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$O(n)$-Spherical Functions on Heisenberg Groups

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ABSTRACT. In a previous paper the authors developed a general calculus for determining the bounded K-spherical functions on the Heisenberg group H_n when the action of $K \subset U(n)$ yields a Gelfand pair. When $K = U(n)$, our methods are related to classical results concerning the generalized Laguerre polynomials and the resulting spherical functions are well known. In this paper, we apply our theory to the action of $K = SO(n, \mathbf{R}) \times \mathbf{T}$ on H_n for $n \geq 3$. The (generic) bounded K-spherical functions are completely determined by a family of orthogonal polynomials in two variables (γ_1, γ_2). These are obtained by suitably ordering the monomials $\gamma_1^k \gamma_2^\ell$ and applying the Gram-Schmidt algorithm using a certain measure.

§1 Introduction.

It is well known that the action of the unitary group $U(n)$ on the $(2n+1)$-dimensional Heisenberg group H_n yields a *Gelfand pair.* That is, the convolution algebra $L^1_{U(n)}(H_n)$ of $U(n)$-invariant L^1-functions on H_n is commutative. It is also the case that the actions of many compact subgroups $K \subset U(n)$ yield Gelfand pairs (K, H_n). In fact, (K, H_n) is a Gelfand pair if and only if the action of K on the holomorphic polynomials $\mathcal{P}(V)$ where $V = \mathbf{C}^n$ is multiplicity free. (See Theorem 3.3 below.) This fact together with results of V. Kac [**K**] yields a classification of the Gelfand pairs (K, H_n) where K is connected and acts irreducibly on V [**BJR1**].

In [**BJR2**] the authors developed a general calculus to determine the bounded K-spherical functions for a Gelfand pair (K, H_n). These can be described in terms of the representation theory of both H_n and K and fall into two classes which we call type 1 and type 2. The type 1 K-spherical functions are generic and are associated with the infinite dimensional irreducible unitary representations of H_n. Type 2 K-spherical functions arise from the characters on H_n. These are non-generic and reflect the abelian side of analysis on H_n.

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We denote the K-invariant polynomials on the underlying real vector space $V_{\mathbf{R}}$ of V by ${}^K\mathcal{P}(V_{\mathbf{R}})$. The type 1 K-spherical functions are completely determined by a set of polynomials $q_\alpha \in {}^K\mathcal{P}(V_{\mathbf{R}})$. (See Equations 3.8 and 3.9 below.) The multiplicity free decomposition $\mathcal{P}(V) = \sum_{\alpha\in\Lambda} P_\alpha$ of $\mathcal{P}(V)$ into K-irreducible components yields a canonical basis $\{p_\alpha : \alpha \in \Lambda\}$ for ${}^K\mathcal{P}(V_{\mathbf{R}})$. (See Equation 3.10 below.) Determining the p_α's is a problem in classical invariant theory. The p_α's in turn determine the q_α's. In [**BJR2**], we describe two methods:

(1) $\{q_\alpha\}$ is obtained from $\{p_\alpha\}$ via an orthogonalization procedure using the measure $e^{-\frac{|z|^2}{2}} dz d\bar{z}$. (See Theorem 3.11 below.)

(2) $\mathcal{M}_{p_\alpha}(e^{-\frac{|z|^2}{2}}) = dim(P_\alpha) q_\alpha(z) e^{-\frac{|z|^2}{2}}$ where $\mathcal{M}_{p_\alpha}$ is a "magic" constant coefficient differential operator obtained in a simple way from p_α. (See Theorem 3.13 below.)

Results of Howe and Umeda [**HU**] imply that

$$ {}^K\mathcal{P}(V_{\mathbf{R}}) = \mathbf{C}[\gamma_1, \dots, \gamma_d] $$

where $\{\gamma_1, \dots, \gamma_d\} \subset \{p_\alpha : \alpha \in \Lambda\}$ is a set of *fundamental invariants*. The polynomials p_α and q_α can be viewed as polynomial functions on the image $\Gamma \subset (\mathbf{R}^+)^d$ of $\gamma = (\gamma_1, \dots, \gamma_d)$. The two computational procedures, (1) and (2) above, can be carried out on the value space Γ to determine "reduced" K-spherical functions. These solve eigenvalue problems for reduced differential operators $L'_1, \dots, L'_d$ obtained from $\gamma_1, \dots, \gamma_d$ via symmetrization.

As far as we know, the only well understood example is $K = U(n)$. Here there is just one fundamental invariant, $\gamma = |z|^2/2$, and $\Gamma = \mathbf{R}^+$. The polynomials $\{q_m : m = 0, 1, 2, \dots\}$ are the generalized Laguerre polynomials $q_m(\gamma) = L_m^{(n-1)}(\gamma)$ of order $n-1$ and degree m. In this case, (1) above reduces to the usual description of the Laguerre polynomials as the solution to an orthogonalization problem. Procedure (2) above is equivalent to the classical Rodrigues' formula for Laguerre polynomials.

The new results in this paper concern the type 1 spherical functions for the action of $K = SO(n, \mathbf{R}) \times \mathbf{T}$ where $n \geq 3$. This is one of the groups on Kac's list and hence (K, H_n) is a Gelfand pair. The decomposition of $\mathcal{P}(V)$ into K-irreducibles involves the classical theory of spherical harmonics. We show that there are two fundamental invariants (γ_1, γ_2) and present a recurrence relation to determine the p_α's as polynomials in γ_1 and γ_2. (See Theorem 5.12 below.) We derive a simple explicit formula for the measure $e^{-\frac{|z|^2}{2}} dz d\bar{z}$ on Γ. (See Theorem 5.23 below.) These results yield the type 1 K-spherical functions via an orthogonalization procedure (1). There is however a further simplification. We show that (by carefully ordering some indices) one can replace the p_α's by the monomials $\gamma_1^k \gamma_2^\ell$ when performing orthogonalization. (See Theorem 5.36 below.) This yields a fairly pleasing and tractable description of the type 1 K-spherical functions. Unfortunately, the formulae for the magic operators $\mathcal{M}_{p_\alpha}$ and for the reduced operators L'_1, L'_2 do not seem illuminating here.

Section 2 presents preliminary material on the Heisenberg group and its representations. Section 3 summarizes the main results in **[BJR2]** and these are applied to the example $K = U(n)$ in Section 4. Section 5 contains a detailed analysis for the example $K = SO(\mathbf{R}, n) \times \mathbf{T}$.

§2 The Heisenberg group and its representations.

Let V denote $\mathbf{C}^n$ with its usual Hermitian inner product $(z, z') = z \cdot \bar{z'}$. The $\mathbf{R}$-valued form $\omega(z, z') = -Im(z, z')$ is a symplectic structure on V. We define the *Heisenberg group* H_n, of dimension $2n+1$, as $H_n := V \times R$ with product

$$(z, t)(z', t') = (z + z', t + t' + \frac{1}{2}\omega(z, z')). \tag{2.1}$$

The unitary group $U(n)$, of automorphisms of $(V, (\cdot, \cdot))$, acts on H_n by

$$k \cdot (z, t) = (kz, t). \tag{2.2}$$

This yields a maximal compact connected group of automorphisms of $Aut(H_n)$ again denoted by $U(n)$.

