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KÄHLER AND SYMPLECTIC STRUCTURES ON NILMANIFOLDS

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1. INTRODUCTION

THERE HAS been recent interest in examples of compact symplectic manifolds which do not admit Kähler structures. Thurston described the first such example in [14] and other examples have appeared in [1], [2], [3], [8], [9], [11], and [16]. With the exception of [11], all of these examples are nilmanifolds. A nilmanifold is a quotient $\Gamma\backslash G$ of a connected simply-connected nilpotent Lie group G by a co-compact discrete subgroup Γ. It is known that such manifolds are the most general compact homogeneous spaces for nilpotent Lie groups [10]. In this paper we study the existence of Kähler structures and symplectic structures on arbitrary nilmanifolds. The non-existence of Kähler structures in the above examples is a special case of our main result:

THEOREM A. *If a nilmanifold $\Gamma\backslash G$ admits a Kähler structure, then G is abelian and $\Gamma\backslash G$ is diffeomorphic to a torus.*

The proof of Theorem A is given in Section 2. One corollary is a closely related topological result.

THEOREM B. *Let M be a compact $K(\Gamma, 1)$-manifold where Γ is a discrete, finitely generated, torsion free, nilpotent group. If M admits a Kähler structure, then Γ is abelian and M has the homotopy type of a torus.*

Theorem A suggests that the nilmanifold setting yields many examples of compact symplectic non-Kähler manifolds. Any non-toral nilmanifold which can be given a symplectic structure is such an example. However, as remarked in [3], it can be difficult to find symplectic nilmanifolds. We address this issue in Section 3.

Suppose that G is any Lie group with Lie algebra $\mathfrak{g}$. It is well known that any orbit O in $\mathfrak{g}^* = (\mathrm{Hom}\,\mathfrak{g}, \mathbb{R})$ under the coadjoint action of G on $\mathfrak{g}^*$ carries a canonical symplectic structure ω_O [6]. We call an orbit O *normal* if the isotropy subgroup of any point in O is normal. In this case O is itself a group and ω_O is left O-invariant (see Section 3 for details). One obtains a symplectic structure ω_O on $\Gamma\backslash O$ for any discrete subgroup Γ of O. In the nilmanifold case, this construction is universal.

THEOREM C. *Let $(\Gamma\backslash G, \omega)$ be a symplectic nilmanifold. There is a normal coadjoint orbit O (for some larger nilpotent Lie group) and a Lie group isomorphism $\phi: O \to G$ such that $\bar{\phi}^*(\omega)$ is cohomologous to ω_O. Here, $\bar{\phi}$ denotes the diffeomorphism $\phi^{-1}(\Gamma)\backslash O \to \Gamma\backslash G$ given by ϕ.*

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Theorem C says that coadjoint orbits provide a systematic way to produce all nilmanifolds which admit symplectic structures. We illustrate this in Section 3 by showing how the examples of Cordero, Fernandez and Gray [3] arise from coadjoint orbits. It is interesting that the examples are not *ad hoc* but fit into a natural framework that has wonderful applications in representation theory [6].

We conclude this section by noting that the most obvious way to construct a symplectic or Kähler structure on $\Gamma\backslash G$ is to begin with a left G-invariant one on G. However, if G has a homogeneous Kähler structure then $\Gamma\backslash G$ is complex parallizable and hence is a torus by a theorem of Wang [15]. Theorem C shows that if $\Gamma\backslash G$ has a symplectic structure then it has one coming from a homogeneous symplectic structure on G. Thus for nilmanifolds, there is a close parallel between the general situation and the homogeneous case.

