+c+d)\,\sqrt{a+b}}{\sqrt{b+c+d}\,(a+2b+c)\,\sqrt{a+b+d}},$$

$$-\frac{\sqrt{a+2b+c+d}\,\sqrt{a+b+c+d+f}\,\sqrt{b+c}\,\sqrt{b+c+d+f}\,\sqrt{c+f}}{\sqrt{a+2b+c}\,\sqrt{b+c+d}},$$

$$\frac{\sqrt{b+c+d+f}\,\sqrt{d}\,\sqrt{a+b+c+d+f}\,\sqrt{a+2b+c+d}}{\sqrt{b+c+d}\,\sqrt{a+b+d}},$$

$$\left.\sqrt{f}\,\sqrt{c+f}\,\sqrt{b+c+d+f}\,\sqrt{a+b+c+d+f}\right]$$

$$[a=0, e=0]$$

$$[h2, h3, h4, h6]$$

$$\left[-\frac{(b+c+d+f)\,\sqrt{b}\,(2b+d+c)\,\sqrt{b+c}}{\sqrt{b+d}\,\sqrt{b+c+d}\,(2b+c)},\right.$$

$$-\frac{\sqrt{b+c}\,\sqrt{2b+d+c}\,(b+c+d+f)\,\sqrt{c+f}}{\sqrt{2b+c}\,\sqrt{b+c+d}},$$

$$\left.\frac{(b+c+d+f)\,\sqrt{d}\,\sqrt{2b+d+c}}{\sqrt{b+d}\,\sqrt{b+c+d}}, \sqrt{f}\,\sqrt{c+f}\,(b+c+d+f)\right]$$

$$[b=0, e=0]$$

$$[h1, h3, h4, h6]$$

$$\left[\frac{\sqrt{a}\,\sqrt{d+a+c}\,\sqrt{d+f+a+c}}{\sqrt{a+c}\,\sqrt{a+d}},\right.$$

$$-\frac{\sqrt{d+a+c}\,\sqrt{d+f+a+c}\,\sqrt{d+f+c}\,\sqrt{c}\,\sqrt{c+f}}{\sqrt{a+c}\,\sqrt{d+c}},$$

$$\frac{\sqrt{d+f+c}\,\sqrt{d}\,\sqrt{d+f+a+c}\,\sqrt{d+a+c}}{\sqrt{d+c}\,\sqrt{a+d}},$$

$$\left.\sqrt{f}\,\sqrt{c+f}\,\sqrt{d+f+c}\,\sqrt{d+f+a+c}\right]$$

$$[a=0, b=0, e=0]$$

$$[h3, h4, h6]$$

$$\left[-(d+f+c)\,\sqrt{c+f}, d+f+c, \sqrt{f}\,\sqrt{c+f}\,(d+f+c)\right]$$

$$[c=0, e=0]$$

$$[h1, h2, h4, h6]$$

$$\left[\frac{\sqrt{a+b}\,\sqrt{a+2b+d}\,\sqrt{a+b+d+f}}{\sqrt{a+2b}\,\sqrt{a+b+d}},\right.$$

$$-\frac{\sqrt{b+d+f}\,\sqrt{b}\,\sqrt{a+b+d+f}\,(a+2b+d)\,\sqrt{a+b}}{\sqrt{b+d}\,\sqrt{a+b+d}\,(a+2b)},$$

$$\left.\frac{\sqrt{b+d+f}\,\sqrt{d}\,\sqrt{a+b+d+f}\,\sqrt{a+2b+d}}{\sqrt{b+d}\,\sqrt{a+b+d}}, f\sqrt{b+d+f}\,\sqrt{a+b+d+f}\right]$$

$$[a=0, c=0, e=0]$$

$$[h2, h4, h6]$$

$$\left[-\frac{1}{2}\,\frac{(2b+d)\,(b+d+f)}{b+d}, \frac{(b+d+f)\,\sqrt{d}\,\sqrt{2b+d}}{b+d}, f\,(b+d+f)\right]$$