Fock space $\mathcal{F}$ consists of entire functions $f : V \to \mathbf{C}$ which are square integrable with respect to $(\frac{1}{2\pi})^n e^{-\frac{|z|^2}{2}} dz d\bar{z}$. $\mathcal{F}$ has a Hilbert space structure given by

$$\langle f, g \rangle := \left(\frac{1}{2\pi}\right)^n \int_V f(z)\overline{g(z)} e^{-\frac{|z|^2}{2}} dz d\bar{z}. \tag{2.3}$$

The holomorphic polynomials $\mathcal{P}(V) = \mathbf{C}[z_1, \ldots, z_n]$ are dense in $\mathcal{F}$ and the monomials $z^\alpha := {z_1}^{\alpha_1} \ldots {z_n}^{\alpha_n}$ are pair-wise orthogonal with

$$\langle z^\alpha, z^\alpha \rangle = 2^{|\alpha|} \alpha!, \tag{2.4}$$

where as usual $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\alpha! := \alpha_1! \ldots \alpha_n!$. The restriction of $\langle \cdot, \cdot \rangle$ to $\mathcal{P}_m(V)$, the homogeneous polynomials of degree m, can be written as

$$\langle p, q \rangle = 2^m p(D) q, \tag{2.5}$$

where $p(D) := p(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n})$.

The infinite dimensional irreducible unitary representations π_λ of H_n are parameterized by non-zero real numbers λ and can be realized on $\mathcal{F}$. One has

$$\pi_\lambda(z, t) = \pi(\sqrt{|\lambda|} z, \lambda t) \tag{2.6}$$

where $\pi = \pi_1$ is given by the formula

$$(\pi(z, t) f)(w) = e^{it - \frac{(w, z)}{2} - \frac{|z|^2}{4}} f(w + z). \tag{2.7}$$

Each point $w \in V$ yields a one-dimensional unitary representation

$$\chi_w(z, t) = e^{iRe(w, z)}. \tag{2.8}$$

The representations π_λ for $\lambda \in \mathbf{R}^*$ together with χ_w for $w \in V$ exhaust $\widehat{H_n}$.

§3 Gelfand pairs and spherical functions.

This section provides a summary of the main results in [**BJR2**]. We refer the reader to [**BJR2**] for proofs and further details.

Let K be a compact subgroup of $U(n)$. We denote by $C_K^\infty(H_n)$ and $L_K^1(H_n)$ the spaces of smooth (respectively L^1-) functions on H_n that are K-invariant. Such functions satisfy $f(kz,t) = f(z,t)$ for all $k \in K$. $\mathbf{D}_K(H_n)$ is the set of differential operators on H_n invariant under both the left action of H_n and the action of K.

DEFINITION 3.1. *A function $\phi \in C_K^\infty(H_n)$ is said to be K-spherical if ϕ is an eigenfunction for each operator $D \in \mathbf{D}_K(H_n)$ and $\phi(0,0) = 1$.*

We can describe the *bounded* K-spherical functions on H_n when (K, H_n) is a *Gelfand pair.*

DEFINITION 3.2. *(K, H_n) is a Gelfand pair if $L_K^1(H_n)$ is a commutative algebra under convolution.*

The following result provides an equivalent representation-theoretic condition [**BJR1**], [**C**].

THEOREM 3.3. *(K, H_n) is a Gelfand pair if and only if the action of K on $\mathcal{P}(V)$ (given by $(k \cdot p)(z) = p(k^{-1}z)$) is multiplicity free.*

Suppose below that (K, H_n) is a Gelfand pair and let

$$\mathcal{P}(V) = \sum_{\alpha \in \Lambda} P_\alpha \tag{3.4}$$

be the multiplicity free decomposition of $\mathcal{P}(V)$ into K-irreducible components. Note that since each $\mathcal{P}_m(V)$ is K-invariant, P_α is a space of homogeneous polynomials. The bounded K-spherical functions are parameterized by the set $\mathcal{M} = (\mathbf{R}^* \times \Lambda) \coprod (V/K)$. The K-spherical function $\phi_{\lambda,\alpha}$ for $(\lambda, \alpha) \in \mathbf{R}^* \times \Lambda$ can be written as

$$\phi_{\lambda,\alpha} = \int_K \langle \pi_\lambda(kz,t)v, v\rangle \, dk \tag{3.5}$$

for any unit vector $v \in P_\alpha \subset \mathcal{F}$, and also as

$$\phi_{\lambda,\alpha} = \frac{1}{dim(P_\alpha)} \sum_{i=1}^{dim(P_\alpha)} \langle \pi_\lambda(kz,t)v_i, v_i\rangle \tag{3.6}$$

where $\{v_i\}$ is any orthonormal basis for P_α. The bounded K-spherical function for $Kw \in V/K$ is

$$\eta_w(z,t) = \int_K \chi_w(kz,t)dk. \tag{3.7}$$

We call $\phi_{\lambda,\alpha}$ and $\eta_w(z,t)$ K-spherical functions of types 1 and 2 respectively.

The type 2 K-spherical functions do not depend on the central variable t. These factor through the abelian group $H_n/\mathbf{R} \simeq V$ and reflect the abelian component of analysis on H_n. The type 2 K-spherical functions occupy a set of measure 0 in the Gelfand space $\widehat{L^1_K(H_n)}$. The type 1 K-spherical functions are our main concern here. Using Formula 2.6 one shows that

$$\phi_{\lambda,\alpha}(z,t) = \begin{cases} \overline{\phi_\alpha(\sqrt{|\lambda|}, |\lambda|t)}, & \text{for } \lambda < 0 \\ \phi_\alpha(\sqrt{\lambda}z, \lambda t), & \text{for } \lambda > 0. \end{cases} \tag{3.8}$$

where $\phi_\alpha := \phi_{1,\alpha}$. Moreover, using Formulas 2.3 and 2.7 one can show that ϕ_α has the general form

$$\phi_\alpha(z,t) = e^{it-\frac{|z|^2}{4}} q_\alpha(z) \tag{3.9}$$

where $q_\alpha(z)$ is a polynomial function in variables $(z, \bar{z})$. For $P_\alpha \subset \mathcal{P}_m(V)$, the degree of q_α is $2m$.

Let ${}^K\mathcal{P}(V_{\mathbf{R}})$ denote the K-invariant polynomials on the underlying real vector space $V_{\mathbf{R}}$ for V. In terms of coordinates one can write

$$\mathcal{P}(V_{\mathbf{R}}) = \mathbf{C}[z_1, \ldots, z_n, \overline{z_1}, \ldots, \overline{z_n}].$$

Note that $q_\alpha \in {}^K\mathcal{P}(V_{\mathbf{R}})$. It is not difficult to describe a basis for ${}^K\mathcal{P}(V_{\mathbf{R}})$ using decomposition 3.4. Let

$$p_\alpha := \sum_{j=1}^{dim(P_\alpha)} v_j(z)\overline{v_j(z)} \tag{3.10}$$

where $\{v_j\}$ is any orthonormal basis for P_α. Note that p_α takes values in $\mathbf{R}^+ = [0, \infty)$. In [**BJR2**], we prove that $\{p_\alpha : \alpha \in \Lambda\}$ is a vector space basis for ${}^K\mathcal{P}(V_{\mathbf{R}})$. It turns out that the q_α's can be obtained from the p_α's by orthogonalization. In particular, the homogeneous component of highest degree in q_α is $\frac{(-1)^{deg(P_\alpha)}}{dim(P_\alpha)} p_\alpha$.

