

Bounded K -Spherical Functions on Heisenberg Groups*

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Let H_n be the $(2n + 1)$ -dimensional Heisenberg group, and let K be a compact subgroup of $\text{Aut}(H_n)$, the group of automorphisms of H_n . The pair (K, H_n) is called a Gelfand pair if $L_K^1(H_n)$, the subalgebra of elements of $L^1(H_n)$ that are invariant under the action of K , is commutative. In this case, the continuous homomorphisms on $L_K^1(H_n)$ are given by integrating against certain K -invariant functions on H_n . These functions are the K -spherical functions associated to the Gelfand pair (K, H_n) . In this paper we show how to compute the bounded K -spherical functions on H_n . © 1992 Academic Press, Inc.

INTRODUCTION

Suppose that G is a Lie group and K is a compact Lie subgroup of $\text{Aut}(G)$. We say that (K, G) is a *Gelfand pair* if the convolution algebra $L_K^1(G)$ of K -invariant L^1 -functions on G is commutative. Equivalently, $K \subset K \rtimes G$ is a Gelfand pair in the usual sense that $L^1(K \rtimes G // K)$, the space of K -bi-invariant L^1 -functions on $K \rtimes G$, is $*$ -commutative.

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In [BJR] the authors considered Gelfand pairs (K, G) with G a connected, solvable Lie group. The classification of such Gelfand pairs was reduced to the classification of Gelfand pairs (K, N) with N a connected, nilpotent Lie group. In order for (K, N) to be a Gelfand pair, it was shown that N could be at most 2-step. Gelfand pairs of the form (K, H_n) , where H_n is the $(2n + 1)$ -dimensional Heisenberg group, play a key role in the more general theory.

Let (K, N) be a Gelfand pair with N nilpotent. We denote the set of differential operators on N that are invariant under the action of K and the left action of N by $\mathcal{U}_K(\mathfrak{n})$. A smooth K -invariant function $\phi: N \rightarrow \mathbf{C}$ is K -spherical if $\phi(e) = 1$ and ϕ is a joint eigenfunction for all $D \in \mathcal{U}_K(\mathfrak{n})$. Reference [BJR] provides a parametrization for the bounded K -spherical functions on N , yielding a description of the Gelfand space for $L_K^1(N)$. We review this below in Section 2. The main focus here is to specialize the theory to Heisenberg groups, obtaining methods to construct, in principle, all bounded K -spherical functions on H_n .

Section 1 begins with some basic facts about Heisenberg groups and their representation theory. We identify H_n with $V \times \mathbf{R}$ where $V \simeq \mathbf{C}^n$. A maximal compact connected group of automorphisms of H_n is given by the unitary group $U(n)$ acting via $k \cdot (z, t) = (kz, t)$. Conjugating by an automorphism of H_n if necessary, we can always assume that a given compact connected group $K \subset \text{Aut}(H_n)$ is contained in $U(n)$. It is well known that $(U(n), H_n)$ is a Gelfand pair (see, for example, [Ko]) and as discussed in Section 1, there are many proper closed subgroups $K \subset U(n)$ for which (K, H_n) is a Gelfand pair.

There is a representation theoretic criterion, due to Carcano [Ca], for (K, N) to be a Gelfand pair. In Section 1, we show that this implies that (K, H_n) is a Gelfand pair if and only if the action of K on the holomorphic polynomials $\mathcal{P}(V)$ is multiplicity free. Let $\mathcal{P}(V) = \sum_{\alpha} P_{\alpha}$ denote the decomposition of $\mathcal{P}(V)$ into K -irreducibles.

The general theory in [BJR] describes the bounded K -spherical functions for a Gelfand pair (K, N) in terms of representation theory. In Section 2 we show that for Gelfand pairs (K, H_n) , there are two distinct classes of bounded K -spherical functions.

1. The first type is parametrized by pairs (λ, P_{α}) where λ is a non-zero real number. These arise from the infinite dimensional representations of H_n and we denote them by $\phi_{\lambda, \alpha}(z, t)$. We will see that $\phi_{\lambda, \alpha}(z, t) = \phi_{1, \alpha}(\lambda^{1/2}z, \lambda t)$ for $\lambda > 0$ and $\phi_{\lambda, \alpha} = \phi_{|\lambda|, \alpha}$ for $\lambda < 0$. Subsequently, we will concentrate on $\phi_{\alpha} := \phi_{1, \alpha}$.

2. The second type arise from the 1-dimensional representations of H_n and are parametrized by V/K , the space of K -orbits in $V \simeq \mathbf{C}^n$. For $\omega \in V$ we write η_{ω} for the associated K -spherical function. One has $\eta_{\omega} = \eta_{\omega'}$

if $K \cdot \omega = K \cdot \omega'$. $\eta_\omega(z, t)$ is independent of t and given by the Fourier transform of the unit mass on $K \cdot \omega$.

The type 2 K -spherical functions reflect the abelian component of analysis on H_n and are non-generic (they have zero Plancherel measure). The main concern here is the study of the type 1 K -spherical functions.

The prescription for the ϕ_α 's derived from [BJR] involves integrating matrix coefficients over K . This is difficult to carry out (at best) except in very simple cases. The main results in this paper (contained in Section 4) provide alternative approaches. We will show that $\phi_\alpha(z, t) = e^{it} q_\alpha(z) e^{-(1/4)|z|^2}$ where $q_\alpha(z)$ is a polynomial function in (z, \bar{z}) . One also has the (real) K -invariant polynomial $p_\alpha(z) = (1/\dim(P_\alpha)) \sum v_i(z) \overline{v_i(z)}$ where $\{v_i\}$ is an orthonormal basis for P_α . This turns out to be (up to sign) the homogeneous component of highest degree in q_α . In fact, the q_α 's are completely determined by the p_α 's. We provide two computational procedures:

Orthogonalization. The q_α 's are obtained (up to normalization) by applying the Gram-Schmidt algorithm using the measure $e^{-(1/2)|z|^2} dz d\bar{z}$ to the p_α 's. (One can order the p_α 's in any way that ensures $\alpha < \beta \Rightarrow \deg(p_\alpha) \leq \deg(p_\beta)$.)

Rodrigues' Formula. There is a constant coefficient differential operator D_{p_α} with the property

$$D_{p_\alpha}(e^{-(1/2)|z|^2}) = q_\alpha(e) e^{-(1/2)|z|^2}.$$

D_{p_α} is essentially the abelian symmetrization of p_α .

These results show that the spherical functions ϕ_α can be written down explicitly once one has the K -invariant polynomials p_α in hand.

Recent results of Howe and Umeda show that $\mathcal{P}(V_{\mathbf{R}})^K$, the K -invariant polynomials on the underlying real vector space $V_{\mathbf{R}}$ of V , is a polynomial ring. In fact, $\mathcal{P}(V_{\mathbf{R}})^K = \mathbf{C}[\gamma_1, \dots, \gamma_d]$ where $\{\gamma_1, \dots, \gamma_d\}$ is an essentially canonical subset of $\{p_\alpha\}$ [HU]. In Section 3, we use $\gamma_1, \dots, \gamma_d$ together with the symmetrization map for H_n to produce two sets of K -invariant differential operators L_1^1, \dots, L_d^1 and L_1^0, \dots, L_d^0 on V . The functions $\{\psi_\alpha(z) := q_\alpha(z) e^{-(1/4)|z|^2}\}$ and $\{\eta_\omega(z)\}$ are simultaneous eigenfunctions for $\{L_j^1\}$ and $\{L_j^0\}$, respectively.

In Section 5, we show how the analysis can be reduced to "radial directions." We can regard the functions $p_\alpha(z)$ and $\psi_\alpha(z)$ as living on $\Gamma \subset (\mathbf{R}^+)^d$, the space of values assumed by the fundamental invariants $\gamma = (\gamma_1, \dots, \gamma_d)$. Let \tilde{p}_α and $\tilde{\psi}_\alpha$ be the functions on Γ characterized by $\tilde{p}_\alpha \circ \gamma = p_\alpha$ and $\tilde{\psi}_\alpha \circ \gamma = \psi_\alpha$. The orthogonalization procedure and Rodrigues' formula can be formulated on Γ to obtain algorithms for computing the $\tilde{\psi}_\alpha$'s from the

\tilde{p}_α 's. The $\tilde{\psi}_\alpha$'s solve eigenvalue problems for differential operators $\tilde{L}_1, \dots, \tilde{L}_d$ obtained from L_1^1, \dots, L_d^1 .

There is an alternative radial reduction when the representation $K|V$ is *polar*. This means that there is a cross section \mathcal{A} meeting every orbit orthogonally [Da]. We show that for Gelfand pairs (K, H_n) with $K|V$ polar, the orthogonalization procedure and the Rodrigues' formula can be reformulated on \mathcal{A} . These results provide algorithms for determining the restrictions $\psi_\alpha|_{\mathcal{A}}$ given $\{p_\alpha|_{\mathcal{A}}\}$.

In Section 6 we apply the general theory to the motivating example $(U(n), H_n)$. Most of the results here have been obtained independently by other authors using different methods and viewpoints. (See [Fa, Ko, HR, St, Str].) The type 1 spherical functions $\phi_\alpha(z, t) = e^{it}\psi_\alpha(z)$ can be expressed in terms of even Hermite functions on a cross section $\mathcal{A} \simeq \mathbf{R}$ and as Laguerre functions on the value space $\Gamma = \mathbf{R}^+$ for the fundamental invariant $\gamma_1(z) = |z|^2/2n$. The orthogonalization procedure and Rodrigues' formula reduce to classical theorems concerning these special functions. Our main results generalize these well known theorems. We also show how an easy abstract identity (Proposition 2.4) can be used to derive non-obvious formulas involving Laguerre polynomials (Theorem 6.16).

The results in this paper reduce the study of K -spherical functions on H_n to problems in classical invariant theory. The type 1 K -spherical functions are computable given the multiplicity free decomposition $\mathcal{P}(V) = \sum_\alpha P_\alpha$. The relevant differential operators and a radial reduction procedure require the fundamental invariants in $\mathcal{P}(V_{\mathbf{R}})$. Determination of the type 2 K -spherical functions involves the dual problem of describing the K -orbits in V . The connected groups K which act irreducibly on V and are multiplicity free on $\mathcal{P}(V)$ have been classified by V. Kac [Ka]. This yields a table (Table 1.8 below) of all possible "irreducible" Gelfand pairs (K, H_n) . The decompositions of $\mathcal{P}(V)$ and the fundamental invariants for all of these groups can be found in [HU]. The authors hope to study the resulting spherical functions in a subsequent paper.

1. GELFAND PAIRS (K, H_n)

We begin with a representation theoretic criterion for (K, H_n) to be a Gelfand pair. We identify H_n with $\mathbf{C}^n \times \mathbf{R}$ with multiplication given by

$$(z, t)(z', t') = (z + z', t + t' + \frac{1}{2}\omega(z, z')), \quad (1.1)$$

where $\omega(z, z') := -\text{Im}\langle z, z' \rangle = -\text{Im}(z \cdot \bar{z}')$ for $z, z' \in \mathbf{C}^n$, and $t, t' \in \mathbf{R}$. The natural action of the group of $n \times n$ unitary matrices on \mathbf{C}^n (which we

denote by $k \cdot z$ for $k \in U(n)$ and $z \in \mathbf{C}^n$) gives rise to a compact subgroup of $\text{Aut}(H_n)$, the group of automorphisms of H_n , via $k \cdot (z, t) = (k \cdot z, t)$. This subgroup, again denoted $U(n)$, is a maximal connected, compact subgroup of $\text{Aut}(H_n)$, and thus any connected, compact subgroup of $\text{Aut}(H_n)$ is the conjugate of a subgroup K of $U(n)$. Since conjugates of K form Gelfand pairs with H_n if, and only if, K does, and produce the same spherical functions, we will always assume that we are dealing with a compact subgroup of $U(n)$.

It will occasionally be more congenial to have a “coordinate free” model for H_n . In such cases, we assume that V is an n -dimensional vector space over \mathbf{C} equipped with a Hermitian inner product $\langle \cdot, \cdot \rangle$. H_n is then identified with $V \times \mathbf{R}$, and the multiplication is given by $(v, t)(v', t') = (v + v', t + t' + (1/2)\omega(v, v'))$ where $\omega(v, v') := -\text{Im}\langle v, v' \rangle$. We will write H_V for the Heisenberg group given by $(V, \langle \cdot, \cdot \rangle)$. $U(V) \simeq U(n)$, the subgroup of $Gl(V)$ preserving $\langle \cdot, \cdot \rangle$, is again naturally identified with a closed subgroup of $\text{Aut}(H_V)$.