2. KÄHLER NILMANIFOLDS

There are a number of cohomological conditions necessary for a compact manifold to admit a Kähler structure. One condition is that the odd Betti numbers must be even; this was used by Thurston to prove non-existence of a Kähler structure in the example cited in Section 1. For a general nilmanifold the Betti numbers can have any parity. We will instead use the following:

HARD LEFSCHETZ THEOREM. (See [4]) *Let M^{2n} be a compact Kähler manifold and let $[\omega]$ denote the cohomology class of its Kähler form ω. Then for each $j=0, 1, \ldots, n$, the map $\cup[\omega]^j : H^{n-j}(M)\to H^{n+j}(M)$ defined by $\alpha\to\alpha\cup[\omega]^j$ is an isomorphism.*

Let $\Gamma\backslash G$ be a (compact) nilmanifold and let $\mathfrak{g}$ be the Lie algebra of G. We denote by $H^*(\mathfrak{g})$ the cohomology ring of $\mathfrak{g}$ with trivial coefficients $\mathbb{R}$. Recall that this is the cohomology of the complex $\wedge(\mathfrak{g}^*)$ of left-invariant forms on G. A theorem of Nomizu [12] states that the standard inclusion $\wedge(\mathfrak{g}^*)\subsetneq\Omega^*(\Gamma\backslash G)$ gives an isomorphism $H^*(\mathfrak{g})\to H^*(\Gamma\backslash G; \mathbb{R})$.

Proof of Theorem A. Suppose that the nilmanifold $\Gamma\backslash G$ admits a Kähler structure. By Nomizu's theorem the Kähler form is cohomologous to a left-invariant form $\omega\in\wedge(\mathfrak{g}^*)$. Since $[\omega]^n$ is non-zero in $H^{2n}(\Gamma\backslash G; \mathbb{R})$, ω is non-degenerate and hence is a symplectic structure. By the Hard Lefschetz Theorem, the map

$$\wedge[\omega]^{n-1} : H^1(\mathfrak{g})\to H^{2n-1}(\mathfrak{g}) \tag{2.1}$$

must be an isomorphism. We assume that $\mathfrak{g}$ is non-abelian and show that for any closed, non-degenerate form ω in $\wedge^2(\mathfrak{g}^*)$, the map (2.1) is not surjective. This contradiction will complete the proof.

Suppose that $\mathfrak{g}$ is $(r+1)$-step nilpotent (note that $r\geqslant 1$ since $\mathfrak{g}$ is non-abelian) and consider the descending central series for $\mathfrak{g}$, $\mathfrak{g}=\mathfrak{g}^{(0)}\supset\mathfrak{g}^{(1)}\supset\cdots\supset\mathfrak{g}^{(r)}\supset\mathfrak{g}^{(r+1)}=\{0\}$ where $\mathfrak{g}^{(i+1)}=[\mathfrak{g}, \mathfrak{g}^{(i)}]$. Note that $\mathfrak{g}^{(r)}$ is contained in the center of $\mathfrak{g}$. Choose a vector space complement $\mathfrak{a}^{(i)}$ of $\mathfrak{g}^{(i+1)}$ in $\mathfrak{g}^{(i)}$:

$$\mathfrak{g}^{(i)}=\mathfrak{g}^{(i+1)}+\mathfrak{a}^{(i)} \tag{2.2}$$

for $i=0, \ldots, r-1$ in such a way that $\mathfrak{a}^{(i)}$ is spanned by elements of the form $[U, V]$ with $V\in\mathfrak{g}^{(i-1)}$. We have

$$\mathfrak{g}=\mathfrak{a}^{(0)}+\mathfrak{a}^{(1)}+\ldots+\mathfrak{a}^{(r)}. \tag{2.3}$$

Next let

$$\mathfrak{b}^{(i)}=\mathfrak{a}^{(0)}+\ldots+\mathfrak{a}^{(i)}. \tag{2.4}$$

We use (2.3) to view $\mathfrak{a}^{(i)*}$ as a subspace of $\mathfrak{g}^*$. Thus (2.3) yields a dual decomposition of the space of forms,

$$\wedge^s(\mathfrak{g}^*)=\sum_{\substack{i_0+\ldots+i_r=s\\ i_j\leqslant n_j}}\wedge^{i_0,\ldots,i_r} \tag{2.5}$$

where $n_j=\dim\mathfrak{a}^{(j)}$ and $\wedge^{i_0,\ldots,i_r}=\wedge^{i_0}(\mathfrak{a}^{(0)*})\wedge\ldots\wedge\wedge^{i_r}(\mathfrak{a}^{(r)*})$.