THEOREM 3.11 (ORTHOGONALIZATION PROCEDURE). *Order Λ in any way that ensures $\alpha < \beta \Longrightarrow deg(p_\alpha) \le deg(p_\beta)$. The sequence $\{q_\alpha\}$ is obtained from $\{p_\alpha\}$ by performing Gram-Schmidt orthogonalization with respect to the measure $e^{-\frac{|z|^2}{2}} dz d\bar{z}$ and normalizing so that $q_\alpha(0) = 1$. Equivalently, one can perform the orthogonalization procedure using the partial order defined by $\alpha < \beta \Longleftrightarrow deg(p_\alpha) < deg(p_\beta)$.*

There is an alternative procedure that can be used to obtain the q_α's from the p_α's. For $p \in \mathcal{P}(V_{\mathbf{R}})$ we define the associated *magic operator* $\mathcal{M}_p$ on $C^\infty(V_{\mathbf{R}})$ by

$$\mathcal{M}_p := p(2\frac{\partial}{\partial \bar{z}}, -2\frac{\partial}{\partial z}). \tag{3.12}$$

([**BJR2**] also contains a coordinate-free description of $\mathcal{M}_p$.) One has

THEOREM 3.13 (RODRIGUES' FORMULA).

$$\mathcal{M}_{p_\alpha}(e^{-\frac{|z|^2}{2}}) = dim(P_\alpha)q_\alpha(z)e^{-\frac{|z|^2}{2}}$$

Theorem 3.3 together with results of Howe and Umeda [**HU**] show that ${}^K\mathcal{P}(V_{\mathbf{R}})$ is a polynomial ring:

$$(3.14) \qquad {}^K\mathcal{P}(V_{\mathbf{R}}) = \mathbf{C}[\gamma_1, \gamma_2, \ldots, \gamma_d]$$

where $\{\gamma_1, \gamma_2, \ldots, \gamma_d\} \subset \{p_\alpha : \alpha \in \Lambda\}$ is an (essentially canonical) set of generators which we will call *fundamental invariants.* The *value space* $\Gamma \subset (\mathbf{R}^+)^d$ is the image of $\gamma := \gamma_1 \times \gamma_2 \times \cdots \times \gamma_d$. We can express any $p \in {}^K\mathcal{P}(V_{\mathbf{R}})$ as a polynomial function on Γ in the variables $(\gamma_1, \ldots, \gamma_d)$. Note that when K acts irreducibly on V, $\mathcal{P}_1(V)$ is one of the K-irreducible components in decomposition 3.4. Formula 2.4 shows this has orthonormal basis $\{z_1/\sqrt{2}, \ldots, z_n/\sqrt{2}\}$ and Formula 3.10 yields the fundamental invariant

$$(3.15) \qquad \gamma_1 := \frac{|z|^2}{2}.$$

The measure $e^{-\frac{|z|^2}{2}}dzd\bar{z}$ and the magic operators $\mathcal{M}_{p_\alpha}$ are K-invariant and hence descend to the value space Γ. Thus, Theorems 3.11 and 3.13 can be reformulated on Γ. We will illustrate this procedure in the examples presented in the remaining sections. When K acts irreducibly on V, the factor $e^{-\frac{|z|^2}{2}}$ appearing in Theorems 3.11 and 3.13 can be written as $e^{-\gamma_1}$ on the value space.

Using the symplectic structure ω on $V_{\mathbf{R}}$ to identify $V_{\mathbf{R}}$ with $V_{\mathbf{R}}^*$, the symmetrization map [**He**] yields a K-equivariant linear map

$$(3.16) \qquad \tilde{\lambda} : \mathcal{P}(V_{\mathbf{R}}) \to \mathbf{D}(H_n).$$

Let

$$(3.17) \qquad L_j := \tilde{\lambda}(\gamma_j).$$

$\mathbf{D}_K(H_n)$ is generated by $\{L_1, \ldots, L_d, \frac{\partial}{\partial t}\}$. Writing $\phi_\alpha(z,t) = e^{it}\psi_\alpha(z)$ where $\psi_\alpha(z) = e^{-\frac{|z|^2}{4}}q_\alpha(z)$, we see that ψ_α is an eigenfunction for the *reduced operators* $L_1', \ldots, L_d'$ obtained by replacing each occurrence of $\frac{\partial}{\partial t}$ in L_j by $\sqrt{-1}$. By K-invariance, each L_j' descends to Γ and $\{\psi_\alpha(\gamma) : \alpha \in \Lambda\}$ are the bounded simultaneous eigenfunctions ψ for these differential operators with $\psi(0) = 1$. The associated eigenvalues can be computed by the recipe

$$(3.18) \qquad L_j'\psi_\alpha = \lambda_\alpha\psi_\alpha$$

where $\pi(L_j)v_\alpha = \lambda_\alpha v_\alpha$ for any $v_\alpha \in P_\alpha$.

§4 $U(n)$-spherical functions.

It is well known that $(U(n), H_n)$ is a Gelfand pair, and the associated spherical functions have been obtained independently by many authors. (See **[Fa]**, **[HR]**, **[Ko]**, **[St]**, **[Str]**.) We present these results here as a relatively straight-forward application of the general theory presented in Section 3. The reader may wish to consult **[BJR2]** for further details.

The homogeneous polynomials $\mathcal{P}_m(V)$ of degree m on V are irreducible $U(n)$-modules. Thus decomposition 3.4 for $K = U(n)$ is just

$$\mathcal{P}(V) = \sum_{m=0}^{\infty} \mathcal{P}_m(V). \tag{4.1}$$

This shows immediately that $(U(n), H_n)$ is indeed a Gelfand pair. In view of Formula 2.4, $\{z^\alpha/\sqrt{2^m \alpha!} : |\alpha| = m\}$ is an orthonormal basis for $\mathcal{P}_m(V)$ and the associated invariant $p_m \in {}^{U(n)}\mathcal{P}(V_{\mathbf{R}})$ is given by

$$p_m(z) = \frac{1}{2^m} \sum_{|\alpha|=m} \frac{|z|^{2\alpha}}{\alpha!} = \frac{1}{m!2^m}|z|^{2m} = \frac{1}{m!}\left(\frac{|z|^2}{2}\right)^m \tag{4.2}$$

Setting $\gamma := \gamma_1 = \frac{|z|^2}{2}$ as in Formula 3.15, we see that ${}^{U(n)}\mathcal{P}(V_{\mathbf{R}}) = \mathbf{C}[\gamma]$. That is, there is just one fundamental invariant here and the value space is $\Gamma = \mathbf{R}^+$. Moreover,

$$p_m = \frac{\gamma^m}{m!} \tag{4.3}$$

is the formula for p_m as a polynomial function on Γ.