The irreducible unitary representations of H_n which are non-trivial on the center, \mathbf{R} , are determined up to equivalence by their central character. The Fock model, for real $\lambda > 0$, is defined on the space \mathcal{F}_λ of holomorphic functions on \mathbf{C}^n which are square integrable with respect to the measure $d\tilde{w}_\lambda = (\lambda/2\pi)^n e^{-(1/2)\lambda|w|^2} dw d\bar{w}$ [Ba]. The space $\mathcal{P}(\mathbf{C}^n)$ of holomorphic polynomials is dense in \mathcal{F}_λ , and contains an orthonormal basis given by $\{u_{\alpha, \lambda} : \alpha \in \mathbf{Z}_+^n\}$, where

$$u_{\alpha, \lambda}(w) = \frac{\lambda^{|\alpha|/2} w^\alpha}{(2^{|\alpha|} \alpha!)^{1/2}} = \frac{w_1^{\alpha_1}}{((2/\lambda)^{\alpha_1} \alpha_1!)^{1/2}} \cdots \frac{w_n^{\alpha_n}}{((2/\lambda)^{\alpha_n} \alpha_n!)^{1/2}}. \tag{1.2}$$

$\mathcal{P}_m(\mathbf{C}^n)$ ($= \mathcal{P}_m$) will denote the subspace of homogeneous polynomials of degree m , and is thus spanned by $\{u_{\alpha, \lambda} : |\alpha| = m\}$. We will write u_α for $u_{\alpha, \lambda}$. For $p \in \mathcal{P}_m$, let $p(D)$ be the constant coefficient differential operator $p(\partial/\partial w_1, \dots, \partial/\partial w_n)$. One has an alternative description of the Hilbert space structure.

LEMMA 1.3. For $p, q \in \mathcal{P}_m$, the inner product in $\mathcal{F} := \mathcal{F}_1$ is given by

$$\langle p, q \rangle = 2^m p(D)q.$$

Proof. For monomials w^α, w^β , with $|\alpha| = |\beta| = m$, the inner product on \mathcal{F} is

$$\langle w^\alpha, w^\beta \rangle = \begin{cases} 0, & \text{if } \alpha \neq \beta; \\ 2^m \alpha!, & \text{if } \alpha = \beta. \end{cases}$$

Remark. In general, for polynomials $p, q \in \mathcal{P}(\mathbf{C}^n)$, we have $\langle p, q \rangle = (p(2D)q)(0)$.

The representation π_λ of H_n of \mathcal{F}_λ is given by

$$\pi_\lambda(z, t) u(w) = e^{i\lambda t - (1/2)\lambda \langle w, z \rangle - (1/4)\lambda |z|^2} u(w + z). \quad (1.4)$$

For $\lambda < 0$, \mathcal{F}_λ consists of antiholomorphic functions which are square integrable with respect to $d\tilde{w}_{|\lambda|}$, and the representation is given by

$$\pi_\lambda(z, t) u(\bar{w}) = e^{i\lambda t + (1/2)\lambda \langle w, z \rangle + (1/4)\lambda |z|^2} u(\overline{w + z}). \quad (1.5)$$

For $k \in U(n)$, the representation $\pi_\lambda^k(z, t) := \pi_\lambda(k \cdot z, t)$ has the same central character as π_λ , and hence is equivalent to π_λ . For $\lambda > 0$ the operator that intertwines these two representations comes from the standard action of $U(n)$ on \mathbf{C}^n . More precisely,

$$[\pi_\lambda(k \cdot z, t)u](k \cdot w) = [\pi_\lambda(z, t)(k^{-1} \cdot u)](w), \quad (1.6)$$

where $k \cdot u(w) = u(k^{-1} \cdot w)$. One has a similar formula for $\lambda < 0$, except that the action of $U(n)$ on antiholomorphic functions is given by $k \cdot u(\bar{w}) = u(\overline{k \cdot w})$.

The following theorem is a special case of a general result proved by G. Carcano [Ca].

THEOREM 1.7. *(K, H_n) is a Gelfand pair if, and only if, the action of K on \mathcal{F}_λ decomposes into irreducible components of multiplicity one for all $\lambda \neq 0$.*

Since $\mathcal{P}(\mathbf{C}^n)$ is dense in \mathcal{F}_λ , $\lambda > 0$, and \mathcal{P}_m is invariant under $U(n)$, for $K \subset U(n)$, the K -irreducible components of \mathcal{F}_λ have multiplicity one if, and only if, the action of K on $\mathcal{P}(\mathbf{C}^n)$ is multiplicity free.

A similar statement holds for the decomposition of \mathcal{F}_λ , $\lambda < 0$, and the action of K on the antiholomorphic polynomials. Thus one concludes that (K, H_n) is a Gelfand pair if, and only if, \mathcal{F} ($= \mathcal{F}_1$) decomposes into K -irreducible components of multiplicity one, which in turn is equivalent to requiring that the action of K on $\mathcal{P}(\mathbf{C}^n)$ be multiplicity free.

Let $K_{\mathbf{C}} \subset Gl(n, \mathbf{C})$ be the complexification of K . Then the irreducible components of $\mathcal{P}(\mathbf{C}^n)$ with respect to K and $K_{\mathbf{C}}$ are identical. The connected groups $K_{\mathbf{C}}$ which act *irreducibly* and without multiplicity have been classified by V. Kac [Ka]. These groups and their representations are given in the following table.

1.8. *Multiplicity Free Representations.*

Group	Acting On	Subject To
$Sl(n, \mathbb{C})$	\mathbb{C}^n	$n \geq 2$
$Gl(n, \mathbb{C})$	\mathbb{C}^n	$n \geq 1$
$Sp(k, \mathbb{C})$	\mathbb{C}^n	$n = 2k$
$\mathbb{C}^* \times Sp(k, \mathbb{C})$	\mathbb{C}^n	$n = 2k$
$\mathbb{C}^* \times SO(n, \mathbb{C})$	\mathbb{C}^n	$n \geq 3$
$Gl(k, \mathbb{C})$	$S^2(\mathbb{C}^k) \simeq \mathbb{C}^n$	$n = k(k+1)/2, k \geq 2$
$Sl(k, \mathbb{C})$	$A^2(\mathbb{C}^k) \simeq \mathbb{C}^n$	$n = \binom{k}{2}$ and k is odd
$Gl(k, \mathbb{C})$	$A^2(\mathbb{C}^k) \simeq \mathbb{C}^n$	$n = \binom{k}{2}$
$Sl(k, \mathbb{C}) \times Sl(l, \mathbb{C})$	$\mathbb{C}^k \otimes \mathbb{C}^l \simeq \mathbb{C}^n$	$n = kl, k \neq l$
$Gl(k, \mathbb{C}) \times Sl(l, \mathbb{C})$	$\mathbb{C}^k \otimes \mathbb{C}^l \simeq \mathbb{C}^n$	$n = kl$
$Gl(2, \mathbb{C}) \times Sp(k, \mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^{2k} \simeq \mathbb{C}^n$	$n = 4k$
$Sl(3, \mathbb{C}) \times Sp(k, \mathbb{C})$	$\mathbb{C}^3 \otimes \mathbb{C}^{2k} \simeq \mathbb{C}^n$	$n = 6k$
$Gl(3, \mathbb{C}) \times Sp(k, \mathbb{C})$	$\mathbb{C}^3 \otimes \mathbb{C}^{2k} \simeq \mathbb{C}^n$	$n = 6k$
$Gl(4, \mathbb{C}) \times Sp(4, \mathbb{C})$	$\mathbb{C}^4 \otimes \mathbb{C}^8 \simeq \mathbb{C}^n$	$n = 32$
$Sl(k, \mathbb{C}) \times Sp(4, \mathbb{C})$	$\mathbb{C}^k \otimes \mathbb{C}^8 \simeq \mathbb{C}^n$	$n = 8k, k > 4$
$Gl(k, \mathbb{C}) \times Sp(4, \mathbb{C})$	$\mathbb{C}^k \otimes \mathbb{C}^8 \simeq \mathbb{C}^n$	$n = 8k, k > 4$
$\mathbb{C}^* \times Spin(7, \mathbb{C})$	\mathbb{C}^n	$n = 8$
$\mathbb{C}^* \times Spin(9, \mathbb{C})$	\mathbb{C}^n	$n = 16$
$Spin(10, \mathbb{C})$	\mathbb{C}^n	$n = 16$
$\mathbb{C}^* \times Spin(10, \mathbb{C})$	\mathbb{C}^n	$n = 16$
$\mathbb{C}^* \times G_2$	\mathbb{C}^n	$n = 7$
$\mathbb{C}^* \times E_6$	\mathbb{C}^n	$n = 27$

2. *K*-SPHERICAL FUNCTIONS

There are several equivalent definitions for the *K*-spherical functions associated to a given Gelfand pair (K, N) . By one definition, a *K*-spherical function associated to such a pair is a smooth *K*-invariant complex-valued function ϕ on *N* such that $\phi(e) = 1$, and ϕ is a joint eigenfunction for all left *N*-and *K*-invariant differential operators on *N*. Equivalently, they can be defined as the homomorphisms of the commutative convolution algebra of compactly supported, *K*-invariant distributions, or the commutative convolution algebra $L^1_K(N)$. (See [He].) In the latter case, the homomorphisms corresponding to bounded *K*-spherical functions are continuous in the L^1 -norm. In [BJR] we give a parametrization for the bounded *K*-spherical functions. Given $\pi \in \hat{N}$, we define

$$K_\pi = \{k \in K: \pi^k \text{ is unitarily equivalent to } \pi\}, \tag{2.1}$$

where $\pi^k := \pi \circ k$.

THEOREM 2.2. *Given a Gelfand pair (K, N) , and an irreducible representation π of *N*, let $\mathbf{H}_\pi = \bigoplus_\alpha P_\alpha$ be the decomposition of the representation*

space into K_π -irreducible subspaces. For each subspace P_α , and each representation π , we have the bounded K -spherical function

$$\phi_{\pi,v}(x) = \int_K \langle \pi(k \cdot x)v, v \rangle dk,$$

for any unit vector $v \in P_\alpha$, and $x \in N$. All bounded K -spherical functions are obtained in this manner, and $\phi_{\pi,v} = \phi_{\pi',v'}$ if, and only if, $\pi' = \pi^k$ for some $k \in K$ and v, v' belong to the same P_α .

Notation. We set $\phi_{\pi,\alpha} = \phi_{\pi,v}$ for any unit vector $v \in P_\alpha$.

The proof of (2.2) given in [BJR] yields an alternative formula for $\phi_{\pi,\alpha}$ in certain cases.

COROLLARY 2.3. *Suppose that $K_\pi = K$ and that $\{v_1, \dots, v_l\}$ is any orthonormal basis for P_α . Then*

$$\phi_{\pi,\alpha}(x) = \frac{1}{l} \sum_{j=1}^l \langle \pi(x)v_j, v_j \rangle.$$

Recall that $\pi_\lambda^k \simeq \pi_\lambda$ for an infinite dimensional representation π_λ of H_n and any $k \in U(n)$. (See Eq. (1.6).) Thus, $K_\pi = K$ for any compact subgroup K of $U(n)$ and (2.3) can be applied to compute the associated K -spherical functions $\phi_{\lambda,\alpha} := \phi_{\pi_\lambda,\alpha}$.

Remark. Although the group K is used explicitly in (2.2) to define the K -spherical functions, one sees from (2.3) that when $K_\pi = K$, the K -spherical functions (as well as the K -invariant polynomials p_α given in (3.9)) depend only upon the decomposition $\mathcal{P}(V) = \bigoplus P_\alpha$, not upon the group K itself. For example, the K -spherical functions on H_n given by π_λ coincide if $K_{\mathbb{C}}$ is any of the first four groups in Table 1.8. These are computed in Section 5. In addition, the \mathbf{C}^* -factors do not affect the decomposition of $\mathcal{P}(V)$. Thus, when two groups in Table 1.8 differ by the addition or removal of \mathbf{C}^* , they yield the same spherical functions.

PROPOSITION 2.4. *Suppose that K' is a compact subgroup of K , and that (K', N) is a Gelfand pair. Let $P_\alpha = \bigoplus_{i=1}^{n_\alpha} P_{\alpha,i}$ be the decomposition of P_α into K' -irreducible subspaces. Let $\phi_{\pi,\alpha,i}$ denote the K' -spherical function corresponding to a unit vector in $P_{\alpha,i}$. Then*

$$\phi_{\pi,\alpha}(x) = \int_{K/K'} \phi_{\pi,\alpha,i}(k \cdot x) dk. \tag{2.5}$$

Moreover, if $K_\pi = K$ then

$$\phi_{\pi,\alpha} = \frac{1}{\dim(P_\alpha)} \sum_{i=1}^{n_\alpha} \dim(P_{\alpha,i}) \phi_{\pi,\alpha,i}. \tag{2.6}$$

Proof. If $K_\pi = K$ then $K'_\pi = K' \cap K_\pi = K'$. Thus, (2.6) follows easily from (2.3) by using an orthonormal basis for P_x that is a union of orthonormal bases for the P_{α_i} 's.