We will use decomposition (2.5) to study $H^1(\mathfrak{g})$, $H^2(\mathfrak{g})$ and $H^{2n-1}(\mathfrak{g})$. Recall that for $\alpha\in\wedge^k(\mathfrak{g}^*)$ and $X_1,\ldots,X_{k+1}\in\mathfrak{g}$, we have

$$d\alpha(X_1,\ldots,X_{k+1})=\sum_{1\leqslant i<j\leqslant k+1}(-1)^{i+j}\alpha([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{k+1}) \tag{2.6}$$

It is clear from (2.6) that

$$H^1(\mathfrak{g})=(\mathfrak{g}^{(1)})^{\perp}=\wedge^{1,0,\ldots,0}. \tag{2.7}$$

Next we consider $H^2(\mathfrak{g})$.

LEMMA 2.8. *Any closed 2-form σ belongs to $\wedge^{1,0,\ldots,1}+\Sigma\wedge^{i_0,\ldots,i_{r-1},0}$.*

Proof. Write $\sigma=\sigma_1+\sigma_2$ with $\sigma_1\in\Sigma\wedge^{i_0,\ldots,i_{r-1},1}+\Sigma\wedge^{i_0,\ldots,i_{r-1},2}$ and $\sigma_2\in\Sigma^{i_0,\ldots,i_{r-1},0}$. For $X,Y\in\mathfrak{g}$, $Z\in\mathfrak{g}^{(r)}$, equation (2.6) yields $d\sigma(X,Y,Z)=-\sigma([X,Y],Z)=-\sigma_1([X,Y],Z)$ since Z is central. Hence if σ is closed, then $\sigma_1(U,Z)=0$ for all $U\in\mathfrak{g}^{(1)}$, $Z\in\mathfrak{g}^{(r)}$. Thus $\sigma_1\in\wedge^{1,0,\ldots,0,1}$ and the lemma is proved. ■

Choose a basis $\lambda_1,\ldots,\lambda_{n_r}$ of $\wedge^{0,\ldots,0,1}$. By Lemma 2.8, the symplectic form ω can be written as

$$\omega=\beta_1\wedge\lambda_1+\ldots+\beta_{n_r}\wedge\lambda_{n_r}\text{ modulo }\Sigma\wedge^{i_0,\ldots,i_{r-1},0}. \tag{2.9}$$

for some $\beta_1,\ldots,\beta_{n_r}\in\wedge^{1,0,\ldots,0}$. Non-degeneracy of ω shows that $\beta_1,\beta_2,\ldots,\beta_{n_r}$ are linearly independent and thus can be extended to a basis

$$\beta_1,\ldots,\beta_{n_r},\ldots,\beta_{n_0} \tag{2.10}$$

for $\wedge^{1,0,\ldots,0}=\mathfrak{a}^{(0)*}$. We note in passing that this discussion shows that $\dim H^1(\mathfrak{g})=n_0\geqslant n_r=\dim\mathfrak{g}^{(r)}$ is a necessary condition for the existence of a symplectic form on an $(r+1)$-step nilpotent Lie algebra $\mathfrak{g}$.

Lemma 2.11 addresses the structure of $H^{2n-1}(\mathfrak{g})$.