Suppose that $p(\gamma)$ is any polynomial on Γ. Using polar coordinates on V we compute

$$\begin{aligned} \int_V p\left(\frac{|z|^2}{2}\right) e^{-\frac{|z|^2}{2}} dz d\bar{z} &= \frac{2\pi^n}{(n-1)!} \int_0^\infty p\left(\frac{r^2}{2}\right) e^{-\frac{r^2}{2}} r^{2n-1} dr \\ &= \frac{(2\pi)^n}{(n-1)!} \int_0^\infty p(\gamma) e^{-\gamma} \gamma^{n-1} d\gamma. \end{aligned} \tag{4.4}$$

This shows that the measure $e^{-\frac{|z|^2}{2}} dz d\bar{z}$ on $V_{\mathbf{R}}$ descends to the measure

$$d\nu = \frac{(2\pi)^n}{(n-1)!} e^{-\gamma} \gamma^{n-1} d\gamma \tag{4.5}$$

on $\Gamma = \mathbf{R}^+$.

Since spherical functions are normalized to have value 1 at 0, we can drop the constants in Formulae 4.3 and 4.5 to obtain the following immediately from Theorem 3.11.

LEMMA 4.6. *The polynomials $q_m(\gamma)$ on $\Gamma = \mathbf{R}^+$ are obtained by Gram-Schmidt orthogonalization of the sequence $\{\gamma^m : m \geq 0\}$ with respect to the measure $e^{-\gamma}\gamma^{n-1}d\gamma$ and normalizing so that $q_m(0) = 1$.*

The orthogonalization problem posed in Lemma 4.6 is classical and yields the *generalized Laguerre polynomials* $L_m^{(n-1)}(\gamma)$ of order $n-1$ (suitably normalized). These can be written explicitly as

$$L_m^{(n-1)}(\gamma) = (n-1)! \sum_{j=0}^{m} \binom{m}{j} \frac{(-\gamma)^j}{(j+n-1)!}. \tag{4.7}$$

We have now proved the following.

THEOREM 4.8. *The bounded $U(n)$-spherical functions of type 1 on H_n are determined by the polynomials $q_m(\gamma) = L_m^{(n-1)}(\gamma)$ on the value space $\Gamma = \mathbf{R}^+$. Explicitly, one has for $\lambda \in \mathbf{R}^+$ and $m = 0, 1, \ldots$*

$$\phi_{\lambda,m} = e^{i\lambda t} L_m^{(n-1)}\left(\frac{|\lambda||z|^2}{2}\right) e^{-\frac{|\lambda||z|^2}{4}}. \tag{4.9}$$

Alternatively, one can prove Theorem 4.8 by using Rodrigues' Formula. The magic operator for p_m is

$$\begin{aligned} \mathcal{M}_{p_m} &= \frac{(-2)^m}{m!} \left(\frac{\partial^2}{\partial z_1 \partial \bar{z}_1} + \cdots + \frac{\partial^2}{\partial z_n \partial \bar{z}_n}\right)^m \\ &= \frac{(-1)^m}{m!} \left(\gamma \frac{d^2}{d\gamma^2} + n\frac{d}{d\gamma}\right)^m, \end{aligned} \tag{4.10}$$

where the second equality requires an application of the chain rule. Since $dim(\mathcal{P}_m(V)) = \binom{m+n-1}{m}$, Theorem 3.13 yields the identity

$$q_m(\gamma) = \frac{(-1)^m (n-1)!}{(m+n-1)!} e^{\gamma} \left(\gamma \frac{d^2}{d\gamma^2} + n\frac{d}{d\gamma}\right)^m (e^{-\gamma}) \tag{4.11}$$

for the polynomials $q_m(\gamma)$ on Γ. One can prove by induction on m that Formula 4.11 is equivalent to

$$q_m(\gamma) = \frac{(n-1)!}{(m+n-1)!} \gamma^{1-n} e^{\gamma} \left(\frac{d}{d\gamma}\right)^m \left(\gamma^{m+n-1} e^{-\gamma}\right), \tag{4.12}$$

which is the classical Rodrigues' formula for the Laguerre polynomials.

$2\tilde{\lambda}(\gamma)$ is the usual Heisenberg sublaplacian. The associated reduced operator is

$$\begin{aligned} L := 2L_1' &= 4\frac{\partial^2}{\partial z \partial \bar{z}} - \frac{|z|^2}{4} \\ &= 2\gamma \frac{d^2}{d\gamma^2} + 2n\frac{d}{d\gamma} - \frac{\gamma}{2}. \end{aligned} \tag{4.13}$$

It can be shown using Formula 3.18 that $\psi_m(\gamma) = q_m(\gamma)e^{-\frac{\gamma}{2}}$ is an eigenfunction for L with eigenvalue $-(2m+n)$. We obtain

$$\gamma q_m''(\gamma) + (n-\gamma)q_m'(\gamma) + mq_m(\gamma) = 0, \tag{4.14}$$

which is the standard differential equation for the generalized Laguerre polynomial of order $n-1$ and degree m.

We have shown that the bounded $U(n)$-spherical functions of type 1 are determined by the solutions to an eigenvalue problem on $\Gamma = \mathbf{R}^+$. This can be solved using an orthogonalization procedure with respect to an explicit measure on Γ or by using Rodrigues' formula. For completeness, we note that the bounded $U(n)$-spherical functions of type 2 are given by

$$\eta_\tau(z,t) = \frac{2^{n-1}(n-1)!}{(\tau|z|)^{n-1}} J_{n-1}(\tau|z|) \tag{4.15}$$

for each $\tau > 0$.

§**5** **($SO(n,\mathbf{R}) \times \mathbf{T}$)-spherical functions.**

We now consider the action of $K = SO(n,\mathbf{R}) \times \mathbf{T}$ on $V = \mathbf{C}^n$ and H_n. Here $SO(n,\mathbf{R})$ is the subgroup of $U(n)$ given by the real matrices with determinant one. The circle $\mathbf{T}$ acts on V by scalar multiplication. When n is even, the representation of K on V is not faithful and has kernel $\mathbf{Z}_2 = \{(I,1),(-I,-1)\}$. We assume throughout that $n \geq 3$. In this case, K acts irreducibly on V. (For $n = 2$, the action is equivalent to the usual reducible action of $\mathbf{T} \times \mathbf{T}$ on $\mathbf{C} \oplus \mathbf{C} = \mathbf{C}^2$.) (K, H_n) is a Gelfand pair since K appears on Kac's list of irreducible multiplicity free representations [**BJR1**]. This fact will be verified directly below by exhibiting the K-decomposition of $\mathcal{P}(V)$. First however we will describe the K-orbits in V.