For the first equation, note that the right hand side of (2.5) is equal to

$$\begin{aligned} \int_K \phi_{\pi, \alpha, i}(k \cdot x) dk &= \int_K \int_{K'} \langle \pi(k'k \cdot x)v, v \rangle dk' dk \\ &= \int_K \langle \pi(k \cdot x)v, v \rangle dk \\ &= \phi_{\pi, \alpha}(x). \end{aligned}$$

We now return to the Heisenberg group, and first determine how the bounded K -spherical functions for different representations π_λ are related. For each $\lambda > 0$ we have an automorphism of H_n ,

$$\delta_\lambda(z, y) = (\lambda^{1/2}z, \lambda t). \quad (2.7)$$

called dilation and an isometry $d_\lambda: \mathcal{F}_1 \rightarrow \mathcal{F}_\lambda$,

$$d_\lambda u(w) = u(\lambda^{1/2}w). \quad (2.8)$$

For $\lambda > 0$, π_1 and π_λ are related by the formula

$$d_\lambda \pi_1(\delta_\lambda(z, t)) = \pi_\lambda(z, t) d_\lambda. \quad (2.9)$$

We temporarily adopt the notation $\langle \cdot, \cdot \rangle_\lambda$ for the inner product on \mathcal{F}_λ given by $d\tilde{w}_\lambda$. For $\lambda > 0$ and $v \in P_\alpha$ with $\langle v, v \rangle_\lambda = 1$, one has the associated K -spherical function,

$$\phi_{\lambda, \alpha}(z, t) = \int_K \langle \pi_\lambda(k \cdot z, t)v, v \rangle_\lambda dk = e^{i\lambda t} \psi_{\lambda, \alpha}(z),$$

where

$$\begin{aligned} \psi_{\lambda, \alpha}(z) &= \int_K \langle \pi_\lambda(k \cdot z, 0)v, v \rangle_\lambda dk \\ &= \int_K \langle d_\lambda \pi_1(\delta_\lambda(k \cdot z, 0)) d_\lambda^{-1}v, v \rangle_\lambda dk \quad (\text{by formula (2.9)}) \\ &= \int_K \langle \pi_1(\delta_\lambda(k \cdot z, 0)) d_\lambda^{-1}v, d_\lambda^{-1}v \rangle_1 dk \\ &= \int_K \langle \pi_1(k \cdot \delta_\lambda(z, 0)) d_\lambda^{-1}v, d_\lambda^{-1}v \rangle_1 dk \\ &= \psi_{1, \alpha}(\lambda^{1/2}z) \end{aligned}$$

(since $\langle d_\lambda^{-1}v, d_\lambda^{-1}v \rangle_1 = 1$).

That is, for $\lambda > 0$

$$\phi_{\lambda,\alpha}(z, t) = \phi_{1,\alpha}(\lambda^{1/2}z, \lambda t) = e^{i\lambda t} \psi_{1,\alpha}(\lambda^{1/2}z). \tag{2.10}$$

For $\lambda < 0$, one has

$$\phi_{\lambda,\alpha} = \overline{\phi_{|\lambda|,\alpha}} = \overline{\phi_{1,\alpha}(|\lambda|^{1/2}z, |\lambda| t)} = e^{i\lambda t} \overline{\psi_{1,\alpha}(|\lambda|^{1/2}z)}. \tag{2.11}$$

Indeed, $\pi_\lambda = \overline{\pi_{|\lambda|}}$ and hence the matrix coefficients of π_λ are obtained via complex conjugation from those for $\pi_{|\lambda|}$.

Thus the K -spherical functions arising from infinite dimensional representations of H_n are all obtained by dilating certain K -spherical functions associated to $\pi (= \pi_1)$. We will focus our attention on this representation. Henceforth $\pi, \mathcal{F}, \phi_\alpha,$ and P_α will be used instead of $\pi_1, \mathcal{F}_1, \phi_{1,\alpha},$ and $P_{1,\alpha}$.

Suppose now that K_i is a compact subgroup of $U(n_i)$ for $i = 1, \dots, d$, and that $K^\times = \prod_{i=1}^d K_i$. Let $n = \sum n_i$. Then $K^\times \subset U(n)$, and from [BJR] one has that (K^\times, H_n) is a Gelfand pair if, and only if, (K_i, H_{n_i}) is a Gelfand pair for $i = 1, \dots, d$. Let $H_n = V \times \mathbf{R}$ and let $V = \bigoplus_{i=1}^d V_i$ be such that $K^\times|_{V_i} = K_i$. Then $\mathcal{P}(V) = \bigotimes_{i=1}^d \mathcal{P}(V_i)$. Let $\mathcal{P}(V_i) = \bigoplus_{j=1}^\infty P_{i,j}$ be the decomposition of $\mathcal{P}(V_i)$ into K_i -irreducible subspaces. Then $\mathcal{P}(V) = \bigoplus_{j_1, \dots, j_d} P^{j_1 \dots j_d}$, where $P^{j_1 \dots j_d} =: P_{1,j_1} \otimes \dots \otimes P_{d,j_d}$, is the decomposition of $\mathcal{P}(V)$ into K^\times -irreducible subspaces. Denote by ϕ_{j_1, \dots, j_d} the K^\times -spherical function associated to π and $P^{j_1 \dots j_d}$, and by ϕ_j^i the K_i -spherical function on H_{n_i} associated to π and $P_{i,j}$, for $i = 1, \dots, d$.

PROPOSITION 2.12. *Let $u_i \in V_i$, for $i = 1, \dots, d$. Then*

$$\phi_{j_1, \dots, j_d}(u_1 + \dots + u_d, t) = e^{it} \prod_{i=1}^d \psi_{j_i}^i(u_i).$$

Proof. By an obvious induction argument we may reduce to the case $d=2$. Now note that since K^\times acts trivially on the center of H_n , $\phi_{j_1, j_2}(z, t) = e^{it} \psi_{j_1, j_2}(z)$. Let $\{v_p^i : 1 \leq p \leq m_i\}$ be a basis for P_{i,j_i} for $i = 1, 2$. By Corollary 2.2,

$$\begin{aligned} \phi_{j_1, j_2}(u_1 + u_2, t) &= e^{it} \psi_{j_1, j_2}(u_1 + u_2) \\ &= \frac{e^{it}}{m_1 m_2} \sum_{p,q} \langle \pi(u_1 + u_2, 0) v_p^1 \otimes v_q^2, v_p \otimes v_q \rangle \\ &= \frac{e^{it}}{m_1 m_2} \sum_{p,q} \langle \pi(u_1, 0) v_p^1 \otimes \pi(u_2, 0) v_q^2, v_p^1 \otimes v_q^2 \rangle \\ &= \frac{e^{it}}{m_1 m_2} \sum_p \langle \pi(u_1, 0) v_p^1, v_p^1 \rangle \sum_q \langle \pi(u_2, 0) v_q^2, v_q^2 \rangle. \end{aligned}$$

The two latter sums are $m_i \psi_{j_i}^i$ for $i = 1, 2$, respectively.

In addition to the infinite dimensional representations π of H_n , one also has infinitely many one dimensional representations corresponding to one-point coadjoint orbits. These are parametrized by \mathbb{C}^n where for $\omega \in \mathbb{C}^n$, $\chi_\omega(z, t) = e^{i \operatorname{Re} \langle \omega, z \rangle}$ for $(z, t) \in H_n$. If (K, H_n) is a Gelfand pair, then for each such character, one has by Theorem 2.1 a K -spherical function, η_ω , that is the K -average of χ_ω , i.e.,

$$\eta_\omega(z, t) = \int_K e^{i \operatorname{Re} \langle \omega, k \cdot z \rangle} dk = \int_K e^{i \operatorname{Re} \langle k^{-1} \cdot \omega, z \rangle} dk.$$

Thus η_ω depends only on the K -orbit through ω , and Theorem 2.1 ensures that if $\omega' \notin K \cdot \omega$ then $\eta_{\omega'} \neq \eta_\omega$.

Let $\mu_{K \cdot \omega}$ denote the unit mass supported on the K -orbit through ω . As a distribution this is given by

$$\langle \mu_{K \cdot \omega}, f \rangle = \int_K f(k^{-1} \cdot \omega, 0) dk,$$

for all $f \in C_c^\infty(H_n)$. We define the (Euclidean) Fourier transform for such a function by

$$\hat{f}(\omega, \tau) = \int_{\mathbb{C}^n \times \mathbb{R}} f(z, t) e^{i(\operatorname{Re} \langle \omega, z \rangle + \tau t)} dz d\bar{z} dt.$$

Note that

$$\begin{aligned} \int_{H_n} \eta_\omega(z, t) f(z, t) dz d\bar{z} dt &= \int_{H_n} \int_K e^{i \operatorname{Re} \langle k^{-1} \cdot \omega, z \rangle} f(z, t) dk dz d\bar{z} dt \\ &= \int_K \hat{f}(k^{-1} \cdot \omega, 0) dk \\ &= \langle \hat{\mu}_{K \cdot \omega}, f \rangle, \end{aligned}$$

for $f \in C_c^\infty(H_n)$. Therefore we have

PROPOSITION 2.13. $\eta_\omega = \hat{\mu}_{K \cdot \omega}$.

Finally we observe that if $K = K_1 \times K_2$, where $K_i \subset U(n_i)$, for $i = 1, 2$, then the one point K -orbits are naturally parametrized by (ω_1, ω_2) , where $\omega_i \in \mathbb{C}^{n_i}$ with $K \cdot (\omega_1, \omega_2) = \{(k_1 \cdot \omega_1, k_2 \cdot \omega_2) : k_i \in K_i\}$. One easily checks that $\eta_{(\omega_1, \omega_2)}(z_1, z_2, t) = \eta_{\omega_1}(z_1) \eta_{\omega_2}(z_2)$, which establishes

PROPOSITION 2.14. $\eta_{\omega_1, \omega_2} = \hat{\mu}_{K_1 \cdot \omega_1} \otimes \hat{\mu}_{K_2 \cdot \omega_2}$.

The Gelfand space of a commutative Banach algebra E is the subspace of E' , the dual of E , consisting of all continuous, complex valued

homomorphisms of E equipped with the weak topology. For a Gelfand pair (K, H_n) , the Gelfand space of $L_K^1(H_n)$ is the point set $(\mathbf{R}^* \times \{\alpha\}) \cup V/K$, where $\{\alpha\}$ is the index set for the decomposition $\mathcal{P}(V) = \bigoplus_{\alpha} P_{\alpha}$ into irreducible K -modules and V/K denotes the K -orbits in V . The topology on the first subset is the product of the usual topology on \mathbf{R}^* with the discrete topology on $\{\alpha\}$, and V/K has the quotient topology. One knows from general principles that V/K is in the closure of $\mathbf{R}^* \times \{\alpha\}$. For example, at $0 = K \cdot 0 \in V/K$, one easily sees from (2.10) and (2.11) that as $\lambda \rightarrow 0$, $\phi_{\lambda, \alpha} \rightarrow 1$ uniformly on compacta, for each α . Thus, each neighborhood of 0 contains (λ, α) for each α and all sufficiently small $\lambda = \lambda(\alpha)$. It would be interesting to know how α must behave as $\lambda \rightarrow 0$ in order to produce convergence to a non-zero point in V/K . For $K = U(n)$, this was studied in [Str].

3. K -INVARIANT POLYNOMIALS AND DIFFERENTIAL OPERATORS

Suppose that N is a nilpotent Lie group with Lie algebra \mathfrak{n} and let K be a compact group of automorphisms of N . Given a differential operator $D \in \mathcal{U}(\mathfrak{n})$, the universal enveloping algebra of \mathfrak{n} , and $k \in K$, we define the differential operator D^k by

$$D^k(f) = D(f \circ k) \circ k^{-1} \quad \text{for } f \in C^{\infty}(N). \quad (3.1)$$

D is said to be K -invariant if $D^k = D$ for all $k \in K$. We write $\mathcal{U}_K(\mathfrak{n})$ for the algebra of K -invariant differential operators.

The derived action of K on \mathfrak{n} is given by

$$\exp(k \cdot X) = k \cdot \exp(X) \quad (3.2)$$

and K acts on $\mathfrak{n}^* = \text{Hom}_{\mathbf{R}}(\mathfrak{n}, \mathbf{R})$ via

$$(k \cdot \alpha)(X) = \alpha(k^{-1} \cdot X) \quad (3.3)$$

and on $\mathcal{P}(\mathfrak{n}^*)$, the \mathbf{C} -valued polynomial functions on the \mathbf{R} -vector space \mathfrak{n}^* via

$$(k \cdot p)(\alpha) = p(k^{-1} \cdot \alpha). \quad (3.4)$$

We identify $\mathcal{P}(\mathfrak{n}^*)$ with the complexified symmetric algebra $S(\mathfrak{n})_{\mathbf{C}}$ so that the symmetric product $X_1 X_2 \cdots X_n$ of $X_1, X_2, \dots, X_n \in \mathfrak{n}$ becomes the polynomial $\mathfrak{n}^* \rightarrow \mathbf{C}$ given by

$$p_{X_1 \dots X_n}(\alpha) = \alpha(X_1) \cdots \alpha(X_n). \quad (3.5)$$

One has

$$\begin{aligned} k \cdot p_{X_1 \dots X_n}(\alpha) &= \alpha(k \cdot X_1) \cdots \alpha(k \cdot X_n) \\ &= p_{(k \cdot X_1) \dots (k \cdot X_n)}(\alpha). \end{aligned} \tag{3.6}$$

The symmetrization map $\lambda: \mathcal{P}(\mathfrak{n}^*) \rightarrow \mathcal{U}(\mathfrak{n})$ is defined by

$$(\lambda(p)f)(n) = p \left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n} \right) \Big|_{t=0} f \left(n \exp \left(\sum_{i=1}^n t_i X_i \right) \right), \tag{3.7}$$

where $\{X_1, \dots, X_n\}$ is any basis for \mathfrak{n} . This definition is independent of the basis chosen and λ is characterized by the fact that $\lambda(X^m) = (\tilde{X})^m$ where \tilde{X} denotes the first order differential operator (left invariant vector field) given by $X \in \mathfrak{n}$. λ is a linear bijection, but not an algebra map. We refer the reader to [He] for these facts about the symmetrization map.