LEMMA 2.11. *Every form $\sigma\in\wedge^{2n-1}(\mathfrak{g}^*)$ is closed. If σ is also exact, then σ is divisible by $\beta_1\wedge\ldots\wedge\beta_{n_0}$.*

Proof. By (2.6), if $\alpha\in(\mathfrak{g}^{(i)})^*=(\mathfrak{a}^{(i)}+\ldots+\mathfrak{a}^{(r)})^*$, then $d\alpha\in\mathfrak{b}^{(i-1)*}$. Hence if $\eta\in\wedge^{i_0,\ldots,i_r}$, then each term of $d\eta$ belongs to a space $\wedge^{k_0,\ldots,k_r}$ where for some $j\geqslant 1$, we have $k_j=i_j-1$ and $k_0+\ldots+k_{j-1}=i_0+\ldots+i_{j-1}+2$. In particular, if $i_0+\ldots+i_r=2n-1$, then $i_0+\ldots+i_{j-1}\geqslant\dim\mathfrak{b}^{(j-1)}-1$, so $k_0+\ldots+k_{j-1}>\dim\mathfrak{b}^{(j-1)}$ and $d\eta=0$. Thus the first statement of the lemma follows form (2.5). Next, if $i_0+\ldots+i_r=n-2$, then any non-zero term of $d\eta$ must satisfy $k_0+\ldots+k_{j-1}=\dim\mathfrak{b}^{(j-1)}$ and, in particular, $k_0=n_0$. Hence $d\eta\in\Sigma\wedge^{n_0,i_1,\ldots,i_r}$ and the lemma follows. ■

To complete the proof of Theorem A, we show that if $\sigma \in \wedge^{2n-1}(\mathfrak{g}^*)$ is divisible by $\lambda_1 \wedge \ldots \wedge \lambda_{n_r}$ but not by $\beta_1 \wedge \ldots \wedge \beta_{n_r}$, then $[\sigma]$ is not in the image of the map $\wedge [\omega]^{n-1}: H^1(\mathfrak{g}) \to H^{2n-1}(\mathfrak{g})$. In view of equation (2.7) and Lemma 2.11, it suffices to verify that for any $\gamma \in \wedge^{1,0,\ldots,0}$, $\gamma \wedge \omega^{n-1}$ does not differ from σ by a form divisible by $\beta_1 \wedge \ldots \wedge \beta_{n_0}$.

Write $\omega^{n-1} = \delta_1 + \delta_2$ where $\delta_1 \in \wedge^{n_0-2, n_1, \ldots, n_r}$ and $\delta_2 \in \Sigma \wedge^{n_0-1, i_1, \ldots, i_r} + \Sigma \wedge^{n_0, i_1, \ldots, i_r}$. Each term of δ_1 is divisible by $\lambda_1 \wedge \ldots \wedge \lambda_{n_r}$ and hence also by $\beta_1 \wedge \ldots \wedge \beta_{n_r}$, in view of equation (2.9). Since $\gamma \wedge \delta_2$ is divisible by $\beta_1 \wedge \ldots \wedge \beta_{n_0}$ we see that $\gamma \wedge \omega^{n-1} = \gamma \wedge \delta_1 + \gamma \wedge \delta_2$ is divisible by $\beta_1 \wedge \ldots \wedge \beta_{n_r}$ and hence can't be cohomologous to σ. ■

Proof of Theorem B. The hypotheses on Γ imply that there is a connected, simply connected, nilpotent Lie group G containing Γ as a discrete cocompact subgroup [10]. The group G is diffeomorphic to a Euclidean space [10] so that $\Gamma \backslash G$ is a $K(\Gamma, 1)$-manifold and hence homotopy equivalent to M. Theorem A shows that Γ is abelian and $\Gamma \backslash G$ is a torus. ■

There are compact manifolds homotopy equivalent to tori but not diffeomorphic to tori [5]. We do not know if any of these fake tori admit Kähler structures.

A nilmanifold $\Gamma \backslash G$ is called 2-step if G is a 2-step group (i.e., $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$). It is shown in [13] that a compact manifold is a 2-step nilmanifold if and only if it is the total space of a principal torus bundle over a torus. This provides a corollary to Theorem A.

COROLLARY 2.12. *Let M be the total space of a non-trivial principal T^n-bundle over T^m. Then M does not admit a Kähler structure.*

The examples described in Section 3 show that many of the manifolds in Corollary 2.12 have symplectic structures.