Consider a point $z = (z_1, \ldots, z_n) \neq 0$ in V and write $z = u + iv$ where $u = Re(z)$, $v = Im(z)$. For $A \in SO(n,\mathbf{R})$, one has $Az = Au + iAv$. This shows that the $SO(n,\mathbf{R})$-action preserves $arg(z_1), \ldots, arg(z_n)$. On the other hand, $\mathbf{T}$ acts by phase shifts in these arguments. Since the actions of $SO(n,\mathbf{R})$ and $\mathbf{T}$ commute, we conclude that

$$K \cdot z \simeq (SO(n,\mathbf{R}) \cdot z) \times \mathbf{T}. \tag{5.1}$$

Identify z with the point $(u,v) \in \mathbf{R}^n \times \mathbf{R}^n$. The corresponding action of $SO(n,\mathbf{R})$ on $\mathbf{R}^n \times \mathbf{R}^n$ is diagonal. The orbit $SO(n,\mathbf{R}) \cdot (u,v)$ is diffeomorphic to the sphere S^{n-1} when one of u, v is a (real) scalar multiple of the other. Otherwise, (u,v) belongs to the *Stiefel manifold* $V'_{n,2}$ of 2-frames in $\mathbf{R}^n$. The action of $SO(n,\mathbf{R})$ preserves $V'_{n,2}$. The orbit through (e_1,e_2), where $\{e_j\}$ is the standard basis for $\mathbf{R}^n$, is $V_{n,2}$, the compact Stiefel manifold of *orthonormal* 2-frames in $\mathbf{R}^n$. Translation of component v to the point on S^{n-1} given by component u of an orthonormal frame (u,v) yields a diffeomorphism between $V_{n,2}$ and the unit tangent bundle $T_1(S^{n-1})$ of S^{n-1} [**Ste**]. Consider the map

$$G : V'_{n,2} \to V_{n,2} \tag{5.2}$$

given by the Gram-Schmidt algorithm. G is $SO(n, \mathbf{R})$-equivariant since $SO(n, \mathbf{R})$ preserves the real inner product on $\mathbf{R}^n$. It is not hard to verify that the restriction of G to any orbit $SO(n, \mathbf{R}) \cdot (u, v)$ is one-to-one and onto. In fact,

$$(5.3)\quad SO(n, \mathbf{R}) \cdot (u, v) = \{(u', v') \in V'_{n,2} : ||u'|| = ||u||, ||v'|| = ||v||, u' \cdot v' = u \cdot v\}$$

These remarks prove the following.

THEOREM 5.4. *The topological structure of the K-orbits in V is given by:*
(1) $K \cdot 0 = \{0\}$.
(2) $K \cdot z \simeq S^{n-1} \times \mathbf{T}$ *for* $\{Re(z), Im(z)\}$ *linearly dependent in* $\mathbf{R}^n$.
(3) $K \cdot z \simeq V_{n,2} \times \mathbf{T}$ *for* $\{Re(z), Im(z)\}$ *linearly independent.*

The generic orbits are given by case (3) in Theorem 5.4. For $(u, v) = (e_1, e_2)$, one has the isotropy group

$$(5.5)\qquad K_{e_1+ie_2} = \begin{cases} SO(n-2), & \text{for } n \text{ odd} \\ \mathbf{Z}_2 \cdot SO(n-2), & \text{for } n \text{ even.} \end{cases}$$

The generic orbit codimension is thus

$$dim(V) - (dim(K) - dim(K_{e_1+ie_2})) = 2n - \left(\frac{n(n-1)}{2} + 1 - \frac{(n-2)(n-3)}{2} \right) = 2.$$

This suggests that the action of K should have two fundamental invariants.

The decomposition of $\mathcal{P}(V)$ into K-irreducible components is based on the classical theory of spherical harmonics. (See for example [**Fa**], [**SW**] or [**Ho**].) Let

$$(5.6)\qquad \varepsilon := z_1^2 + \cdots + z_n^2 \qquad \text{and} \qquad \Delta := \varepsilon(D) = \frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2}.$$

The *harmonic polynomials* are $\mathcal{H} := \mathrm{Ker}(\Delta : \mathcal{P}(V) \to \mathcal{P}(V))$ and we define

$$(5.7)\qquad \mathcal{H}_m := \mathcal{H} \cap \mathcal{P}_m(V).$$

$\mathcal{H}_m$ is $SO(n, \mathbf{R})$-invariant by invariance of Δ. In fact, $\mathcal{H}_m$ is $SO(n, \mathbf{R})$-irreducible and

$$(5.8)\qquad \mathcal{P}_m = \mathcal{H}_m \oplus \varepsilon \mathcal{P}_{m-2}$$

as $SO(n, \mathbf{R})$-modules. In particular, we see that

$$dim(\mathcal{H}_m) = dim(\mathcal{P}_m) - dim(\mathcal{P}_{m-2}) = \binom{m+n-1}{m} - \binom{m+n-3}{m-2} \text{ for } m \geq 2.$$

Formula 5.8 yields the inductive decomposition

$$(5.9)\qquad \mathcal{P}_m = \sum_{k+2\ell=m} P_{k,\ell}$$

where $P_{k,\ell} := \mathcal{H}_k \varepsilon^\ell$. Each $P_{k,\ell}$ is $SO(n, \mathbf{R})$-irreducible and the modules $\{P_{k,\ell} : k + 2\ell = m\}$ are inequivalent since the dimensions differ. The modules $P_{k,\ell}$ are also $\mathbf{T}$-invariant since $\mathbf{T}$ acts on V by scalars. Thus, decomposition 3.4 is in the present case

$$\mathcal{P}(V) = \sum_{k,\ell} P_{k,\ell}. \tag{5.10}$$

If $k = k'$ and $\ell \neq \ell'$ then $P_{k,\ell}$ and $P_{k',\ell'}$ are inequivalent K-modules since T acts on $\mathcal{P}_m$ by the character $t \mapsto t^{-m}$. This shows (5.10) to be multiplicity free and hence (K, H_n) is a Gelfand pair. Note that $(SO(n,\mathbf{R}), H_n)$ fails to be a Gelfand pair because ε is $SO(n,\mathbf{R})$-invariant. (ie. $P_{0,0}$ and $P_{0,1}$ are equivalent $SO(n,\mathbf{R})$-modules.)

Write $p_{k,\ell}$ for the K-invariant polynomial on $V_{\mathbf{R}}$ determined by $P_{k,\ell}$. Note that $P_{1,0} = \mathcal{H}_1 = \mathcal{P}_1(V)$ and $P_{0,1} = \mathbf{C}\varepsilon$ yield $p_{1,0} = \frac{|z|^2}{2}$ and $p_{0,1} = \frac{|\varepsilon|^2}{||\varepsilon||^2} = \frac{|\varepsilon|^2}{4n}$ respectively. We define

$$\gamma_1 := p_{1,0} = \frac{|z|^2}{2} \quad \text{and} \quad \gamma_2 := np_{0,1} = \frac{|\varepsilon|^2}{4}. \tag{5.11}$$

The following result shows that (γ_1, γ_2) are fundamental invariants for the action of K by providing an inductive method to express each $p_{k,\ell}$ as a polynomial in γ_1 and γ_2. (See also [**HU**].)

THEOREM 5.12. *Writing p_k for $p_{k,0}$ one has*

(1) $p_{k,\ell} = \frac{\gamma_2^\ell}{c_{k,\ell}} p_k$ *where* $c_{k,\ell} = 4^\ell (\ell!)^2 \binom{k+\frac{n}{2}+\ell-1}{\ell}$

(2) p_k *is determined by the recurrence relation*

$$p_k = \frac{\gamma_1^k}{k!} - \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \frac{\gamma_2^j p_{k-2j}}{c_{k-2j,j}} \qquad \text{for } k \geq 2,$$

with initial conditions $p_0 = 1$, $p_1 = \gamma_1$.