LEMMA 3.8. *Let (K, N) be a Gelfand pair. Then $\lambda(k \cdot p) = \lambda(p)^k$ for all $k \in K$, $p \in \mathcal{P}(\mathfrak{n}^*)$. Hence λ yields a linear isomorphism between $\mathcal{P}(\mathfrak{n}^*)^K \simeq S(\mathfrak{n})_{\mathbb{C}}^K$, the K -invariant elements in $\mathcal{P}(\mathfrak{n}^*) \simeq S(\mathfrak{n})_{\mathbb{C}}$, and $\mathcal{U}_K(\mathfrak{n})$.*

Proof. The result follows immediately from the fact that λ is natural and hence equivariant with respect to Lie algebra automorphisms.

We turn now to the Heisenberg group $H_n = V \times \mathbb{R}$ and the action of a compact subgroup $K \subset U(n)$ for which (K, H_n) is a Gelfand pair. As before, $\mathcal{P}(V) = \bigoplus_{\alpha} P_{\alpha}$ denotes the decomposition of the holomorphic polynomials into K -irreducible components of multiplicity one. Since the constant polynomials $\mathcal{P}_0(V)$ are the component for the trivial representation of K , we must have that $\mathcal{P}(V)^K = \mathbb{C}$. That is, the only K -invariant holomorphic polynomials $V \rightarrow \mathbb{C}$ are constant. However, one can also consider the larger space $\mathcal{P}(V_{\mathbb{R}})$ of \mathbb{C} -valued polynomial functions on the underlying real vector space $V_{\mathbb{R}}$ of V . As a \mathbb{C} -vector space, $\mathcal{P}(V_{\mathbb{R}}) = \mathcal{P}(V) \otimes_{\mathbb{C}} \overline{\mathcal{P}(V)}$ (and identifying V with \mathbb{C}^n one can write $\mathcal{P}(V) = \mathbb{C}[z_1, \dots, z_n]$, $\mathcal{P}(V_{\mathbb{R}}) = \mathbb{C}[z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n]$). K acts on $\mathcal{P}(V_{\mathbb{R}})$ via $\tilde{v}(k) = v(k) \otimes \overline{v(k)}$, where $v(k)$ denotes the standard action on $\mathcal{P}(V)$. We have the following description of the K -invariant elements $\mathcal{P}(V_{\mathbb{R}})^K$ in $\mathcal{P}(V_{\mathbb{R}})$.

PROPOSITION 3.9. *Let (K, H_n) be a Gelfand pair and let $\sum_{\alpha} P_{\alpha}$ be the decomposition of $\mathcal{P}(V)$ (or equivalently \mathcal{F}) into K -irreducible subspaces. Let $\{v_1, \dots, v_l\}$ be an orthonormal basis for P_{α} . Then*

$$p_{\alpha} = \frac{1}{l} \sum_{i=1}^l v_i \bar{v}_i$$

is a K -invariant element of $\mathcal{P}(V_{\mathbf{R}})$. Moreover, $\{p_{\alpha}\}$ is a vector space basis for $\mathcal{P}(V_{\mathbf{R}})^K$.

Proof. One has $\mathcal{P}(V_{\mathbf{R}}) = \mathcal{P}(V) \otimes \overline{\mathcal{P}(V)} = \sum_{\alpha, \beta} P_{\alpha} \otimes \overline{P_{\beta}}$. Decompose a given $v \in \mathcal{P}(V_{\mathbf{R}})^K$ as $v = \sum_{\alpha, \beta} v_{\alpha, \beta}$, where $v_{\alpha, \beta} \in P_{\alpha} \otimes \overline{P_{\beta}}$. (This must be a finite sum, i.e., $v_{\alpha, \beta} = 0$ for a.e. (α, β) .) Since the $P_{\alpha} \otimes \overline{P_{\beta}}$'s are K -invariant and pair-wise orthogonal, each $v_{\alpha, \beta}$ must also be K -invariant. Thus, the K -invariant elements in each $P_{\alpha} \otimes \overline{P_{\beta}}$ span $\mathcal{P}(V_{\mathbf{R}})^K$.

As $\mathcal{P}(V)$ is multiplicity free, an application of Schur's Lemma shows that $P_{\alpha} \otimes \overline{P_{\beta}}$ has no K -fixed vectors when $\alpha \neq \beta$ and that $P_{\alpha} \otimes \overline{P_{\alpha}}$ has a one-dimensional space of K -fixed vectors which is spanned by p_{α} .

Finally, the set $\{p_{\alpha}\}$ is linearly independent since there are no non-trivial linear dependence relations between the subspaces $P_{\alpha} \otimes \overline{P_{\alpha}}$ of $\mathcal{P}(V_{\mathbf{R}})$.

Remark. The polynomial p_{α} is homogeneous of degree $2 \deg(P_{\alpha})$. Thus, all invariant polynomials have even degree. If q is an invariant polynomial of degree $2m$ then we must have

$$q \in \text{Span}(p_{\alpha} : \deg(P_{\alpha}) \leq m).$$

This fact is used later in the proof of Proposition 4.2.

Proposition 3.9 yields a vector space basis for $\mathcal{P}(V_{\mathbf{R}})^K$. Since K is compact, $\mathcal{P}(V_{\mathbf{R}})^K$ is finitely generated. Recent results of Howe and Umeda show that $\mathcal{P}(V_{\mathbf{R}})^K$ is in fact a polynomial algebra. We summarize some ideas from [HU].

If $f(z)$ and $g(z)$ are K -highest weight vectors in $\mathcal{P}(V)$ (with respect to some fixed choice of a Cartan subalgebra and set of positive roots for K) then so is $f(z)g(z)$. A highest weight vector in $\mathcal{P}(V)$ is called *primitive* if it can't be written as a product of highest weight vectors in this way. These will be finite in number. Let $P_{\alpha_1}, \dots, P_{\alpha_d}$ be the irreducible components in $\mathcal{P}(V)$ that contain primitive highest weight vectors and let $\gamma_i := p_{\alpha_i}$. We call $\gamma_1, \dots, \gamma_d$ the *fundamental invariants*.

THEOREM 3.10 (See [HU]). $\mathcal{P}(V_{\mathbf{R}})^K = \mathbf{C}[\gamma_1, \dots, \gamma_d]$.

Remark. When V is K -irreducible, $V^* \subset \mathcal{P}(V)$ is one of the primitive K -irreducible components. We will always let γ_1 be the corresponding invariant in $\mathcal{P}(V_{\mathbf{R}})$. One has $\gamma_1(z) = |z|^2/2n$ (where $n = \dim_{\mathbf{C}}(V)$).

The symplectic structure ω on $V_{\mathbf{R}}$ sets up an isomorphism $V_{\mathbf{R}} \simeq V_{\mathbf{R}}^*$ and an induced algebra isomorphism

$$\Omega : \mathcal{P}(V_{\mathbf{R}}) \rightarrow \mathcal{P}(V_{\mathbf{R}}^*). \tag{3.11}$$

Define $\tilde{\lambda}: \mathcal{P}(V_{\mathbf{R}}) \rightarrow \mathcal{U}(\mathfrak{h}_n)$ by

$$\tilde{\lambda} := \lambda \circ j \circ \Omega, \tag{3.12}$$

where j is the obvious inclusion $\mathcal{P}(V_{\mathbf{R}}^*) \hookrightarrow \mathcal{P}(\mathfrak{h}_n^*)$ and λ is the symmetrization map for H_n . Let (z_1, z_2, \dots, z_n) be coordinates on V with respect to any $\langle \cdot, \cdot \rangle$ -orthonormal basis. We obtain isomorphisms $V \simeq \mathbf{C}^n$ and $\mathcal{P}(V_{\mathbf{R}}) \simeq \mathbf{C}[z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n]$ and $\tilde{\lambda}$ is given by

LEMMA 3.13.

$$(\tilde{\lambda}(p)f)(z, t) = p \left(2 \frac{\partial}{\partial \bar{\zeta}}, 2 \frac{\partial}{\partial \zeta} \right) f \left(z + \zeta, t + \frac{1}{2} \omega(z, \zeta) \right) \Big|_{\zeta=0}.$$

Proof. Write $z_j = x_j + iy_j$, with corresponding basis $\{X_j, Y_j\}$ for the Lie algebra \mathfrak{h}_n of H_n . Then by (3.7), for $p \in \mathcal{P}(\mathfrak{h}_n^*)$,

$$\begin{aligned} (\lambda(p)f)(z, t) &= p \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right) f \left((z, t) \exp \left(\sum u_j X_j + \sum v_j Y_j \right) \right) \Big|_{u=v=0} \\ &= p \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right) f \left(z + (u + iv), t + \frac{1}{2} \omega(z, u + iv) \right) \Big|_{u=v=0}. \end{aligned}$$

Let $\zeta = u + iv$. Then for $p \in \mathbf{C}[z, \bar{z}]$,

$$\begin{aligned} (\tilde{\lambda}(p)f)(z, t) &= p \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v}, \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \\ &\quad f \left(z + (u + iv), t + \frac{1}{2} \omega(z, u + iv) \right) \Big|_{u=v=0} \\ &= p \left(2 \frac{\partial}{\partial \bar{\zeta}}, 2 \frac{\partial}{\partial \zeta} \right) f \left(z + \zeta, t + \frac{1}{2} \omega(z, \zeta) \right) \Big|_{\zeta=0}. \end{aligned}$$

We have $\tilde{\lambda}(\mathcal{P}(V_{\mathbf{R}})^K) \subset \mathcal{U}_K(\mathfrak{h}_n)$ since λ, j , and Ω are all K -equivariant. In fact, $\mathcal{U}_K(\mathfrak{h}_n)$ is generated by $\tilde{\lambda}(\mathcal{P}(V_{\mathbf{R}})^K)$ and the operator $\partial/\partial t$. Let $L_1, \dots, L_d \in \mathcal{U}_K(\mathfrak{h}_n)$ be defined by

$$L_j := \tilde{\lambda}(\gamma_j). \tag{3.14}$$

LEMMA 3.15. *A smooth K -invariant function $\phi: H_n \rightarrow \mathbf{C}$ with $\phi(e) = 1$ is K -spherical if and only if ϕ is a joint eigenfunction for L_1, \dots, L_d and $\partial/\partial t$.*

Indeed, $\tilde{\lambda}$ induces an algebra map on the associated graded algebras and

one can use induction to show that eigenfunctions of L_1, \dots, L_d and $\partial/\partial t$ are eigenfunctions of any element in $\mathcal{U}_K(\mathfrak{h}_n)$.

We know (see (2.10)) that $\phi_{\lambda, \alpha}(z, t)$ has the form $\phi_{\lambda, \alpha}(z, t) = e^{i\lambda t} \psi_{\lambda, \alpha}(z)$. (Here $\phi_{0, * } := \eta_{*}$.) Thus,

$$L_j(\phi_{\lambda, \alpha}) = e^{i\lambda t} L_j^\lambda(\psi_\alpha), \tag{3.16}$$

where $L_j^\lambda \in \mathcal{U}_K(\mathfrak{h}_n)$ is the operator obtained by replacing each copy of $\partial/\partial t$ in L_j by $i\lambda$.

THEOREM 3.17. *The K -spherical functions $\{\phi_\alpha(z, t)\}$ are obtained from the bounded K -invariant simultaneous eigenfunctions $\{\psi_\alpha(z)\}$ of L_1^1, \dots, L_d^1 with $\psi_\alpha(0) = 1$ via $\phi_\alpha(z, t) = e^{it} \psi_\alpha(z)$. The K -spherical functions $\{\eta_\omega(z)\}$ are the bounded K -invariant eigenfunctions for L_1^0, \dots, L_d^0 with $\eta_\omega(0) = 1$.*

Remark. For K acting irreducibly on V , $\gamma_1 = |z|^2/2n$ and one computes

$$L_1 = \frac{1}{2n} \left(4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} + iz \frac{\partial}{\partial z} \frac{\partial}{\partial t} - i\bar{z} \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial t} + \frac{1}{4} |z|^2 \frac{\partial^2}{\partial t^2} \right) \tag{3.18}$$

and

$$L_1^1 = \frac{1}{2n} \left(4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} - z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} - \frac{1}{4} |z|^2 \right) \tag{3.19}$$

$$L_1^0 = \frac{4}{2n} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}.$$

L_1 is a scalar multiple of the usual Heisenberg sub-Laplacian.