3. SYMPLECTIC NILMANIFOLDS

A Lie group G acts on $\mathfrak{g}^*$, the dual of its Lie algebra, by the coadjoint action

$$Ad^*(g)f = f \circ Ad(g^{-1}) \tag{3.1}$$

for $g \in G, f \in \mathfrak{g}^*$. The orbit $O = O_f$ through a given $f \in \mathfrak{g}^*$ can be identified with G/G_f where $G_f = \{g \in G \mid Ad^*(g)f = f\}$. If f_0 and f_1 lie in the same orbit then G_{f_0} and G_{f_1} are conjugate subgroups of G. We call an orbit O *normal* if G_f is a normal subgroup of G for any (hence all) $f \in O$. In this case, O is a Lie group in a natural way and the identification $O \leftrightarrow G/G_f$ is canonical (independent of $f \in O$). Moreover, if G is connected, simply connected and nilpotent then so is O. We denote by $\pi_f: G \to O = O_f$ the projection given by $\pi_f(g) = Ad^*(g)f$. There is a homogeneous symplectic structure ω_O on O characterized by

$$\pi_f^*(\omega_O) = -df.$$

When O is a normal orbit, ω_O is left O-invariant. For further details on the orbit construction, we refer the reader to [6].

If a normal orbit O is rational then O will have a discrete cocompact subgroup Γ [10] and $\Gamma \backslash O$ inherits the symplectic structure ω_O from O. Theorem C states that these are the only nilmanifolds admitting symplectic structures. Before proving this, we will discuss some examples.

Example 3.3. The $(2n+1)$ dimensional Heisenberg group H_n has a Lie algebra $\mathfrak{g}$ with basis elements $X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z$ satisfying $[X_i, Y_i]=Z$ (with other brackets vanishing). The dual basis for $\mathfrak{g}^*$ will be denoted by $\mu_1, \ldots, \mu_n, v_1, \ldots, v_n, \lambda$. The orbit $O=O_\lambda$ is a normal orbit and $O \cong \mathbb{R}^{2n}$. To see this, note that the Lie algebra $\mathfrak{g}_\lambda$ of G_λ is $\mathfrak{g}_\lambda=\{v\in\mathfrak{g} \mid \lambda[v, -]=0\}=\mathrm{Span}(Z)$. This is an ideal in $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{g}_\lambda$ is abelian. The symplectic structure ω_O is $\omega_O=-d\lambda=\sum_{i=1}^{n} \mu_i \wedge v_i$. This shows how the torus T^{2n} with its usual symplectic structure can be obtained from an orbit.

Example 3.4. Let $G_{p,q}$ be the connected simply-connected group with Lie algebra given by $\mathfrak{g}=\mathrm{Span}(X_1, \ldots, X_p, Y, Z_1, \ldots, Z_p, X'_1, \ldots, X'_q, Y', Z'_1, \ldots, Z'_q, W)$ where the non-zero brackets are

$$\begin{aligned} [X_i, Y]&=Z_i\\ [X'_i, Y']&=Z'_i\\ [X_i, Z_i]&=W\\ [X'_i, Z'_i]&=W\\ [Y, Y']&=W \end{aligned} \tag{3.5}$$

We write $\mu_1, \ldots, \mu_p, v, \lambda_1, \ldots, \lambda_p, \mu'_1, \ldots, \mu'_q, v', \lambda'_q, \ldots, \lambda'_q, \gamma$ for the dual basis. One computes that $\mathfrak{g}_\gamma=\mathrm{Span}(W)$, which is an ideal (the center) in $\mathfrak{g}$ so that O_γ is a normal orbit. The Lie algebra of O_γ is $\mathfrak{g}/\mathfrak{g}_\gamma$ which is isomorphic to $\mathrm{Span}(\vec{X}, Y, \vec{Z}, \vec{X}', Y', \vec{Z}')$ where $[X_i, Y]=Z_i$ and $[X'_i, Y']=Z'_i$. The symplectic structure ω_O on $O_{p,q}=O_\gamma$ is given by

$$\omega_O=-d\gamma=\sum_{i=1}^{p} \mu_i \wedge \lambda_i+\sum_{i=1}^{q} \mu'_i \wedge \lambda'_i+v \wedge v'. \tag{3.6}$$