PROOF. Formula 2.5 shows that for $g, h \in \mathcal{H}_k$,

$$\left\langle \varepsilon^\ell g, \varepsilon^\ell h \right\rangle = 2^{2\ell} \left\langle g, \Delta^\ell \left(\varepsilon^\ell h\right) \right\rangle. \tag{5.13}$$

We compute that $\Delta(\varepsilon^\ell h) = 4\ell(k + \frac{n}{2} + \ell - 1)\varepsilon^{\ell-1} h$ and hence by induction that

$$\Delta^\ell \left(\varepsilon^\ell h\right) = \frac{4^\ell \ell!(k + \frac{n}{2} + \ell - 1)!}{(k + \frac{n}{2} - 1)!} h = c_{k,\ell} h. \tag{5.14}$$

(For $N \in \mathbf{N}$, $(N + \frac{1}{2})!$ means $(N + \frac{1}{2})(N - \frac{1}{2}) \dots \frac{1}{2} = \frac{\Gamma(N+3/2)}{2\Gamma(3/2)}$.) Thus we have

$$\left\langle \varepsilon^\ell g, \varepsilon^\ell h \right\rangle = 2^{2\ell} c_{k,\ell} \left\langle g, h \right\rangle. \tag{5.15}$$

Let $\{v_1, \dots, v_s\}$ be an orthonormal basis for $P_{k,0} = \mathcal{H}_k$. By Formula 5.15,

$$\left\{\varepsilon^\ell v_1 / \left(2^\ell \sqrt{c_{k,\ell}}\right), \dots, \varepsilon^\ell v_s / \left(2^\ell \sqrt{c_{k,\ell}}\right)\right\}$$

is an orthonormal basis for $P_{k,\ell}$ and thus

$$\begin{aligned} p_{k,\ell} &= \sum_{j=1}^{s} \left(\frac{\varepsilon^\ell v_j}{2^\ell \sqrt{c_{k,\ell}}}\right) \overline{\left(\frac{\varepsilon^\ell v_j}{2^\ell \sqrt{c_{k,\ell}}}\right)} \\ &= \frac{1}{c_{k,\ell}} \left(\frac{|\varepsilon|^2}{4}\right)^\ell \sum_{j=1}^{s} v_j \bar{v}_j \\ &= \frac{\gamma_2^\ell p_k}{c_{k,\ell}}. \end{aligned}$$

Next recall that the $U(n)$-invariant polynomial obtained from $\mathcal{P}_m(V)$ is $\gamma_1^m/m!$. (See Formula 4.3.) In view of (5.9), we conclude that

$$\sum_{k+2\ell=m} p_{k,\ell} = \frac{\gamma_1^m}{m!} \tag{5.16}$$

since one can form an orthonormal basis for $\mathcal{P}_m(V)$ by taking the union of orthonormal bases for $P_{m,0}, P_{m-2,1}, \ldots, P_{m(mod\ 2),\lfloor \frac{m}{2} \rfloor}$. Using the first identity of Theorem 5.12, we obtain

$$p_m = p_{m,0} = \frac{\gamma_1^m}{m!} - \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} \frac{\gamma_2^j p_{m-2j}}{c_{m-2j,j}}$$

as claimed. □

LEMMA 5.17. *For all* $0 \leq N \leq m$ *one has*

$$Span\{p_{k,\ell} : k+2\ell = m \quad and \quad k \leq N\} = Span\{\gamma_1^k \gamma_2^\ell : k+2\ell = m \quad and \quad k \leq N\}.$$

PROOF. We use induction on m. Note that for $m = 0$ and 1 one has $p_{0,0} = 1$ and $p_{1,0} = \gamma_1$. Assume inductively that $Span\{p_{k,\ell} : k + 2\ell = m \text{ and } k \leq N\} = Span\{\gamma_1^k \gamma_2^\ell : k + 2\ell = m \text{ and } k \leq N\}$ holds for all $0 \leq N \leq m$. In view of Theorem 5.12, we have

$$p_{k,\ell+1} = \frac{c_{k,\ell}}{c_{k,\ell+1}} \gamma_2 p_{k,\ell}. \tag{5.18}$$

Thus for $0 \leq N \leq m$,

$$\begin{aligned} &Span\{p_{k,\ell} : k + 2\ell = m + 2,\ \ell > 0 \text{ and } k \leq N\} \\ &\quad = Span\{\gamma_2 p_{k,\ell} : k + 2\ell = m,\ \ell > 0 \text{ and } k \leq N\} \\ &\quad = Span\{\gamma_1^k \gamma_2^{\ell+1} : k + 2\ell = m,\ \ell > 0 \text{ and } k \leq N\} \quad \text{(by hypothesis)} \\ &\quad = Span\{\gamma_1^k \gamma_2^\ell : k + 2\ell = m + 2,\ \ell > 0 \text{ and } k \leq N\}. \end{aligned}$$

To complete the induction step note that in addition,

$$p_{m+2,0} = \frac{\gamma_1^{m+2}}{(m+2)!} - p_{m,1} - p_{m-2,2} - \cdots - p_{m(mod\ 2),\lfloor \frac{m+2}{2} \rfloor}. \tag{5.19}$$

Thus the result holds for all $0 \leq N \leq m + 2$. □

The fundamental invariants (γ_1, γ_2) yield a map $\gamma : V \to (\mathbf{R}^+)^2$ whose image is the value space

$$\Gamma = \{(\gamma_1, \gamma_2) \in (\mathbf{R}^+)^2 : \gamma_2 \leq \gamma_1^2\}. \tag{5.20}$$

We also introduce *rational* invariants

$$\rho = \gamma_1 = \frac{|z|^2}{2}, \qquad \sigma = \frac{\gamma_2}{\gamma_1^2} = \frac{|\varepsilon|^2}{|z|^4} \tag{5.21}$$

and the map $\gamma' : V\backslash\{0\} \to (\mathbf{R}^+)^2$, $z \mapsto (\rho(z), \sigma(z))$ with image

$$\Gamma' = \mathbf{R}^+ \times [0,1]. \tag{5.22}$$

A K-invariant polynomial on V can be regarded as a polynomial $p : \Gamma \to \mathbf{C}$ in the variables (γ_1, γ_2) and as a polynomial $p' : \Gamma' \to \mathbf{C}$ in the variables (ρ, σ) for which each non-zero term $c_{i,j}\rho^i\sigma^j$ has $i \geq 2j$.

In order to formulate the orthogonalization procedure on Γ (or on Γ') one needs to find the measure $d\nu$ on Γ (respectively $d\nu'$ on Γ') for which $\gamma^*(d\nu) = e^{-\frac{|z|^2}{2}} dz d\bar{z}$ (respectively $\gamma'^*(d\nu') = e^{-\frac{|z|^2}{2}} dz d\bar{z}$). The following theorem describes these measures.