$$[b=0, c=0, e=0]$$

$$[h1, h4, h6]$$

$$\left[\sqrt{d+f+a}, \sqrt{d+f}\,\sqrt{d+f+a}, f\sqrt{d+f}\,\sqrt{d+f+a}\right]$$

$$[a=0, b=0, c=0, e=0]$$

$$[h4, h6]$$

$$[d+f, f\,(d+f)]$$

$$[d=0, e=0]$$

$$[h1, h2, h3, h6]$$

$$\left[\sqrt{a+b+c+f}, -\sqrt{b+c+f}\,\sqrt{a+b+c+f}, -\sqrt{a+b+c+f}\,\sqrt{b+c+f}\,\sqrt{c+f},\right.$$

$$\left.\sqrt{f}\,\sqrt{c+f}\,\sqrt{b+c+f}\,\sqrt{a+b+c+f}\right]$$

$$[a=0, d=0, e=0]$$

$$[h2, h3, h6]$$

$$\left[-b-c-f, -(b+c+f)\,\sqrt{c+f}, \sqrt{f}\,\sqrt{c+f}\,(b+c+f)\right]$$

$$[b=0, d=0, e=0]$$

$$[h1, h3, h6]$$

$$\left[\sqrt{f+a+c}, -\sqrt{f+a+c}\,(c+f), \sqrt{f}\,(c+f)\,\sqrt{f+a+c}\right]$$

$$[a=0, b=0, d=0, e=0]$$

$$[h3, h6]$$

$$\left[-(c+f)^{3/2}, \sqrt{f}\,(c+f)^{3/2}\right]$$

$$[c=0, d=0, e=0]$$

$$[h1, h2, h6]$$

$$\left[\sqrt{a+b+f}, -\sqrt{b+f}\,\sqrt{a+b+f}, f\sqrt{b+f}\,\sqrt{a+b+f}\right]$$

$$[a=0, c=0, d=0, e=0]$$

$$[h2, h6]$$

$$[-b-f, f\,(b+f)\,]$$

$$[b=0, c=0, d=0, e=0]$$

$$[h1, h6]$$

$$\left[\sqrt{f+a}, f^{3/2}\,\sqrt{f+a}\right]$$

$$[a=0, b=0, c=0, d=0, e=0]$$

$$[h6]$$

$$[f^2]$$

$$[f=0]$$

$$[h1, h2, h3, h4, h5]$$

$$\left[\left(\sqrt{a}\,\sqrt{a+b+e}\,\sqrt{a+c+e}\,\sqrt{a+b+c+2e}\,\sqrt{a+b+c+d+e}\,\sqrt{a+2b+c+d+2e}\right)\Big/\left(\sqrt{a+c}\,\sqrt{a+e}\,\sqrt{a+b+c+e}\,\sqrt{a+b+d+e}\,\sqrt{a+2b+c+2e}\right),\right.$$

$$-\left(\sqrt{b+d+e}\,\sqrt{b+c+e}\,\sqrt{b}\,(a+2b+c+d+2e)\,\sqrt{a+b+c+d+e}\,\sqrt{a+b+c+2e}\,\sqrt{a+b+e}\right)\Big/\left(\sqrt{b+e}\,\sqrt{b+d}\,(a+2b+c+2e)\,\sqrt{a+b+d+e}\,\sqrt{a+b+c+e}\right),$$

$$-\frac{1}{\sqrt{a+c}\,\sqrt{a+2b+c+2e}\,\sqrt{a+b+c+e}}\left(\sqrt{a+c+e}\,\sqrt{a+b+c+2e}\,\sqrt{a+b+c+d+e}\,\sqrt{a+2b+c+d+2e}\,\sqrt{b+c+e}\,\sqrt{c}\right),$$

$$\frac{\sqrt{b+d+e}\,\sqrt{d}\,\sqrt{a+2b+c+d+2e}\,\sqrt{a+b+c+d+e}}{\sqrt{b+d}\,\sqrt{a+b+d+e}},$$