We conclude this section with the following observation on the spectrum of an element $D \in \mathcal{U}_K(\mathfrak{h}_n)$.

PROPOSITION 3.20. *Let (K, H_n) be a Gelfand pair, and let $\mathcal{F} = \bigoplus P_\alpha$ be the decomposition of Fock space into K -irreducible subspaces. Then for $D \in \mathcal{U}_K(\mathfrak{h}_n)$, $D\phi_{\pi, \alpha} = \lambda_D \phi_{\pi, \alpha}$, where the eigenvalue λ_D is obtained from the equation $\pi(D)v_\alpha = \lambda_D v_\alpha$ for any $v_\alpha \in P_\alpha$.*

Proof. Note that D is a K -invariant distribution supported at the identity, and hence one can find a sequence $\{g_n\}$ of smooth, compactly supported, K -invariant functions on H_n which converge weakly to D . Then $\langle \pi(g_n)u, v \rangle \rightarrow \langle \pi(D)u, v \rangle$ for all $u, v \in \mathcal{F}$. Since each g_n is K -invariant, $\pi(g_n)$ commutes with the action of K on \mathcal{F} . Since the representations of K on the various P_α 's are inequivalent, $\pi(g_n)$ preserves each P_α . Thus, by

Schur's Lemma, $\pi(g_n)$ is constant on each P_α . It follows that $\pi(D)$ also is a constant multiple of the identity on each P_α , say $\lambda_\alpha I_{P_\alpha}$. Then

$$\begin{aligned} D\phi_{\pi,\alpha}(x) &= D_x \left(\int_K \langle \pi(k \cdot x) v_\alpha, v_\alpha \rangle dk \right) \\ &= \int_K \langle \pi(k \cdot x) \pi(D) v_\alpha, v_\alpha \rangle dk \\ &= \lambda_\alpha \phi_{\pi,\alpha}(x), \end{aligned}$$

and the result follows.

4. *K*-INVARIANT POLYNOMIALS AND *K*-SPHERICAL FUNCTIONS

We now fix a Gelfand pair (K, H_n) (with $H_n = V \times \mathbf{R}$) and decompose the space $\mathcal{P}(V) = \bigoplus P_\alpha$ into *K*-irreducible subspaces. For a given P_α with orthonormal basis $\{v_1, \dots, v_l\}$, we have

- (i) The *K*-invariant polynomial $p_\alpha = \frac{1}{l} \sum_{j=1}^l v_j \bar{v}_j$.
 - (ii) The *K*-spherical function $\phi_\alpha(x) = \frac{1}{l} \sum_{j=1}^l \langle \pi(x) v_j, v_j \rangle$.
- (4.1)

PROPOSITION 4.2 (Orthogonalization Procedure). $\phi_\alpha(z, t) = e^{it} q_\alpha(z) e^{-(1/4)|z|^2}$, where q_α is a polynomial with homogeneous component of highest degree $(-1)^m p_\alpha$ where $m = (1/2) \deg(p_\alpha)$. More specifically, the polynomials $\{q_\alpha\}$ are obtained, up to scalar multiples, by applying Gram-Schmidt orthogonalization with respect to the measure $d\tilde{z} = e^{-(1/2)|z|^2} dz d\bar{z}$ to the polynomials $\{p_\alpha\}$ and normalizing so that $q_\alpha(0) = 1$. The sequence $\{p_\alpha\}$ may be ordered in any way that ensures $(\alpha < \beta) \Rightarrow \deg(p_\alpha) \leq \deg(p_\beta)$.

Proof. First note that since $\mathcal{P}_m(V)$, the homogeneous polynomials of degree m on V , is invariant under $U(n)$ and $K \subset U(n)$, $P_\alpha \subset \mathcal{P}_m(V)$ for some m . Then

$$\begin{aligned} \langle \pi(z, t) v, v \rangle &= \int [\pi(z, t) v](w) \bar{v}(w) d\tilde{w} \\ &= \int e^{it - (1/2)\langle w, z \rangle - (1/4)|z|^2} v(w+z) \bar{v}(w) d\tilde{w} \\ &= e^{it} q(z) e^{-(1/4)|z|^2}, \end{aligned}$$

where

$$q(z) = \int e^{-(1/2)\langle w, z \rangle} v(w+z) \bar{v}(w) d\tilde{w}.$$

We use two expansions:

$$e^{-(1/2)\langle w, z \rangle} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{1}{2} \langle w, z \rangle \right)^k,$$

and the Taylor's series

$$v(w+z) = v(w) + \cdots + v(z).$$

Thus,

$$q(z) = \sum_{r=0}^{\infty} \frac{1}{r!} \left(-\frac{1}{2} \right)^r \int (w \cdot \bar{z})^r (v(w) + \cdots + v(z)) \overline{v(w)} d\tilde{w}.$$

Note that distinct monomials in $\mathcal{P}(V)$ are orthogonal with respect to $d\tilde{w}$, so that in the integrand only terms of degree m in w will give a non-zero integral against $\overline{v(w)}$. (In particular, the integral will be zero for $r > m$.) Thus q is a polynomial in z and \bar{z} . The highest order term of q is obtained from

$$\frac{1}{m!} \left(-\frac{1}{2} \right)^m \int (w \cdot \bar{z})^m v(z) \overline{v(w)} d\tilde{w}.$$

Since for $r \neq m$

$$\int (w \cdot \bar{z})^r \overline{v(w)} d\tilde{w} = 0,$$

we have

$$\begin{aligned} \frac{1}{m!} \left(-\frac{1}{2} \right)^m \int (w \cdot \bar{z})^m v(\bar{w}) d\tilde{w} &= (-1)^m \sum_{r=0}^{\infty} \frac{1}{r! 2^r} \int (w \cdot \bar{z})^r \overline{v(w)} d\tilde{w} \\ &= (-1)^m \int e^{(1/2)w \cdot \bar{z}} \overline{v(w)} d\tilde{w} \\ &= (-1)^m \overline{v(z)}, \end{aligned}$$

since $e^{(1/2)w \cdot \bar{z}}$ is the reproducing kernel in Fock space [Ba]. Thus,

$$q(z) = (-1)^m v(z) \bar{v}(z) + \text{lower order terms.}$$

Applying this result to ϕ_α , we obtain

$$\begin{aligned} \phi_\alpha(z, t) &= \frac{1}{l} \sum \langle \pi(z, t) v_\alpha, v_\alpha \rangle \\ &= e^{it} q_\alpha(z) e^{-(1/4)|z|^2}, \end{aligned}$$

where

$$q_\alpha(z) = \frac{(-1)^m}{l} \sum_{j=1}^l v_j(z) \bar{v}_j(z) + \text{lower order terms.}$$

To complete the determination of the polynomials $\{q_\alpha\}$ we next note that π is a square integrable representation modulo the centre of H_n , so that for $v_\alpha \in P_\alpha$ and $v_{\alpha'} \in P_{\alpha'}$,

$$\int \langle \pi(z, 0) v_\alpha, v_\alpha \rangle \overline{\langle \pi(z, 0) v_{\alpha'}, v_{\alpha'} \rangle} dz d\bar{z} = \text{Const } |\langle v_\alpha, v_{\alpha'} \rangle|^2 = 0,$$

if $\alpha \neq \alpha'$ (cf. [Ki]). Thus if $\alpha \neq \alpha'$,

$$\begin{aligned} &\int q_\alpha(z) \overline{q_{\alpha'}(z)} e^{-(1/2)|z|^2} dz d\bar{z} \\ &= \int \phi_\alpha(z, 0) \overline{\phi_{\alpha'}(z, 0)} dz d\bar{z} \\ &= \int \left(\int_K \langle \pi(k \cdot z, 0) v_\alpha, v_\alpha \rangle dk \right) \overline{\left(\int_K \langle \pi(k' \cdot z, 0) v_{\alpha'}, v_{\alpha'} \rangle dk' \right)} dz d\bar{z} \\ &= \int_K \int_K \left(\int \langle \pi(z, 0) k \cdot v_\alpha, k \cdot v_\alpha \rangle \overline{\langle \pi(z, 0) k' \cdot v_{\alpha'}, k' \cdot v_{\alpha'} \rangle} dz d\bar{z} \right) dk dk' \\ &= \text{Const} \int_K \int_K |\langle k \cdot v_\alpha, k' \cdot v_{\alpha'} \rangle|^2 dk dk' \\ &= 0. \end{aligned}$$

To ease the notation, take for the discrete index set $\{\alpha\}$ the numbers $\{0, 1, 2, 3, \dots\}$ and suppose that the various P_j 's have been ordered so that $i < j \Rightarrow \text{deg}(P_i) \leq \text{deg}(P_j)$. So $P_0 = \mathcal{P}_0(V)$; $P_1 + \dots + P_{l_1} = \mathcal{P}_1(V)$; $P_{l_1+1} + \dots + P_{l_2} = \mathcal{P}_2(V)$; etc.

Up to normalization constants, there is a *unique* sequence of non-zero, pair-wise orthogonal polynomials r_0, r_1, r_2, \dots with the property that

$$\text{Span}(r_0, \dots, r_d) = \text{Span}(p_0, \dots, p_d) \quad \text{for all } d. \tag{4.3}$$

This sequence can be obtained from $\{p_0, p_1, \dots\}$ by applying Gram-Schmidt (using the measure $d\bar{z}$).

In view of the orthogonality established above, we need only show that Condition (4.3) holds with r_0, \dots, r_d replaced by q_0, \dots, q_d . This is clear for $d=0$ since $q_0(z)$ and $p_0(z)$ are both constants. Assume inductively that (4.3) holds. Then

$$\begin{aligned} \text{Span}(q_0, \dots, q_{d+1}) &= \text{Span}(p_0, \dots, p_d, q_{d+1}) \\ &= \text{Span}(p_0, \dots, p_d, p_{d+1} + l_{d+1}), \end{aligned}$$

where $l_{d+1}(z)$ denotes the lower order terms in $(-1)^{\deg(p_{d+1})}q_{d+1}$. As q_{d+1} and p_{d+1} both belong to $\mathcal{P}(V_{\mathbf{R}})^K$, so does their difference l_{d+1} . Thus, by (3.9) $l_{d+1} \in \text{Span}(p_j: \deg p_j < \deg p_{d+1}) \subset \text{Span}(p_0, \dots, p_d)$ in view of our ordering on the P_j 's. Therefore we have $\text{Span}(q_0, \dots, q_{d+1}) = \text{Span}(p_0, \dots, p_{d+1})$.

Remark. The polynomials q_α are normalized so that $q_\alpha(0) = 1$. It follows in particular that since each polynomial p_α , as well as the measure $d\bar{z}$, is real, each q_α is real. Combining this observation with Eqs. (2.10) and (2.11) shows that for $\lambda \neq 0$,

$$\phi_{\lambda, \alpha}(z, t) = e^{i\lambda t} e^{-(1/4)|\lambda| |z|^2} q_\alpha(|\lambda|^{1/2}z).$$

Proposition 4.2 shows that the polynomials q_α , which determine the spherical functions ϕ_α , are obtained from the invariants p_α by applying the Gram-Schmidt algorithm. This involves computing certain integrals. We now present an alternative method which requires only the taking of derivatives!

Given $p(z, \bar{z}) = p(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n) \in \mathbf{C}[z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n]$, let D_p denote the constant coefficient differential operator

$$D_p := p \left(2 \frac{\partial}{\partial \bar{z}}, -2 \frac{\partial}{\partial z} \right). \tag{4.4}$$

That is, replace each occurrence of z_j in p by $2(\partial/\partial \bar{z}_j) = \partial/\partial x_j + i(\partial/\partial y_j)$ and each occurrence of \bar{z}_j by $-2(\partial/\partial z_j) = -\partial/\partial x_j + i(\partial/\partial y_j)$.

The map $p \mapsto D_p$ can be given a coordinate free description. Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional Hermitian vector space. Then $\omega(\cdot, \cdot) := \text{Im} \langle \cdot, \cdot \rangle$ is a symplectic structure on the underlying real vector space $V_{\mathbf{R}}$. The pairing ω sets up an isomorphism $V_{\mathbf{R}} \simeq V_{\mathbf{R}}^*$ and we obtain an induced isomorphism

$$\Omega: \mathcal{P}(V_{\mathbf{R}}) \rightarrow \mathcal{P}(V_{\mathbf{R}}^*). \tag{4.5}$$

LEMMA 4.6. For $p \in \mathcal{P}(V_{\mathbf{R}})$ homogeneous of degree m , $i^m D_p = (-1)^m \lambda_{\nu}(\Omega(p))$ where λ_{ν} denotes the symmetrization map (3.7) for the abelian group $V_{\mathbf{R}}$ (see (3.7)).