THEOREM 5.23. *Let $p : \Gamma \to \mathbf{C}$, $p' : \Gamma' \to \mathbf{C}$ be corresponding K-invariant polynomials. Then*

$$\begin{aligned}\int_V p(\gamma(z)) e^{-\frac{|z|^2}{2}} dz d\bar{z} &= c_n \int_0^\infty \int_0^1 p'(\rho, \sigma) \rho^{n-1} e^{-\rho} (1-\sigma)^{\frac{n-3}{2}} d\sigma d\rho \\ &= c_n \int_0^\infty \int_0^{\gamma_1^2} p(\gamma_1, \gamma_2) e^{-\gamma_1} (\gamma_1^2 - \gamma_2)^{\frac{n-3}{2}} d\gamma_2 d\gamma_1\end{aligned}$$

where $c_n = \frac{(2\pi)^n}{2(n-2)!}$.

The proof requires the following Lemma.

LEMMA 5.24.

$$\int_V \gamma_1(z)^\ell \gamma_2(z)^m e^{-\frac{|z|^2}{2}} dz d\bar{z} = (2\pi)^n 4^m \ell! (m!)^2 \binom{2m+n+\ell-1}{\ell} \binom{m+\frac{n}{2}-1}{m}.$$

PROOF.

$$\begin{aligned}\gamma_1^\ell \gamma_2^m &= \left(\frac{z_1\bar{z}_1 + \cdots + z_n\bar{z}_n}{2}\right)^\ell \frac{1}{4^m} \left(z_1^2 + \cdots + z_n^2\right)^m \left(\bar{z}_1^2 + \cdots + \bar{z}_n^2\right)^m \\ &= \frac{1}{4^m 2^\ell} \left(\sum_{|\alpha|=\ell} \frac{\ell!}{\alpha!} z^\alpha \bar{z}^\alpha\right) \left(\sum_{|\beta|=m} \frac{m!}{\beta!} z^{2\beta}\right) \left(\sum_{|\nu|=m} \frac{m!}{\nu!} \bar{z}^{2\nu}\right) \\ &= \frac{1}{2^{\ell+2m}} \sum_{\substack{|\alpha|=\ell \\ |\beta|=m \\ |\nu|=m}} \frac{\ell!(m!)^2}{\alpha!\beta!\nu!} z^{\alpha+2\beta} \bar{z}^{\alpha+2\nu}.\end{aligned}$$

Thus,

$$\int_V \gamma_1^\ell \gamma_2^m e^{-\frac{|z|^2}{2}}\,dz d\bar{z} = \frac{(2\pi)^n \ell!(m!)^2}{2^{\ell+2m}} \sum_{\substack{|\alpha|=\ell\\|\beta|=m\\|\nu|=m}} \frac{\langle z^{\alpha+2\beta}, z^{\alpha+2\nu}\rangle}{\alpha!\beta!\nu!}$$

$$= \frac{(2\pi)^n \ell!(m!)^2}{2^{\ell+2m}} \sum_{\substack{|\alpha|=\ell\\|\beta|=m}} \frac{2^{\ell+2m}(\alpha+2\beta)!}{\alpha!(\beta!)^2}$$

$$= (2\pi)^n \ell!(m!)^2 \sum_{\substack{|\alpha|=\ell\\|\beta|=m}} \frac{(\alpha+2\beta)!}{\alpha!(\beta!)^2}$$

To complete the proof of the lemma we will show that

$$\sum_{\substack{|\alpha|=\ell\\|\beta|=m}} \frac{(\alpha+2\beta)!}{\alpha!(\beta!)^2} = 4^m \binom{2m+n+\ell-1}{\ell}\binom{m+\frac{n}{2}-1}{m}. \tag{5.25}$$

To see this, first note that

$$\left(\frac{1}{1-x}\right)^{2m+n} = \sum_{j=0}^{\infty} \binom{2m+n+j-1}{j} x^j, \tag{5.26}$$

and that for $|\beta| = m$,

$$\left(\frac{1}{1-x}\right)^{2m+n} = \left(\frac{1}{1-x}\right)^{2|\beta|+n} = \left(\frac{1}{1-x}\right)^{2\beta_1+1} \cdots \left(\frac{1}{1-x}\right)^{2\beta_n+1}$$

$$= \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \binom{\alpha_1+2\beta_1}{\alpha_1} \cdots \binom{\alpha_n+2\beta_n}{\alpha_n} x^j$$

$$= \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \binom{\alpha+2\beta}{\alpha} x^j.$$

Thus, $\sum_{|\alpha|=j} \binom{\alpha+2\beta}{\alpha} = \binom{2m+n+j-1}{j}$ holds for any β with $|\beta| = m$. This shows

$$\sum_{\substack{|\alpha|=\ell\\|\beta|=m}} \frac{(\alpha+2\beta)!}{\alpha!(\beta!)^2} = \sum_{\substack{|\alpha|=\ell\\|\beta|=m}} \binom{\alpha+2\beta}{\alpha} \frac{(2\beta)!}{(\beta!)^2} = \binom{2m+n+\ell-1}{\ell} \sum_{|\beta|=m} \binom{2\beta}{\beta}. \tag{5.27}$$

Moreover,

$$\left(\frac{1}{1-4x}\right)^{\frac{1}{2}} = \sum_{m=0}^{\infty} \binom{m-\frac{1}{2}}{m}(4x)^m = \sum_{m=0}^{\infty} \frac{1}{4^m}\binom{2m}{m}(4x)^m = \sum_{m=0}^{\infty} \binom{2m}{m} x^m$$

and hence

$$\left(\frac{1}{1-4x}\right)^{\frac{n}{2}} = \sum_{m=0}^{\infty} \sum_{|\beta|=m} \binom{2\beta}{\beta} x^m. \tag{5.28}$$

On the other hand,

$$\left(\frac{1}{1-4x}\right)^{\frac{n}{2}} = \sum_{m=0}^{\infty} \binom{m+\frac{n}{2}-1}{m} (4x)^m . \tag{5.29}$$

Thus we must have,

$$\sum_{|\beta|=m} \binom{2\beta}{\beta} = 4^m \binom{m+\frac{n}{2}-1}{m}. \tag{5.30}$$

Substituting Formula 5.30 in 5.27 yields Formula 5.25 and completes the proof of the Lemma. □

Proof of Theorem 5.23. It suffices to consider monomials $p = \gamma_1^\ell \gamma_2^m$ or equivalently $p' = \rho^{\ell+2m}\sigma^m$. We compute

$$\begin{aligned}(5.31)\quad \int_0^\infty \int_0^1 \rho^{\ell+2m}\sigma^m \rho^{n-1} e^{-\rho} (1-\sigma)^{\frac{n-3}{2}}\, d\sigma d\rho \\ = (\ell+2m+n-1)! \int_0^1 \sigma^m (1-\sigma)^{\frac{n}{2}-\frac{3}{2}}\, d\sigma\end{aligned}$$

and

(5.32)

$$\begin{aligned}\int_0^1 \sigma^m (1-\sigma)^{\frac{n}{2}-\frac{3}{2}}\, d\sigma &= \frac{m}{\left(\frac{n}{2}-\frac{1}{2}\right)} \int_0^1 \sigma^{m-1} (1-\sigma)^{\frac{n}{2}-\frac{1}{2}}\, d\sigma \\ &= \cdots = \frac{m(m-1)\cdots 1}{\left(\frac{n}{2}-\frac{1}{2}\right)\left(\frac{n}{2}+\frac{1}{2}\right)\cdots\left(\frac{n}{2}+m-\frac{3}{2}\right)} \int_0^1 (1-\sigma)^{\frac{n}{2}+m-\frac{3}{2}}\, d\sigma \\ &\qquad \text{(via integration by parts)} \\ &= \frac{m!}{\left(\frac{n}{2}-\frac{1}{2}\right)\left(\frac{n}{2}+\frac{1}{2}\right)\cdots\left(\frac{n}{2}+m-\frac{1}{2}\right)} = \frac{m!\left(\frac{n}{2}-\frac{3}{2}\right)!}{\left(\frac{n}{2}+m-\frac{1}{2}\right)!}.\end{aligned}$$