$$-\left(\sqrt{e}\,\sqrt{b+d+e}\,\sqrt{b+c+e}\,(a+2b+c+d+2e)\,\sqrt{a+b+c+d+e}\,(a+b+c+2e)\,\sqrt{a+c+e}\,\sqrt{a+b+e}\right)\Big/\left(\sqrt{b+e}\,\sqrt{a+e}\,\sqrt{a+b+d+e}\,\sqrt{a+b+c+e}\,(a+2b+c+2e)\right)\Big]$$

$$[a=0, f=0]$$

$$[h2, h3, h4, h5]$$

$$\left[-\frac{\sqrt{b+c+d+e}\,\sqrt{b}\,(2b+d+2e+c)\,\sqrt{b+c+2e}}{\sqrt{b+d}\,(2b+c+2e)},\right.$$

$$-\frac{\sqrt{b+c+2e}\,\sqrt{2b+d+2e+c}\,\sqrt{b+c+d+e}\,\sqrt{c+e}}{\sqrt{2b+c+2e}},$$

$$\frac{\sqrt{b+c+d+e}\,\sqrt{d}\,\sqrt{2b+d+2e+c}}{\sqrt{b+d}},$$

$$\left.-\frac{\sqrt{c+e}\,\sqrt{b+c+d+e}\,(2b+d+2e+c)\,(b+c+2e)}{2b+c+2e}\right]$$

$$[b=0, f=0]$$

$$[h1, h3, h4, h5]$$

$$\left[\frac{\sqrt{a}\,\sqrt{d+e+a+c}\,\sqrt{a+c+d+2e}}{\sqrt{a+c}\,\sqrt{a+d+e}},\right.$$

$$-\frac{\sqrt{d+e+a+c}\,\sqrt{a+c+d+2e}\,\sqrt{c}\,\sqrt{c+e}}{\sqrt{a+c}},$$

$$\frac{\sqrt{d+e}\,\sqrt{a+c+d+2e}\,\sqrt{d+e+a+c}}{\sqrt{a+d+e}},$$

$$\left.-\frac{\sqrt{c+e}\,\sqrt{d+e}\,(a+c+d+2e)\,\sqrt{d+e+a+c}}{\sqrt{a+d+e}}\right]$$

$$[a=0, b=0, f=0]$$

$$[h3, h4, h5]$$

$$\left[-\sqrt{d+2e+c}\,\sqrt{d+c+e}\,\sqrt{c+e}, \sqrt{d+c+e}\,\sqrt{d+2e+c}, -\sqrt{c+e}\,\sqrt{d+c+e}\,(d+2e+c)\right]$$