Proof. Let $\mathcal{B} = \{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ be a basis for $V_{\mathbf{R}}$ which is both ω -symplectic (i.e., $\omega(X_i, Y_j) = \delta_{i,j}$, etc.) and orthonormal with respect to the real inner product $(\cdot, \cdot) = \text{Re}\langle \cdot, \cdot \rangle$. Setting $Z_j = X_j + iY_j$, $\{Z_1, \dots, Z_n\}$ is a $\langle \cdot, \cdot \rangle$ -orthonormal basis for V . Let $\mathcal{B}' = \{X'_1, \dots, X'_n, Y'_1, \dots, Y'_n\}$ be the dual basis for $V_{\mathbf{R}}^*$ and let $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$ and $(x', y') = (x'_1, \dots, x'_n, y'_1, \dots, y'_n)$ be coordinates on $V_{\mathbf{R}}$ and $V_{\mathbf{R}}^*$ with respect to \mathcal{B} and \mathcal{B}' .

The isomorphism $V_{\mathbf{R}} \simeq V_{\mathbf{R}}^*$ given by ω is $v \leftrightarrow v^* := \omega(v, \cdot)$. We have $X_j^* = Y'_j$ and $Y_j^* = -X'_j$. Regarding the coordinate functions x_j, y_j and x'_j, y'_j as polynomials on $V_{\mathbf{R}}$ and $V_{\mathbf{R}}^*$ one sees that $\Omega(x_j) = -y'_j$ and $\Omega(y_j) = x'_j$. Indeed,

$$\begin{aligned} \Omega(x_j)(x', y') &= \Omega(x_j)(x'_1 X'_1 + \dots + x'_n X'_n + y'_1 Y'_1 + \dots + y'_n Y'_n) \\ &= \Omega(x_j)(-y'_1 X_1^* - \dots - y'_n X_n^* + x'_1 Y_1^* + \dots + x'_n Y_n^*) \\ &= x_j(-y'_1 X_1^* - \dots - y'_n X_n^* + x'_1 Y_1^* + \dots + x'_n Y_n^*) \\ &= -y_j, \end{aligned}$$

and the second identity is similar.

The abelian symmetrization map $\lambda_{\nu}: \mathcal{P}(V_{\mathbf{R}}^*) \rightarrow \mathcal{U}(V_{\mathbf{R}})$ is multiplicative and characterized by $\lambda_{\nu}(x'_j) = \partial/\partial x_j$ and $\lambda_{\nu}(y'_j) = \partial/\partial y_j$ on the generators. Thus,

$$(\lambda_{\nu} \circ \Omega)(x_j) = -\frac{\partial}{\partial y_j} \quad \text{and} \quad (\lambda_{\nu} \circ \Omega)(y_j) = \frac{\partial}{\partial x_j}.$$

Now let $z_j = x_j + iy_j$ so that (z_1, \dots, z_n) are coordinates for V with respect to $\{Z_1, \dots, Z_n\}$. We have

$$\begin{aligned} (\lambda_{\nu} \circ \Omega)(z_j) &= -\frac{\partial}{\partial y_j} + i \frac{\partial}{\partial x_j} \\ &= i \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) \\ &= 2i \frac{\partial}{\partial \bar{z}_j} \end{aligned}$$

and similarly

$$(\lambda_{\nu} \circ \Omega)(\bar{z}_j) = -2i \frac{\partial}{\partial z_j}.$$

Since Ω and λ_V are algebra maps ($V_{\mathbf{R}}$ is abelian), we conclude that

$$\begin{aligned} (\lambda_V \circ \Omega)(p(z, \bar{z})) &= p\left(2i \frac{\partial}{\partial \bar{z}}, -2i \frac{\partial}{\partial z}\right) \\ &= i^m p\left(2 \frac{\partial}{\partial \bar{z}}, -2 \frac{\partial}{\partial z}\right) \end{aligned}$$

when p is homogeneous of degree m in (z, \bar{z}) .

Since λ_V is K -equivariant and the action of K on V preserves ω , Lemma 4.6 shows that D_p is K -invariant for $p \in \mathcal{P}(V_{\mathbf{R}})^K$. In particular, the D_{p_α} 's are K -invariant differential operators on V .

With notation as in Proposition 4.2, one has

PROPOSITION 4.7 (Rodrigues' Formula).

$$D_{p_\alpha}(e^{-(1/2)|z|^2}) = q_\alpha(z, \bar{z}) e^{-(1/2)|z|^2}$$

and hence

$$\phi_\alpha(z, t) = e^{it} D_{p_\alpha}(e^{-(1/2)|z|^2}) e^{(1/4)|z|^2}.$$

Proof. We will identify H_n with its Lie algebra \mathfrak{h}_n via the exponential map. The Lie bracket is given by $[(z, t), (z', t')] = (0, \omega(z, z'))$. The left and right derivations given by $A \in \mathfrak{h}_n$ are

$$(L_A f)(z, t) := \left. \frac{d}{ds} \right|_{s=0} f((-sA)(z, t))$$

and

$$(R_A f)(z, t) := \left. \frac{d}{ds} \right|_{s=0} f((z, t)(sA)).$$

(R_A is the left invariant vector field given by A .) Let $\{e_1, \dots, e_n\}$ be the usual basis for \mathbf{C}^n and $X_j := (e_j, 0)$, $Y_j := (ie_j, 0)$, $Z_j := X_j + iY_j$, and $\bar{Z}_j := X_j - iY_j$. We compute

$$\begin{aligned} L_{Z_j} &= -2 \frac{\partial}{\partial \bar{z}_j} + \frac{iz_j}{2} \frac{\partial}{\partial t} \\ L_{\bar{Z}_j} &= -2 \frac{\partial}{\partial z_j} - \frac{i\bar{z}_j}{2} \frac{\partial}{\partial t} \\ R_{Z_j} &= 2 \frac{\partial}{\partial \bar{z}_j} + \frac{iz_j}{2} \frac{\partial}{\partial t} \\ R_{\bar{Z}_j} &= 2 \frac{\partial}{\partial z_j} - \frac{i\bar{z}_j}{2} \frac{\partial}{\partial t}. \end{aligned} \tag{4.8}$$

The Fourier transforms in the t -direction at 1 are given by replacing “ $\partial/\partial t$ ” by i in these equations.

$$\begin{aligned}
 L_{z_j}^1 &= -2 \frac{\partial}{\partial \bar{z}_j} - \frac{z_j}{2} \\
 L_{\bar{z}_j}^1 &= -2 \frac{\partial}{\partial z_j} + \frac{\bar{z}_j}{2} \\
 R_{z_j}^1 &= 2 \frac{\partial}{\partial \bar{z}_j} - \frac{z_j}{2} \\
 R_{\bar{z}_j}^1 &= 2 \frac{\partial}{\partial z_j} + \frac{\bar{z}_j}{2}.
 \end{aligned}
 \tag{4.9}$$

Note that, for example, $L_{z_j}(e^{it}\psi(z)) = e^{it}(L_{z_j}^1\psi)(z)$.

Let $M: C^\infty(\mathbb{C}^n) \rightarrow C^\infty(\mathbb{C}^n)$ be given by $(Mf)(z) := e^{-|z|^2/4} f(z)$. One has,

$$\begin{aligned}
 L_{z_j}^1 &= -2M^{-1} \circ \frac{\partial}{\partial z_j} \circ M \\
 R_{z_j}^1 &= 2M^{-1} \circ \frac{\partial}{\partial \bar{z}_j} \circ M.
 \end{aligned}
 \tag{4.10}$$

We also obtain differential operators from the derived representation of π on Fock space \mathcal{F} . For a smooth vector $f \in \mathcal{F}$, one has

$$(\pi(A)f)(w) = \left. \frac{d}{ds} \right|_{s=0} (\pi(sA)f)(w).$$

Using (1.4), one computes that

$$\begin{aligned}
 (\pi(Z_j)f)(w) &= -w_j f(w) \\
 (\pi(\bar{Z}_j)f)(w) &= 2 \frac{\partial f}{\partial w_j}.
 \end{aligned}
 \tag{4.11}$$

Extending π to $\mathcal{U}(\mathfrak{h}_n)$, we see that for a homogeneous polynomial $v(z_1, \dots, z_n)$ of degree m ,

$$\pi(v(Z))1 := \pi(v(Z_1, \dots, Z_n))1 = (-1)^m v(w_1, \dots, w_n) = (-1)^m v(w). \tag{4.12}$$

Moreover

$$\begin{aligned}
 \pi(Z_j)^* &= \pi(X_j)^* - i\pi(Y_j)^* \\
 &= -\pi(X_j) + i\pi(Y_j) \\
 &= \pi(-\bar{Z}_j) = -\pi(\bar{Z}_j)
 \end{aligned}
 \tag{4.13}$$

so that

$$\pi(v(Z))^* = \pi(\bar{v}(-\bar{Z})) = (-1)^m \pi(\bar{v}(\bar{Z})), \tag{4.14}$$

where \bar{v} is the polynomial whose coefficients are the complex conjugates of those in v .

Let $\{v_1, \dots, v_l\}$ be an orthonormal basis for a K -irreducible subspace $P_\alpha \subset \mathcal{P}_m(\mathbb{C}^n)$. We have

$$\begin{aligned} \phi_\alpha(z, t) &= e^{it} e^{-(1/4)|z|^2} q_\alpha(z) \\ &= \frac{1}{l} \sum_{j=1}^l \langle \pi(z, t) v_j, v_j \rangle_{\mathcal{F}} \\ &= \frac{1}{l} \sum_{j=1}^l \langle \pi(z, t) \pi(v_j(Z))1, \pi(v_j(Z))1 \rangle_{\mathcal{F}} \quad \text{by (4.12)} \\ &= \frac{1}{l} \sum_{j=1}^l \langle \pi(v_j(Z))^* \pi(z, t) \pi(v_j(Z))1, 1 \rangle_{\mathcal{F}} \\ &= \frac{1}{l} \sum_{j=1}^l \langle \pi(\bar{v}_j(-\bar{Z})) \pi(z, t) \pi(v_j(Z))1, 1 \rangle_{\mathcal{F}} \quad \text{by (4.14)} \\ &= \frac{1}{l} \sum_{j=1}^l v_j(R_Z) \bar{v}_j(L_{\bar{Z}}) (\langle \pi(z, t)1, 1 \rangle_{\mathcal{F}}). \end{aligned}$$

Here $\langle \pi(z, t)1, 1 \rangle_{\mathcal{F}} = e^{it} e^{-(1/4)|z|^2}$, so

$$\begin{aligned} e^{it} e^{-(1/4)|z|^2} q_\alpha(z) &= \frac{e^{it}}{l} \sum_{j=1}^l (v_j(R_Z^1) \bar{v}_j(L_{\bar{Z}}^1)) (e^{-(1/4)|z|^2}) \\ &= \frac{e^{it}}{l} \sum_{j=1}^l \left(v_j \left(2M^{-1} \frac{\partial}{\partial \bar{z}_j} M \right) \bar{v}_j \left(-2M^{-1} \frac{\partial}{\partial z_j} M \right) \right) (e^{-(1/4)|z|^2}) \quad \text{by (4.10)}. \end{aligned}$$

Hence,

$$q_\alpha(z) e^{-(1/4)|z|^2} = M^{-1} \left(\frac{1}{l} \sum_{j=1}^l v_j \left(2 \frac{\partial}{\partial \bar{z}_j} \right) \bar{v}_j \left(-2 \frac{\partial}{\partial z_j} \right) \right) (M e^{-(1/4)|z|^2})$$

so that,

$$q_\alpha(z) e^{-(1/2)|z|^2} = \left(\frac{1}{l} \sum_{j=1}^l v_j \left(2 \frac{\partial}{\partial \bar{z}_j} \right) \bar{v}_j \left(-2 \frac{\partial}{\partial z_j} \right) \right) (e^{-(1/2)|z|^2}) = D_{p_\alpha} (e^{-(1/2)|z|^2})$$

as claimed.

Proposition 4.7 can be reformulated using the *symplectic Fourier transform* on $V \simeq \mathbb{C}^n$ defined by

$$\begin{aligned} \widehat{f}^\omega(\zeta) &= \int f(z) e^{-i \operatorname{Im} \langle z, \zeta \rangle} dz \\ &= \int f(x, y) e^{-i(x\eta - y\xi)} dx dy \end{aligned} \tag{4.15}$$

for $\zeta = \xi + i\eta$, $z = x + iy$. One has that

$$\begin{aligned} \widehat{z_j f}^\omega &= 2 \frac{\partial}{\partial \bar{\zeta}_j} \widehat{f}^\omega \\ \widehat{\bar{z}_j f}^\omega &= -2 \frac{\partial}{\partial \zeta_j} \widehat{f}^\omega \end{aligned} \tag{4.16}$$

and that $e^{-(1/2)|z|^2}$ is fixed by \wedge^ω .

COROLLARY 4.17. $(p_\alpha(z, \bar{z}) e^{-(1/2)|z|^2})^\wedge = q_\alpha(\zeta, \bar{\zeta}) e^{-(1/2)|\zeta|^2}$.