The resulting nilmanifolds $\Gamma\backslash O_{p,q}$ are the symplectic non-Kähler manifolds in [2] and [3]. In particular, $O_{1,0}\cong H_1\times\mathbb{R}$ gives Thurston's example [14]. Note that $O_{p,q}$ is a 2-step group so that one can view $\Gamma\backslash O_{p,q}$ as a principal torus bundle over a torus.

Proof of Theorem C. Let $(\Gamma\backslash G, \omega)$ be a given symplectic nilmanifold. As in the proof of Theorem A, Nomizu's Theorem shows that ω is cohomologous to a homogeneous symplectic structure $\omega_0\in \wedge^2(\mathfrak{g}^*)$. We will find a normal coadjoint orbit O and an isomorphim $\phi: O\to G$ with $\phi^*(\omega_0)=\omega_O$.

Let $\tilde{\mathfrak{g}}=\mathfrak{g}+\mathrm{Span}(W)$ where $\mathfrak{g}$ is the Lie algebra of G and W is a new generator. We define $[\,,\,]^\sim$ on $\tilde{\mathfrak{g}}$ in terms of ω_0 and $[\,,\,]$ on $\mathfrak{g}$ by

$$\begin{gathered}[X, W]^\sim=0 \quad \text{for all} \quad X\in\tilde{\mathfrak{g}}\\ \text{and } [X, Y]^\sim=[X, Y]+\omega_0(X, Y)W \quad \text{for} \quad X, Y\in\mathfrak{g}.\end{gathered} \tag{3.7}$$

It is not difficult to verify that $[\,,\,]^\sim$ satisfies the Jacobi identity by using the Jacobi identity for $[\,,\,]$ and the fact that $d\omega_0=0$. Moreover, $\tilde{\mathfrak{g}}$ is nilpotent and non-degeneracy of ω_0 shows that $\mathrm{Span}(W)$ is the center $\mathfrak{z}$ of $\tilde{\mathfrak{g}}$. One has an extension

$$0\to\mathfrak{z}\to\tilde{\mathfrak{g}}\to\mathfrak{g}\to 0. \tag{3.8}$$

Let $\tilde{G}$ be the connected simply connected Lie group with Lie algebra $\tilde{\mathfrak{g}}$. Let $f:\tilde{\mathfrak{g}}\to\mathbb{R}$ be the functional with $f(W)=1$ and $f(\mathfrak{g})=0$. From equations (3.7) one sees that $\tilde{\mathfrak{g}}_f=\{v\in\tilde{\mathfrak{g}} \mid f[v, -]=0\}=\mathfrak{z}$. Hence $O=O_f$ is a normal orbit with Lie algebra $\tilde{\mathfrak{g}}/\mathfrak{z}$. The canonical isomorphism $\tilde{\mathfrak{g}}/\mathfrak{z}\to\mathfrak{g}$ given by (3.8) can be exponentiated to obtain a Lie group isomorphism $\phi: O\to G$ (both O and G are simply connected).

For $X, Y \in \mathfrak{g}$ one has

$$\begin{aligned} df(X, Y) &= -f([X, Y]^{\sim}) \\ &= -f([X, Y] + \omega_0(X, Y)W) \\ &= -\omega_0(X, Y). \end{aligned}$$

It follows that $\phi^*(\omega_0) = \omega_0$ as claimed. ■

Homological algebra provides an abstract viewpoint on the above proof. The central extension (3.8) corresponds to $[\omega_0]$ under the isomorphism $\mathrm{Ext}^1(\mathfrak{g}, \mathbb{R}) \cong H^2(\mathfrak{g})$. We also remark that Theorem C can be established by appealing to Theorem 5.4.1 of [7], which classifies certain simply connected homogeneous symplectic manifolds.

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