$$[c=0, f=0]$$

$$[h1, h2, h4, h5]$$

$$\left[\frac{\sqrt{a+b+2e}\,\sqrt{a+2b+d+2e}}{\sqrt{a+2b+2e}},\right.$$
$$-\frac{\sqrt{b+d+e}\,\sqrt{b}\,(a+2b+d+2e)\,\sqrt{a+b+2e}}{\sqrt{b+d}\,(a+2b+2e)},$$
$$\frac{\sqrt{b+d+e}\,\sqrt{d}\,\sqrt{a+2b+d+2e}}{\sqrt{b+d}},$$
$$\left.-\frac{\sqrt{e}\,\sqrt{b+d+e}\,(a+2b+d+2e)\,(a+b+2e)}{a+2b+2e}\right]$$
$$[a=0, c=0, f=0]$$
$$[h2, h4, h5]$$
$$\left[-\frac{\sqrt{b+d+e}\,\sqrt{b}\,(2b+d+2e)\,\sqrt{b+2e}}{\sqrt{b+d}\,(2b+2e)}, \frac{\sqrt{b+d+e}\,\sqrt{d}\,\sqrt{2b+d+2e}}{\sqrt{b+d}},\right.$$
$$\left.-\frac{\sqrt{e}\,\sqrt{b+d+e}\,(2b+d+2e)\,(b+2e)}{2b+2e}\right]$$
$$[b=0, c=0, f=0]$$
$$[h1, h4, h5]$$
$$\left[\sqrt{d+2e+a}, \sqrt{d+e}\,\sqrt{d+2e+a}, -\sqrt{e}\,\sqrt{d+e}\,(d+2e+a)\right]$$
$$[a=0, b=0, c=0, f=0]$$
$$[h4, h5]$$
$$\left[\sqrt{d+e}\,\sqrt{d+2e}, -\sqrt{e}\,\sqrt{d+e}\,(d+2e)\right]$$
$$[d=0, f=0]$$
$$[h1, h2, h3, h5]$$
$$\left[\frac{\sqrt{a}\,\sqrt{a+c+e}\,\sqrt{a+b+c+2e}}{\sqrt{a+c}\,\sqrt{a+e}}, -\sqrt{b+c+e}\,\sqrt{a+b+c+2e},\right.$$
$$-\frac{\sqrt{a+c+e}\,\sqrt{a+b+c+2e}\,\sqrt{b+c+e}\,\sqrt{c}}{\sqrt{a+c}},$$
$$\left.-\frac{\sqrt{e}\,\sqrt{b+c+e}\,(a+b+c+2e)\,\sqrt{a+c+e}}{\sqrt{a+e}}\right]$$
$$[a=0, d=0, f=0]$$
$$[h2, h3, h5]$$
$$\left[-\sqrt{b+c+e}\,\sqrt{b+c+2e}, -\sqrt{b+c+2e}\,\sqrt{b+c+e}\,\sqrt{c+e},\right.$$

$$-\sqrt{c+e}\,\sqrt{b+c+e}\,(b+c+2e)\Big]$$

$$[b=0, d=0, f=0]$$

$$[h1, h3, h5]$$

$$\left[\frac{\sqrt{a}\,\sqrt{a+c+e}\,\sqrt{a+2e+c}}{\sqrt{a+c}\,\sqrt{a+e}}, -\frac{\sqrt{a+c+e}\,\sqrt{a+2e+c}\,\sqrt{c+e}\,\sqrt{c}}{\sqrt{a+c}},\right.$$

$$\left.-\frac{\sqrt{c+e}\,\sqrt{e}\,(a+2e+c)\,\sqrt{a+c+e}}{\sqrt{a+e}}\right]$$

$$[a=0, b=0, d=0, f=0]$$

$$[h3, h5]$$

$$\left[-\sqrt{c+2e}\,(c+e), -(c+e)\,(c+2e)\right]$$

$$[c=0, d=0, f=0]$$

$$[h1, h2, h5]$$

$$\left[\sqrt{a+b+2e}, -\sqrt{b+e}\,\sqrt{a+b+2e}, -\sqrt{e}\,\sqrt{b+e}\,(a+b+2e)\right]$$

$$[a=0, c=0, d=0, f=0]$$

$$[h2, h5]$$

$$\left[-\sqrt{b+e}\,\sqrt{b+2e}, -\sqrt{e}\,\sqrt{b+e}\,(b+2e)\right]$$

$$[b=0, c=0, d=0, f=0]$$

$$[h1, h5]$$

$$\left[\sqrt{a+2e}, -e\,(a+2e)\right]$$

$$[a=0, b=0, c=0, d=0, f=0]$$

$$[h5]$$

$$\left[-2e^2\right]$$

$$[e=0, f=0]$$

$$[h1, h2, h3, h4]$$

$$\left[\frac{\sqrt{a+b}\,\sqrt{a+2b+c+d}\,\sqrt{a+b+c+d}}{\sqrt{a+b+d}\,\sqrt{a+2b+c}},\right.$$

$$-\frac{\sqrt{b+c}\,\sqrt{a+b+c+d}\,(a+2b+c+d)\,\sqrt{a+b}}{(a+2b+c)\,\sqrt{a+b+d}},$$