5. THE VALUE SPACE AND POLAR REPRESENTATIONS

Suppose that (K, H_V) is a Gelfand pair with associated fundamental invariants $\gamma_1, \dots, \gamma_d$ as introduced in Section 3. By construction each γ_i (and indeed each p_α) is a homogeneous polynomial on $V_{\mathbb{R}}$ that takes on only non-negative real values. Thus the range of γ_i is \mathbb{R}^+ . Let $\gamma := \gamma_1 \times \dots \times \gamma_d$ and

$$\Gamma = \gamma(V) \subset (\mathbb{R}^+)^d. \tag{5.1}$$

Any polynomial $q \in \mathcal{P}(V_{\mathbb{R}})^K$ will factor through the projection $\gamma: V \rightarrow \Gamma$, and we obtain a polynomial \tilde{q} on Γ defined by

$$\tilde{q} \circ \gamma = q. \tag{5.2}$$

We may also transplant K -invariant differential operators L on V to obtain operators \tilde{L} on Γ by the rule

$$\gamma^*(\tilde{L}) = L. \tag{5.3}$$

We see that the functions $p_\alpha(z)$, $q_\alpha(z)$, and $\psi_\alpha(z)$ correspond to functions \tilde{p}_α , \tilde{q}_α , and $\tilde{\psi}_\alpha$ on Γ . This provides a mechanism for reducing the analysis

of K -spherical functions to “radial directions.” In particular, the $\tilde{\psi}_\alpha$'s are precisely the bounded simultaneous eigenfunctions for $\tilde{L}_1^1, \dots, \tilde{L}_d^1$ which satisfy $\tilde{\psi}_\alpha(0) = 1$ (where L_j^1 is given by (3.16)).

Thus, K -spherical functions on H_n can be obtained from solutions to eigenvalue problems on Γ . The orthogonalization procedure (Proposition 4.2) and Rodrigues' formula (Proposition 4.7) both descend to Γ . In principle, these results provide algorithms for identifying solutions.

THEOREM 5.4. *Let ν be the measure on Γ defined by $\gamma^*(\nu) = e^{-|z|^2/2} dz d\bar{z}$. The polynomials \tilde{q}_α on Γ are obtained by applying the Gram–Schmidt algorithm using $d\nu$ to the \tilde{p}_α 's (ordered as before) and normalizing so that $\tilde{q}_\alpha(0) = 1$.*

The Rodrigues' formula can be recast on Γ most clearly when K acts irreducibly on V . Recall that in this case, the first fundamental invariant is $\gamma_1(z) = |z|^2/2n$. Also note that the operators D_{p_α} are K -invariant.

THEOREM 5.5. *Let K act irreducibly on V . The polynomials \tilde{q}_α on Γ are given by $\tilde{D}_{p_\alpha}(e^{-n\gamma_1}) = \tilde{q}_\alpha(\gamma) e^{-n\gamma_1}$.*

Polar Representations

There is an alternative and more geometric radial reduction which can sometimes be performed. Consider the Hermitian vector space $(V, \langle \cdot, \cdot \rangle)$ with $K \subset U(V)$. A point $v_0 \in V$ is called *regular* if the K -orbit $K \cdot v_0$ has maximal dimension. The real part (\cdot, \cdot) of $\langle \cdot, \cdot \rangle$ is an inner product on the underlying real vector space $V_{\mathbf{R}}$. We define

$$\mathcal{A}_{v_0} := (\mathbf{k} \cdot v_0)^\perp = \{u \in V \mid (u, X \cdot v_0) = 0 \text{ for all } X \in \mathbf{k}\}. \quad (5.6)$$

$\mathcal{A} = \mathcal{A}_{v_0}$ meets every K -orbit in V (possibly more than once) [Da]. The representation V of K is *polar* if

$$(\mathbf{k} \cdot u, \mathcal{A}) = 0 \quad \text{for all } u \in \mathcal{A}. \quad (5.7)$$

That is, each K -orbit meets the cross section \mathcal{A} orthogonally at each point of intersection.

The representation V of K may or may not be polar. For example, direct computations show that the first ten entries in Table 1.8 are polar but that the next six are not. For the remainder of this section, we assume that the representation V of K is polar.

Polar representations of compact groups are studied in [Da]. It is shown there that

LEMMA 5.8. *V is a polar representation of K if and only if the translated tangent spaces $\mathfrak{k} \cdot v_0$ and $\mathfrak{k} \cdot v_1$ to the K-orbits through any two regular points v_0, v_1 differ by the action of K.*

It follows that the cross section \mathcal{A} is unique modulo the action of K on V.

As a K-orbit is compact and orthogonal to \mathcal{A} , it meets \mathcal{A} in a finite set of points. The Weyl group for \mathcal{A} is defined as

$$W(\mathcal{A}) := N(\mathcal{A})/Z(\mathcal{A}), \tag{5.9}$$

where $N(\mathcal{A}) = \{k \in K \mid k \cdot \mathcal{A} = \mathcal{A}\}$ (the normalizer of \mathcal{A}) and $Z(\mathcal{A}) = \{k \in K \mid k|_{\mathcal{A}} = id\}$ (the centralizer of \mathcal{A}). It is shown in [Da] that

LEMMA 5.10. *$W := W(\mathcal{A})$ is a finite group with $(K \cdot u) \cap \mathcal{A} = W \cdot u$ for all $u \in \mathcal{A}$.*

Thus we obtain correspondences

$$\mathcal{A}/W \simeq V/K \quad \text{and} \quad C^\infty(V)^K \simeq C^\infty(\mathcal{A})^W \tag{5.11}$$

(via the inclusion $\mathcal{A} \subset V$ and restriction).

We see that the $p_\alpha(z)$'s and the polynomial components $q_\alpha(z)$ of the spherical functions can be viewed as W-invariant polynomials on \mathcal{A} via restriction.

The radial part $D_{\mathcal{A}}$ of a K-invariant differential operator D on V is characterized by

$$D_{\mathcal{A}}(f|_{\mathcal{A}}) = (Df)|_{\mathcal{A}} \quad \text{for all } f \in C^\infty(V)^K. \tag{5.12}$$

(See [He].) As an immediate consequence of Theorem 3.16, the functions $\psi_\alpha(a) = e^{-|a|^2/2} q_\alpha(a)$ on \mathcal{A} are precisely the bounded W-invariant simultaneous eigenfunctions for $(L_1^1)_{\mathcal{A}}, \dots, (L_d^1)_{\mathcal{A}}$ that satisfy $\psi_\alpha(0) = 1$. Similarly, the K-spherical functions $\eta_\omega(a)$ on \mathcal{A} are the bounded W-invariant simultaneous eigenfunctions for $(L_1^0)_{\mathcal{A}}, \dots, (L_d^0)_{\mathcal{A}}$ satisfying $\eta_\omega(0) = 1$.

The orthogonalization procedure can be formulated explicitly in this setting. For $p \in \mathcal{P}(V_{\mathbf{R}})^K$,

$$\int_V p(z) e^{-|z|^2/2} dz d\bar{z} = \frac{1}{|W|} \int_{\mathcal{A}} p(a) e^{-|a|^2/2} J_K(a) da, \tag{5.13}$$

where $J_K(a) := \text{Vol}(K \cdot a)$. Propositions 4.2 and 4.7 yield the following two theorems.

THEOREM 5.14. *The polynomials $q_\alpha \in \mathcal{P}(\mathcal{A})^W$ are obtained by applying the Gram–Schmidt algorithm to the polynomials $p_\alpha|_{\mathcal{A}} \in \mathcal{P}(\mathcal{A})^W$ using the measure*

$$d\mu(a) = \frac{1}{|W|} e^{-|a|^2/2} J_K(a) da$$

and normalizing so that $q_\alpha(0) = 1$. The p_α 's are ordered as in Proposition 4.2.

THEOREM 5.15. $(D_{p_\alpha})_{\mathcal{A}}(e^{-|a|^2/2}) = q_\alpha(a) e^{-|a|^2/2}$.

Remarks. 1. In practice, the operators \widetilde{L}_j^1 on the value space Γ can be computed directly, using the chain rule for $\gamma: V \rightarrow \Gamma$. In the first ten cases in Table 1.8, where V is a polar representation of K , $\gamma|_{\mathcal{A}}$ is a local diffeomorphism, so the operators $(L_j^1)_{\mathcal{A}}$ are pull-backs of the \widetilde{L}_j^1 's via γ .

2. As the spherical functions are normalized by their values at 0, one can ignore any constant factor in $J_K(a) = \text{Vol}(K \cdot a)$. Again, if $\gamma: \mathcal{A} \rightarrow \Gamma$ is a local diffeomorphism, the measure $d\mu$ can be pushed down to Γ .

In the next section, we provide explicit computations for the groups $K = T(n)$ and $K = U(n)$ on H_n to illustrate these results. Further examples from Table 1.8 will appear elsewhere.

6. THE GELFAND PAIRS $(T(n), H_n)$ AND $(U(n), H_n)$

We begin with the pair $(T(n), H_n)$, where $T(n)$ is the n -torus,

$$\{e^{i\theta} = (e^{i\theta_1}, \dots, e^{i\theta_n}) : \theta = (\theta_1, \dots, \theta_n) \in \mathbf{R}^n\}.$$

It is an easy consequence of (1.7) that $(T(n), H_n)$ is a Gelfand pair. One also has a straightforward proof, found in [HR, FS], of this fact: Given any $T(n)$ -invariant function f on H_n , and any representation π_λ of H_n , we see that $\pi_\lambda(f)$ commutes with the action of $T(n)$ on Fock space \mathcal{F}_λ . Since the eigenvalues for the action of $T(n)$ are distinct on each monomial in \mathcal{F}_λ , $\pi_\lambda(f)$ must be diagonalized by the monomials. Thus $\pi_\lambda(f)$ and $\pi_\lambda(g)$ commute for all $f, g \in L^1_{T(n)}(H_n)$, and all $0 \neq \lambda \in \mathbf{R}$. By the Plancherel theorem, one concludes that $f * h = g * f$.

Let $w^\alpha = w_1^{\alpha_1} \cdots w_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$. Then \mathcal{F} decomposes into the $T(n)$ -invariant subspaces $\mathbf{C}w^\alpha$. The corresponding invariant polynomials are $p_\alpha = |z|^{2\alpha} / 2^{|\alpha|} \alpha!$. The spherical functions may be found by direct computation as in [HR], or by using Theorem (4.2). One has

PROPOSITION 6.1. *The bounded spherical functions for the Gelfand pair $(T(n), H_n)$ associated to π are*

$$\phi_\alpha(z, t) = e^{it} e^{-(1/4)|z|^2} \prod_{j=1}^n L_{\alpha_j}(\frac{1}{2}|z_j|^2),$$

where

$$L_k(x) = \sum_{j=0}^k \binom{k}{j} \frac{(-x)^j}{j!}$$

is the k th Laguerre polynomial (of order 0).

We now turn to the pair $(U(n), H_n)$. The irreducible subspaces in \mathcal{F} are the full subspaces $\mathcal{P}_m(\mathbb{C}^n)$, with orthonormal basis $\{w^\alpha/\sqrt{2^m \alpha!} : |\alpha| = m\}$.

PROPOSITION 6.2. *The invariant polynomials for the pair $(U(n), H_n)$ are spanned by*

$$\left\{ p_m(w) = \frac{(n-1)!}{2^m(m+n-1)!} |w|^{2m} : m \in \mathbb{Z}^+ \right\}.$$

The corresponding bounded spherical functions associated to π are

$$\phi_m(z, t) = e^{it} L_m^{(n-1)}(\frac{1}{2}|z|^2) e^{-(1/4)|z|^2},$$

where $L_m^{(n-1)}$ is the generalized Laguerre polynomial of order $n-1$ (normalized to be 1 at 0), namely

$$L_m^{(n-1)}(x) = (n-1)! \sum_{j=0}^m \binom{m}{j} \frac{(-x)^j}{(j+n-1)!}.$$

Proof. First note that

$$\begin{aligned} p_m(z) &= \frac{m!(n-1)!}{(m+n-1)!} \sum_{|\alpha|=m} \frac{1}{2^m \alpha!} z^\alpha \bar{z}^\alpha \\ &= \frac{(n-1)!}{2^m(m+n-1)!} \sum_{\alpha_1 + \dots + \alpha_n = m} \frac{m!}{\alpha_1! \dots \alpha_n!} |z_1|^{2\alpha_1} \dots |z_n|^{2\alpha_n} \\ &= \frac{(n-1)!}{2^m(m+n-1)!} |z|^{2m}. \end{aligned}$$

Now we need to obtain an orthonormal sequence from the p_m 's with respect to the measure $d\bar{z} = e^{-(1/2)|z|^2} dz d\bar{z}$. If f is a function of $|z|^2$, then

$$\int f(z) e^{-(1/2)|z|^2} dz d\bar{z} = c_n \int f(r^2) e^{-(1/2)r^2} r^{2n-1} dr, \tag{6.3}$$

where c_n is the volume of the sphere S^{2n-1} . Substituting $s = r^2/2$, (6.3) becomes

$$\int f(2s) e^{-s} s^{n-1} ds.$$

Let $\|\cdot\|$ denote the L^2 -norm on C^n with respect to the measure $d\bar{z}$, and let $|\|\cdot\||$ denote the L^2 -norm on $[0, \infty)$ with respect to the measure $d\bar{s} := e^{-s} s^{n-1} ds$. For $f \in L^2([0, \infty), d\bar{s})$ define \tilde{f} on C^n by $\tilde{f}(z) = f(|z|^2/2)$. Then, by the above calculation, $\|\tilde{f}\|^2 = c_n 2^{n-1} |\|f\||^2$.