Thus

(5.33)

$$\int_0^\infty \int_0^1 \rho^{\ell+2m}\sigma^m \rho^{n-1} e^{-\rho} (1-\sigma)^{\frac{n-3}{2}}\, d\sigma d\rho = \frac{(2m+n+\ell-1)!m!\left(\frac{n-3}{2}\right)!}{\left(m+\frac{n-1}{2}\right)!}.$$

On the other hand, the polynomial $p' = \rho^{\ell+2m}\sigma^m$ on Γ' corresponds to $\gamma_1(z)^\ell \gamma_2(z)^m$ on V. Lemma 5.24 together with Formula 5.33 yields

$$\begin{aligned}\int_V p(\gamma(z)) e^{-\frac{|z|^2}{2}}\, dz d\bar{z} &= \frac{2\pi^n (2m+n+\ell-1)!m!}{\left(\frac{n-2}{2}\right)!\left(m+\left(\frac{n-1}{2}\right)\right)!} \\ &= \frac{2\pi^n}{\left(\frac{n-2}{2}\right)!\left(\frac{n-3}{2}\right)!} \int_0^\infty \int_0^1 \rho^{\ell+2m}\sigma^m \rho^{n-1} e^{-\rho} (1-\sigma)^{\frac{n-3}{2}}\, d\sigma d\rho \\ &= c_n \int_0^\infty \int_0^1 \rho^{\ell+2m}\sigma^m \rho^{n-1} e^{-\rho} (1-\sigma)^{\frac{n-3}{2}}\, d\sigma d\rho\end{aligned}$$

as desired since $\left(\frac{n-2}{2}\right)!\left(\frac{n-3}{2}\right)! = \frac{(n-2)!}{2^{n-2}}$.

The measure on Γ is obtained via the change of variables $\rho = \gamma_1$, $\sigma = \gamma_2/\gamma_1^2$ which has Jacobian

$$\left|\begin{matrix} \frac{\partial\rho}{\partial\gamma_1} & \frac{\partial\rho}{\partial\gamma_2} \\ \frac{\partial\sigma}{\partial\gamma_1} & \frac{\partial\sigma}{\partial\gamma_2} \end{matrix}\right| = \left|\begin{matrix} 1 & 0 \\ -\frac{2\gamma_2}{\gamma_1^3} & \frac{1}{\gamma_1^2} \end{matrix}\right| = \frac{1}{\gamma_1^2}. \tag{5.34}$$

We have

$$\begin{aligned} c_n \int_0^\infty \int_0^1 p'(\rho,\sigma)\rho^{n-1}e^{-\rho}(1-\sigma)^{\frac{n-3}{2}}\, d\sigma d\rho \\ &= c_n \int_0^\infty \int_0^{\gamma_1^2} p(\gamma_1,\gamma_2)\gamma_1^{n-1}e^{-\gamma_1}\left(1-\frac{\gamma_2}{\gamma_1^2}\right)^{\frac{n-3}{2}} \frac{1}{\gamma_1^2} d\gamma_2 d\gamma_1 \\ &= c_n \int_0^\infty \int_0^{\gamma_1^2} p(\gamma_1,\gamma_2)e^{-\gamma_1}(\gamma_1^2-\gamma_2)^{\frac{n-3}{2}} d\gamma_2 d\gamma_1 \end{aligned}$$

as stated. □

We introduce the total ordering $<$ on pairs $(k,\ell) \in (\mathbf{N}\cup\{0\})\times(\mathbf{N}\cup\{0\})$ given by

(5.35)

$$(k,\ell) < (k',\ell') \iff k+2\ell < k'+2\ell' \quad \text{or} \quad k+2\ell = k'+2\ell' \quad \text{and} \quad k<k'.$$

Recall that the type 1 bounded K-spherical functions on H_n are completely determined by a sequence of polynomials $q_{k,\ell}(\gamma_1,\gamma_2)$ on Γ.

THEOREM 5.36. *The polynomials $\{q_{k,\ell}(\gamma_1,\gamma_2) : k,\ell \in \mathbf{N}\cup\{0\}\}$ are obtained by applying the Gram-Schmidt algorithm with respect to the measure $e^{-\gamma_1}(\gamma_1^2 - \gamma_2)^{\frac{n-3}{2}}d\gamma_2 d\gamma_1$ on Γ to the sequence of monomials $\{\gamma_1^k\gamma_2^\ell\}$ ordered by (5.35) and normalizing so that $q_{k,\ell}(0,0)=1$.*

PROOF. Theorem 3.11 shows that $\{q_{k,\ell}\}$ is obtained (up to normalization) by applying the Gram-Schmidt algorithm to the sequence $\{p_{k,\ell}\}$. The $p_{k,\ell}$'s can be ordered in any way that ensures that $p_{k,\ell}$ precedes $p_{k',\ell'}$ if $deg(p_{k,\ell}) < deg(p_{k',\ell'})$. Condition 5.35 above gives one such ordering.

$\{q_{k,\ell}\}$ is characterized by the conditions (see [**BJR2**])

(1) $Span\{q_{k'\ell'} : (k',\ell') \leq (k,l)\} = Span\{p_{k'\ell'} : (k',\ell') \leq (k,l)\}$,
(2) $q_{k,\ell} \perp Span\{p_{k'\ell'} : (k',\ell') < (k,l)\}$,
(3) $q_{k,\ell}(0,0) = 1$.

An application of Lemma 5.17 shows that

$$p_{k,\ell} = \frac{\gamma_1^k\gamma_2^\ell}{k!c_{k,\ell}} \bmod Span\{p_{k'\ell'} : (k',\ell') < (k,l)\}. \tag{5.37}$$

This implies that we can replace $p_{k,\ell}$ by $\gamma_1^k\gamma_2^\ell$ when carrying out the Gram-Schmidt algorithm. □

We remark that Theorem 5.36 can be reformulated on Γ'. The polynomials $\{q'_{k,\ell}(\rho,\sigma)\}$ can be obtained by applying the Gram-Schmidt algorithm with the measure

$$\rho^{n-1}e^{-\rho}(1-\sigma)^{\frac{n-3}{2}}\, d\sigma d\rho \tag{5.38}$$

to the monomials $\rho^k\sigma^\ell$ for which $\ell \geq 2k$. These are ordered so that $\rho^k\sigma^\ell < \rho^{k'}\sigma^{\ell'}$ if either $k < k'$ or $k = k'$ and $\ell' > \ell$.

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