$$-\frac{\sqrt{a+2b+c+d}\,\sqrt{a+b+c+d}\,\sqrt{b+c}\,\sqrt{c}}{\sqrt{a+2b+c}},$$

$$\frac{\sqrt{d}\,\sqrt{a+b+c+d}\,\sqrt{a+2\,b+c+d}}{\sqrt{a+b+d}}\Bigg]$$

$$[a=0, e=0, f=0]$$

$$[h2, h3, h4]$$

$$\Bigg[-\frac{\sqrt{b+c+d}\,\sqrt{b}\,(2\,b+d+c)\,\sqrt{b+c}}{\sqrt{b+d}\,(2\,b+c)}, -\frac{\sqrt{b+c}\,\sqrt{2\,b+d+c}\,\sqrt{b+c+d}\,\sqrt{c}}{\sqrt{2\,b+c}},$$

$$\frac{\sqrt{b+c+d}\,\sqrt{d}\,\sqrt{2\,b+d+c}}{\sqrt{b+d}}\Bigg]$$

$$[b=0, e=0, f=0]$$

$$[h1, h3, h4]$$

$$\left[\frac{\sqrt{a}\,(d+a+c)}{\sqrt{a+c}\,\sqrt{a+d}}, -\frac{(d+a+c)\,c}{\sqrt{a+c}}, \frac{\sqrt{d}\,(d+a+c)}{\sqrt{a+d}}\right]$$

$$[a=0, b=0, e=0, f=0]$$

$$[h3, h4]$$

$$\left[-(d+c)\,\sqrt{c}, d+c\right]$$

$$[c=0, e=0, f=0]$$

$$[h1, h2, h4]$$

$$\left[\frac{\sqrt{a+b}\,\sqrt{a+2\,b+d}}{\sqrt{a+2\,b}}, -\frac{\sqrt{b}\,(a+2\,b+d)\,\sqrt{a+b}}{a+2\,b}, \sqrt{d}\,\sqrt{a+2\,b+d}\right]$$

$$[a=0, c=0, e=0, f=0]$$

$$[h2, h4]$$

$$\left[-b-\frac{1}{2}\,d, \sqrt{d}\,\sqrt{2\,b+d}\right]$$

$$[b=0, c=0, e=0, f=0]$$

$$[h1, h4]$$

$$\left[\sqrt{a+d}, \sqrt{d}\,\sqrt{a+d}\right]$$

$$[a=0, b=0, c=0, e=0, f=0]$$

$$[h4]$$

$$[d]$$

$$[d=0, e=0, f=0]$$

$$[h1, h2, h3]$$

$$\left[\sqrt{a+b+c},\ -\sqrt{b+c}\,\sqrt{a+b+c},\ -\sqrt{a+b+c}\,\sqrt{b+c}\,\sqrt{c}\right]$$

$$[a=0,\ d=0,\ e=0,\ f=0]$$

$$[h2,\ h3]$$

$$\left[-b-c,\ -(b+c)\sqrt{c}\right]$$

$$[b=0,\ d=0,\ e=0,\ f=0]$$

$$[h1,\ h3]$$

$$\left[\sqrt{a+c},\ -\sqrt{a+c}\,c\right]$$

$$[a=0,\ b=0,\ d=0,\ e=0,\ f=0]$$

$$[h3]$$

$$\left[-c^{3/2}\right]$$

$$[c=0,\ d=0,\ e=0,\ f=0]$$

$$[h1,\ h2]$$

$$\left[\sqrt{a+b},\ -\sqrt{b}\,\sqrt{a+b}\right]$$

$$[a=0,\ c=0,\ d=0,\ e=0,\ f=0]$$

$$[h2]$$

$$[-b]$$

$$[b=0,\ c=0,\ d=0,\ e=0,\ f=0]$$

$$[h1]$$

$$\left[\sqrt{a}\right]$$

$$[a=0,\ b=0,\ c=0,\ d=0,\ e=0,\ f=0]$$

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