For $m \in \mathbb{Z}^+$ set $e_m(s) = s^m$ for $s \in [0, \infty)$. Then $\tilde{e}_m = m! p_m$. The generalized Laguerre polynomials, $\{L_m^{(n-1)}\}$, are obtained by applying the Gram-Schmidt process to $\{e_m\}$ with respect to $d\bar{s}$ (cf., e.g., [Sz]). Thus $\{\tilde{L}_m^{(n-1)}\}$ gives the desired polynomials, as claimed.

Consider now the bounded K -spherical functions associated to the one-dimensional representations. For $K = U(n)$ the distinct K -orbits are parametrized by real $\tau \geq 0$. For $\tau = 0$, one has the trivial character and the K -spherical function $\phi_0(z, t) \equiv 1$. For each $\tau > 0$, the sphere S_τ of radius τ in C^n is a K -orbit and we obtain from (2.13) the K -spherical function $\eta_\tau = \hat{\mu}_{S_\tau}$. The Fourier transform of the unit mass on the $(2n-1)$ -sphere is given in terms of the Bessel function J_{n-1} . (See, e.g., [Ey].) After normalization one has

$$\eta_\tau(z, t) = \frac{2^{n-1}(n-1)!}{(\tau|z|)^{n-1}} J_{n-1}(\tau|z|), \tag{6.4}$$

for each $\tau > 0$.

Alternatively, by (3.19), $2nL_1^0 = 4\Delta$ where Δ is the usual Euclidean Laplace operator. Eigenfunctions of the form $e^{i\omega \cdot z}$ with $|\omega| = 1$ give eigenvalues $-\tau^2$. Averaging these functions over K -orbits $|z| = \text{const}$, we obtain (6.4).

For $K = T(n) = \prod_{i=1}^n U(1)$, the K -orbits in C^n are parametrized by real, non-negative n -tuples, $\tau = (\tau_1, \dots, \tau_n)$. By (2.14) and the above result, the associated K -spherical functions on H_n are given by $\eta_\tau(z, t) = \prod_{i=1}^n J_0(\tau_i |z_i|)$.

Lemma 3.15 shows that the $U(n)$ -invariant differential operators on H_n are generated by $L_1 = \tilde{\lambda}(\gamma_1)$ and $\partial/\partial t$. Here we consider the more usual

$$2nL_1 = \frac{1}{2} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j). \tag{6.5}$$

We let $L := 2nL_1^1$ where L_j^1 is as in (3.16).

LEMMA 6.6. $L = 4\Delta - |z|^2/4$.

Proof. We have

$$L = \frac{1}{2} \sum_{j=1}^n \left\{ \left(2 \frac{\partial}{\partial \bar{z}_j} + \frac{iz_j}{2} \frac{\partial}{\partial t} \right) \left(2 \frac{\partial}{\partial z_j} - \frac{i\bar{z}_j}{2} \frac{\partial}{\partial t} \right) + \left(2 \frac{\partial}{\partial z_j} - \frac{i\bar{z}_j}{2} \frac{\partial}{\partial t} \right) \left(2 \frac{\partial}{\partial \bar{z}_j} + \frac{iz_j}{2} \frac{\partial}{\partial t} \right) \right\}.$$

As $\phi_x(z, t) = e^{it}\psi_x(z)$, when applied to ϕ_x ,

$$L = 4 \sum_{j=1}^n \frac{\partial}{\partial \bar{z}_j} \frac{\partial}{\partial z_j} + \frac{1}{2} \sum_{j=1}^n \left(\bar{z}_j \frac{\partial}{\partial \bar{z}_j} - z_j \frac{\partial}{\partial z_j} \right) - |z|^2/4.$$

Note that $\bar{z}_j(\partial/\partial \bar{z}_j) f(|z|^2) = z_j(\partial/\partial z_j) f(|z|^2)$.

PROPOSITION 6.7. *The functions $\psi_m(z) = L_m^{(n-1)}(|z|^2/2) e^{-(1/4)|z|^2}$ are “radial” eigenfunctions of the operator $L = 4\Delta - |z|^2/4$, with eigenvalues $-(2m + n)$.*

Proof. This is a direct consequence of Lemma 3.15 together with an application of Proposition 3.20. Choose $v_m = z_1^m$ in \mathcal{P}_m . Then

$$\begin{aligned} \pi \left(\frac{1}{2} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j) \right) v_m &= \frac{1}{2} \sum_{j=1}^n \left(-2z_j \frac{\partial}{\partial z_j} - 2 \frac{\partial}{\partial z_j} z_j \right) z_1^m \\ &= -(2m + n) z_1^m. \end{aligned}$$

As the action of $U(n)$ on \mathbf{C}^n is polar, we can apply the techniques of Section 5. For $v_0 = e_1 = (1, 0, \dots, 0)$,

$$\mathbf{k}v_0 = i\mathbf{R}e_1 \oplus \mathbf{C}^{n-1}, \quad \text{and} \quad \mathcal{A} = (\mathbf{k}v_0)^\perp = \mathbf{R}e_1. \tag{6.8}$$

The value space Γ is \mathbf{R}^+ , with $\gamma: \mathbf{C}^n \rightarrow \mathbf{R}^+$ given by

$$\gamma(z) = |z|^2/2. \tag{6.9}$$

(Actually, $p_1(z) = |z|^2/2n$, but the analysis is simplified by taking $\gamma = np_1$.) Note that $\gamma: \mathcal{A} \rightarrow \Gamma$ is just

$$\gamma(ae_1) = a^2/2. \tag{6.10}$$

The proof of Proposition 6.2 used reduction to the value space via γ . The measure μ on \mathbf{R}^+ appearing in Theorem 5.4 is $\mu = d\tilde{s} = s^{-s} s^{n-1} ds$ in this case and the resulting orthogonal polynomials are $\tilde{q}_m(s) = L_m^{(n-1)}(s)$.

An alternative proof for Proposition 6.2 uses Proposition 4.7 and Theorem 5.5. One needs to verify that

$$e^{(1/2)|z|^2} D_{p_m}(e^{-(1/2)|z|^2}) = L_m^{(n-1)}(\frac{1}{2}|z|^2). \quad (6.11)$$

Using γ to reduce to the value space requires making the change of variables $s = |z|^2/2$ in (6.11). An exercise with the chain rule yields

$$\tilde{D}_{p_m}(e^{-s}) = \frac{(n-1)!}{(m+n-1)!} (-1)^m \left(s \frac{d^2}{ds^2} + n \frac{d}{ds} \right)^m e^{-s} = L_m^{(n-1)}(s) e^{-s}. \quad (6.12)$$

One can show by a painful induction on m that (6.12) is equivalent to

$$\frac{(n-1)!}{(m+n-1)!} s^{1-n} e^s \left(\frac{d}{ds} \right)^m (s^{m+n-1} e^{-s}) = L_m^{(n-1)}(s), \quad (6.13)$$

which is the classical Rodrigues' formula.

A direct calculation shows:

LEMMA 6.14.

$$\tilde{L} = 2\gamma \frac{d^2}{d\gamma^2} + 2n \frac{d}{d\gamma} - \frac{\gamma}{2} \quad \text{on } \Gamma$$

and

$$L_{\mathcal{A}} = \frac{d^2}{da^2} + \left(\frac{2n-1}{a} \right) \frac{d}{da} - \frac{a^2}{4} \quad \text{on } \mathcal{A}.$$

As $\tilde{L}(\tilde{q}_m(\gamma) e^{-\gamma/2}) = -(2m+n) \tilde{q}_m(\gamma) e^{-\gamma/2}$, we can derive the equation

$$\left(2\gamma \frac{d^2}{d\gamma^2} + (2n-2\gamma) \frac{d}{d\gamma} - n \right) \tilde{q}_m(\gamma) = -(2m+n) \tilde{q}_m(\gamma).$$

That is,

$$\left(\gamma \frac{d^2}{d\gamma^2} + (n-\gamma) \frac{d}{d\gamma} + m \right) \tilde{q}_m(\gamma) = 0 \quad (6.15)$$

which is the standard differential equation for the generalized Laguerre polynomial of order $n-1$ and degree m .

As $\mathcal{A} = \mathbf{Re}_1$, $N(\mathcal{A}) = \{\pm 1\} \times U(n-1)$ and $Z(\mathcal{A}) = \{1\} \times U(n-1)$. Thus $W = \{\pm 1\}$, and hence the polynomials $q_m|_{\mathcal{A}}$ are even eigenfunctions

of $L_{\mathcal{A}}$. If we take the map $a \mapsto a^2$ from \mathcal{A} to Γ , (6.15) becomes the equation for *Hermite* polynomials of even order.

Finally, an application of Proposition 2.4 yields some rather nice results on Laguerre polynomials. Part (b) of Theorem 6.16 can be found in [St, Sc].

THEOREM 6.16. *Let $L_m^{(k)}$ be the generalized Laguerre polynomial of degree m and order k .*

- (a) $\sum_{|\alpha|=m} \prod_{j=1}^m L_{\alpha_j}(u_j) = \binom{m+n-1}{m} L_m^{(n-1)}(u)$
- (b) $u^{n-1} L_m^{(n-1)}(u) = \int_0^{u_{n-1}} \cdots \int_0^{u_2} L_{\alpha_1}(u_1) L_{\alpha_2}(u_2 - u_1) \cdots L_{\alpha_n}(u - u_{n-1}) du_1 \cdots du_{n-1}$, where $L_k := L_k^{(0)}$ and $m = |\alpha|$.

Proof. We apply Proposition 2.4 with $K' = T(n)$ and $K = U(n)$. Then $\mathcal{P}_m = \sum_{|\alpha|=m} P_\alpha$, where $P_\alpha = \mathbb{C}w^\alpha$. We have $q_m(z) = L_m^{(n-1)}(|z|^2/2)$ and $q_\alpha(z) = \prod_{j=1}^n L_{\alpha_j}(|z_j|^2/2)$. By (2.6),

$$\psi_m = \frac{1}{\dim \mathcal{P}_m} \sum_{|\alpha|=m} \dim(P_\alpha) \psi_\alpha,$$

so

$$\binom{m+n-1}{m} q_m(z) e^{-|z|^2/4} = \sum_{|\alpha|=m} q_\alpha(z) e^{-|z|^2/4},$$

and (a) follows.

By (2.5), for any α with $|\alpha| = m$,

$$\psi_m(z) = \int_{U(n)/T(n)} \psi_\alpha(k \cdot z) dk.$$

Thus, for $f \in L_{U(n)}(\mathbb{C}^n)$,

$$\begin{aligned} & \int f(|z|^2) \psi_\alpha(z) dz d\bar{z} \\ &= \int f(|z|^2) \prod_{i=1}^n L_{\alpha_i}(\tfrac{1}{2} |z_i|^2) e^{-(1/4)|z|^2} dz d\bar{z} \\ &= a \int f\left(\sum_{i=1}^n r_i^2\right) \prod_{i=1}^n L_{\alpha_i}(\tfrac{1}{2} r_i^2) e^{-(1/4)\sum r_i^2} r_1 \cdots r_n dr_1 \cdots dr_n \\ &= b \int f\left(\sum_{i=1}^n 2s_i\right) \prod_{i=1}^n L_{\alpha_i}(s_i) e^{-(1/2)\sum s_i} ds_1 \cdots ds_n \end{aligned}$$

$$\begin{aligned}
&= b \int_0^\infty \int_0^{u_{n-1}} \cdots \int_0^{u_2} f(2u_n) L_{\alpha_1}(u_1) \\
&\quad \times L_{\alpha_2}(u_2 - u_1) \cdots L_{\alpha_n}(u_n - u_{n-1}) e^{-(1/2)u_n} du_1 \cdots du_n \\
&= \int f(|z|^2) \psi_m(z) dz d\bar{z} \\
&= c \int_0^\infty f(r^2) L_m^{(n-1)}(\frac{1}{2}r^2) e^{-(1/4)|r|^2} r^{2n-1} dr \\
&= d \int_0^\infty f(2u) L_m^{(n-1)}(u) e^{-(1/2)u} u^{n-1} du.
\end{aligned}$$

Comparing normalizations, one concludes that for $|\alpha| = m$,

$$\begin{aligned}
u^{n-1} L_m^{(n-1)}(u) &= \int_0^{u_{n-1}} \cdots \int_0^{u_2} L_{\alpha_1}(u_1) \\
&\quad \times L_{\alpha_2}(u_2 - u_1) \cdots L_{\alpha_n}(u - u_{n-1}) du_1 \cdots du_{n-1}
\end{aligned}$$

as claimed.

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